Modeling and solving the Euclidean Steiner Problem

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Work done in collaboration with:
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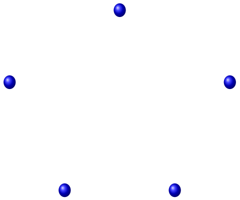
Problem Definition

Given p points in $\mathbb{R}^n$.

Euclidean Steiner Tree Problem

Find a tree with minimal Euclidean length that spans these points using or not extra points, which are called Steiner points.



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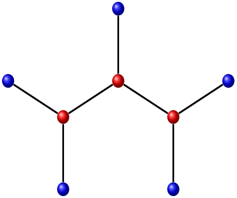
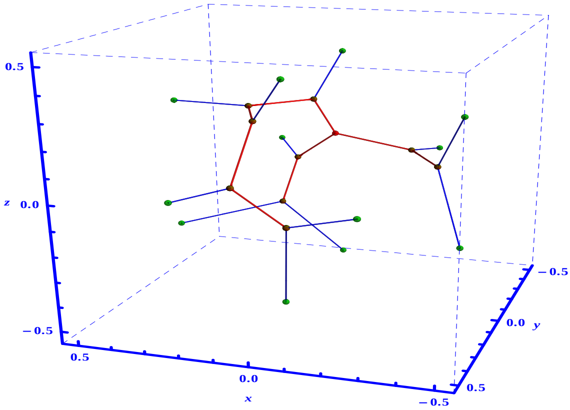


Example in $\mathbb{R}^3$

Terminals are the 12 vertices of an icosahedron



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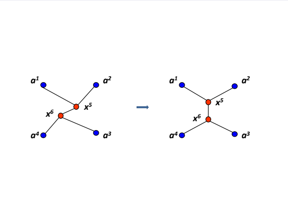
Determine the *Steiner Minimal Tree* (SMT):

- The number of Steiner points to be used on the minimal tree.
- The edges of the tree.
- Geometric position of the Steiner points.

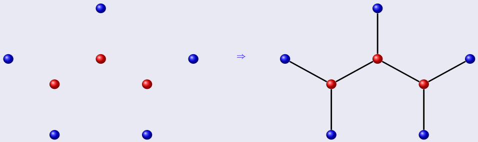
The Euclidean Steiner Problem for a Given Topology



$$\text{Minimize } ||a^1 - x^5|| + ||a^2 - x^5|| + ||x^5 - x^6|| + ||a^3 - x^5|| + ||a^4 - x^6||$$
$$\text{subject to: } x^5, x^6 \in \mathbb{R}^n$$

The shortest tree for a given topology $\mathcal{T}$ is called a *Relatively Minimal Tree* (RMT) for $\mathcal{T}$

The minimum spanning tree



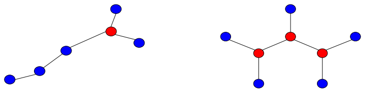
Properties:

In a SMT for p given terminals:

- No angle between edges is < 120 degrees.
- Each terminal has degree between 1 and 3.
- Each Steiner point has degree equal to 3.
- The Steiner point and its 3 adjacent nodes lie in a plane.
- The number of Steiner points is no more than $p - 2$.
- If the number of Steiner points is exactly $p - 2$, each terminal has degree equal to 1.

Definition:

- A *Full Steiner Topology* (FST) for p terminals is a topology with $p - 2$ Steiner points, where each Steiner point has degree 3 and each terminal has degree 1.



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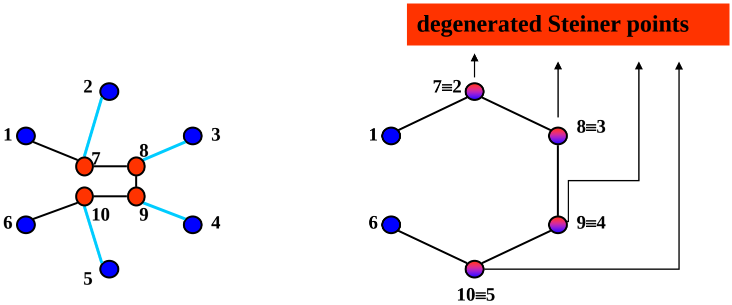
Degenerate Steiner Topologies

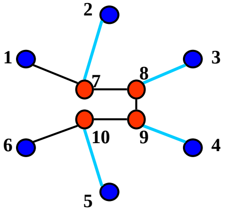
A topology is called a degeneracy of another if the former can be obtained from the latter by shrinking edges.

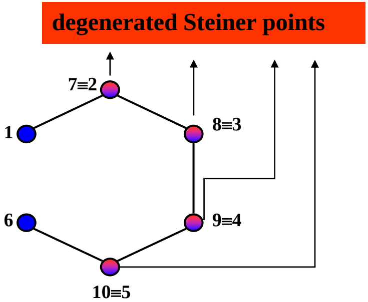


Degenerate Steiner Topologies

A topology is called a degeneracy of another if the former can be obtained from the latter by shrinking edges.

Fact: Given an SMT, an FST always exists such that the SMT is an RMT for it.





Gilbert and Pollak (1968)

- Find all the full Steiner topologies on the p given terminals.
- For each topology optimize the coordinates of the Steiner points.
- Output: the shortest tree found.

E. N. Gilbert and H. O. Pollak, *Steiner minimal trees*, SIAM J. Applied Math., vol. 16, pp. 1-29, 1968.

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The number of full Steiner topologies for p terminals is

$$t(p) := 1 \cdot 3 \cdot 5 \cdot 7 \ldots (2p-5) = (2p-5)!!$$

$t(2) = 1,\ t(4) = 3,\ t(6) = 105,\ t(8) = 10395,\ t(10) = 2,027,025,\ t(12) = 654,729,075$

Exact Algorithms (general n-space)

Smith (1992)

An implicit enumeration scheme to generate full Steiner topologies.

W. D. Smith, *How to find Steiner minimal trees in Euclidean d-space*, Algorithmica, vol. 7, pp. 137-177,1992.

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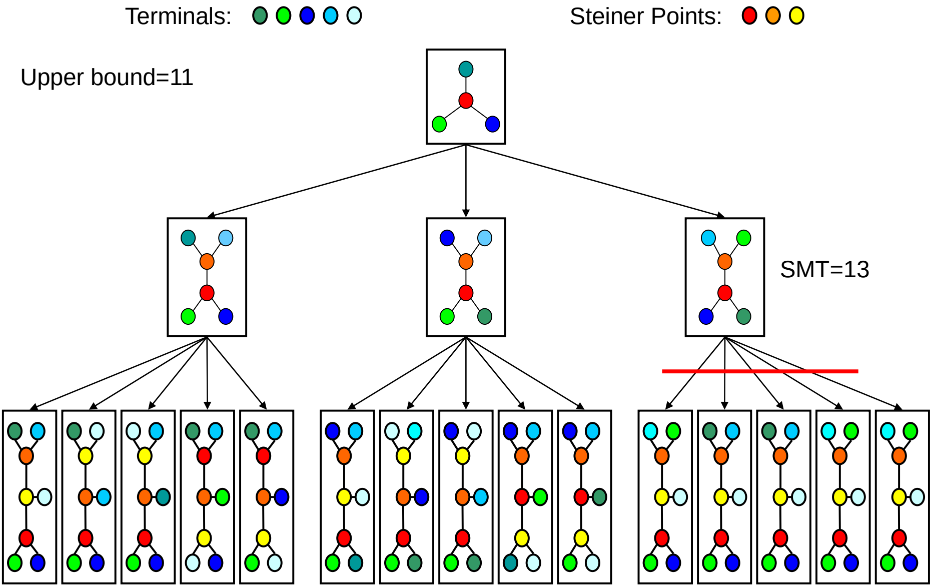
Terminals: Steiner Points:

Upper bound=11



- Use of a conic formulation for the subproblems of locating the Steiner points
- Implementation of a strong branching strategy
- New order in which the terminals are added

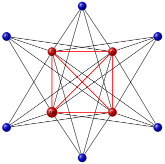
M. Fampa and Kurt M. Anstreicher, *An improved algorithm for computing Steiner minimal trees in Euclidean d-space*, Discrete Optimization 5(2), 530-540, 2008.

- Good: Enumeration scheme.
- Bad: The pruning criterion.
  - Lower bounds for the subproblems are given by the length of the RMTs corresponding to FSTs on only a subset of the terminals - weak bounds.
  - Few nodes are pruned until deep down in the B&B tree.
  - Growth of tree is super-exponential with depth.
  - Subproblems get larger at deeper levels.

Maculan, Michelon, Xavier (2000)

Given p points in $\mathbb{R}^n$ $(a^1, \ldots, a^n)$, let $G = (V, E)$ be the graph where

- $P := \{1, 2, \ldots, p-1, p\}$ is the set of indices associated with p terminals;
- $S := \{p+1, p+2, \ldots, 2p-3, 2p-2\}$ is the set of indices associated with $p-2$ Steiner points;
- $V := P \cup S$;
- $E_1 := \{[i,j] | i \in P, j \in S\}$;
- $E_2 := \{[i,j] | i \in S, j \in S\}$;
- $E := E_1 \cup E_2$.



Maculan, Michelon, Xavier (2000)

(MMX): Minimize $\sum_{[i,j]\in E_1} ||a^i - x^j|| y_{ij} + \sum_{[i,j]\in E_2} ||x^i - x^j|| y_{ij}$
subject to

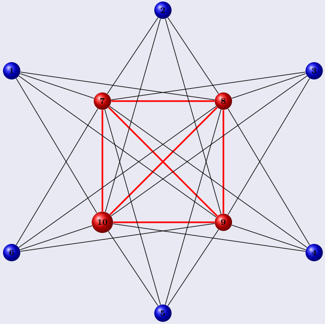
$$\begin{aligned} \sum_{j\in S} y_{ij} &= 1, \quad i \in P \\ \sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} &= 3, \quad j \in S \\ \sum_{k<j, k\in S} y_{kj} &= 1, \quad j \in S - \{p+1\} \end{aligned}$$

$$x^i \in \mathbb{R}^n,\ i \in S, \quad y_{ij} \in \{0,1\}, \quad [i,j] \in E$$

N. Maculan, P. Michelon and A. E. Xavier, *The Euclidean Steiner problem in $\mathbb{R}^n$: A mathematical programming formulation*, Annals of Operations Research, vol. 96, pp. 209-220, 2000.

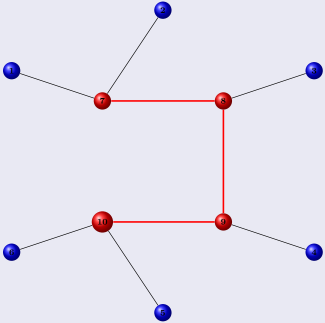
$G(V, E)$ for $p = 6$

- 6 terminals, 4 Steiner points, all possible edges;



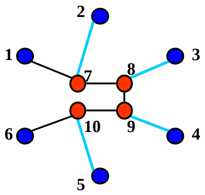
$G(V, E)$ for $p = 6$

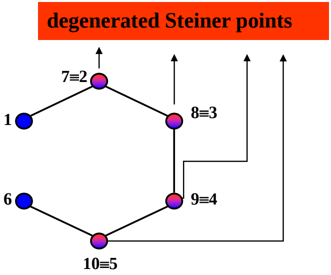
- 6 terminals, 4 Steiner points, all possible edges;
- a feasible solution;



$G(V, E)$ for $p = 6$

- 6 terminals, 4 Steiner points, all possible edges;
- a feasible solution;
- the optimal solution;





- Weak lower bounds for subproblems.
- Nondifferentiability at points where the FSTs degenerate - problem for algorithms that require the functions to be twice continuously differentiable.

C. D'Ambrosio, M. Fampa, J. Lee, and S. Vigerske. *On a nonconvex MINLP formulation of the Euclidean Steiner tree problems in n-space*, LNCS (SEA 2015), vol. 9125, pp. 122-133, 2015.

Non-combinatorial cuts

Let

- η_i be the distance from terminal a^i to the nearest other terminal.

Theorem

$$y_{ik}\|x^k - a^i\| \leq \eta_i, \text{ for all } i \in P,\ k \in S.$$

Theorem (Extend to the case where x^k is adjacent to fewer than two terminals)

$$y_{ik}y_{jk}\left(\|x^k - a^i\| + \|x^k - a^j\|\right) \leq 2\|a^i - a^j\|/\sqrt{3},$$
$$\text{for all } i,j \in P,\ i < j,\ k \in S.$$

Theorem (Extend to $n > 3$)

For $n = 3$ and $i,j \in P,\ i < j,\ k,l \in S, k < l$

$$y_{ik}y_{jk}y_{kl} \cdot \det \begin{bmatrix} a^i & a^j & x^k & x^l \\ 1 & 1 & 1 & 1 \end{bmatrix} = 0.$$

Combinatorial cuts

Let

- η_i be the distance from terminal a^i to the nearest other terminal.
- T be a minimum-length spanning tree on the terminals.
- β_{ij} be the length on the longest edge on the path between a^i and a^j in T.

Theorem

For $i, j \in P,\; i < j$,

$$\text{If } \; \|a^i - a^j\| > \eta_i + \eta_j, \;\; \textit{then} \;\; y_{ik} + y_{jk} \leq 1, \text{ for all } k \in S.$$

Lemma - well known

An SMT contains no edge of length greater than β_{ij} on the path between a^i and a^j, for all $i, j \in P$.

Corollary

For $i, j \in P,\; i \neq j$,

$$\text{If } \|a^i - a^j\| > \eta_i + \eta_j + \beta_{ij}, \;\; \textit{then} \;\; y_{ik} + y_{kl} + y_{jl} \leq 2, \text{ for all } k, l \in S, \; k < l.$$

Remarks on Nonconvex models/Global solvers

- Good:
  - Capacity and facility to formulate geometric cuts and other valid inequalities to the model.
  - The valid cuts proposed were effective in reducing the number of nodes in the B&B tree and the computational time on small instances.
- Bad: With nonconvex constraints we lose a nice feature of the problem - *it can be efficiently solved once the topology (binary variables) is fixed.*

Fampa and Maculan (2004)

$$(FM): \quad \text{Minimize} \sum_{[i,j]\in E} d_{ij} \text{ subject to}$$

$$\begin{aligned}
d_{ij} &\geqslant ||a^i - x^j|| - M(1-y_{ij}), \ [i,j] \in E_1 \\
d_{ij} &\geqslant ||x^i - x^j|| - M(1-y_{ij}), \ [i,j] \in E_2 \\
\sum_{j\in S} y_{ij} &= 1, \quad i \in P \\
\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} &= 3, \quad j \in S \\
\sum_{i<j, i\in S} y_{kj} &= 1, \quad j \in S - \{p+1\}
\end{aligned}$$

$$x^i \in \mathbb{R}^n, \ i \in S, \ \ y_{ij} \in \{0,1\}, \ \ [i,j] \in E, \ \ d_{ij} \in \mathbb{R}_+, \ \ [i,j] \in E$$

M. Fampa and N. Maculan, *Using a conic formulation for finding Steiner minimal trees*, Numerical Algorithms, vol. 35, pp. 315-330, 2004.

Drawbacks in applying MINLP solvers to FM

- Weak lower bounds for subproblems.
- Isomorphic subproblems.

M. Fampa, J. Lee, and W. Melo. *A specialized branch-and-bound algorithm for the Euclidean Steiner tree problem in n-space*, Computational Optimization and Applications, vol. 63(2) DOI 10.1007/s10589-016-9835-z, 2016.

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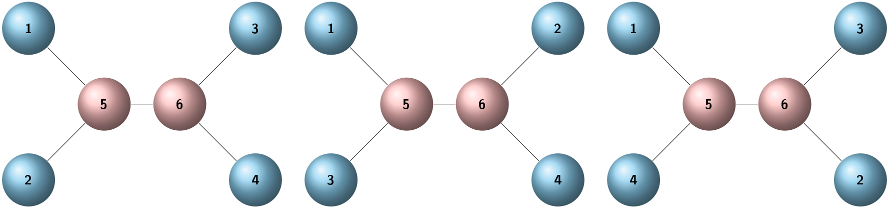
Create a set of representative FSTs, with one topology saved for each group of isomorphic FSTs.



Later, during the B&B execution, we only solve subproblems corresponding to these representative topologies, pruning all the others.

Number of Representative FSTs vs Number of FSTs in the feasible set of FM

p	Rep. FSTs	FSTs in FM	Reduction (%)
4	3	6	50
5	15	30	50
6	105	450	76
7	945	9450	90
8	10395	264600	96
9	135135	9525600	98
10	2027025	428652000	99

Discarding representative FSTs based on geometric conditions satisfied by SMTs

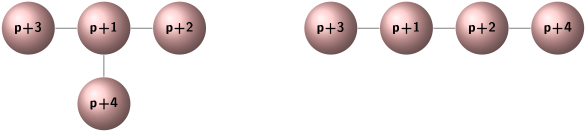
Let

- η_i be the minimum Euclidean distance between a^i and all other terminals.
- T be a minimum-length spanning tree on the terminals.
- β_{ij} be the length on the longest edge on the path between a^i and a^j in T.

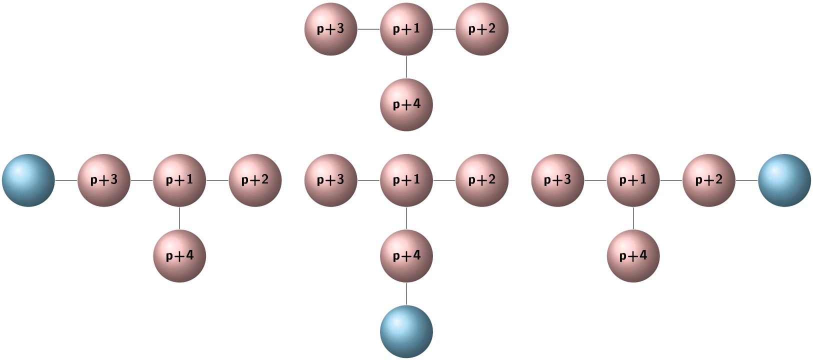
Then

1. Two terminals a^i and a^j may be connected to a common Steiner point only if
$$\|a^i - a^j\| \leq \eta_i + \eta_j.$$
2. Two terminals a^i and a^j may be connected by two or fewer Steiner points only if
$$\|a^i - a^j\| \leq \eta_i + \eta_j + \beta_{ij}.$$

B&B enumeration scheme starts, at its first level, with one node corresponding to each representative of spanning trees of Steiner points only.



Each node on the enumeration tree has at most $p-2$ children. A terminal is connected to a different Steiner point (with degree less than 3) at each child of the node.

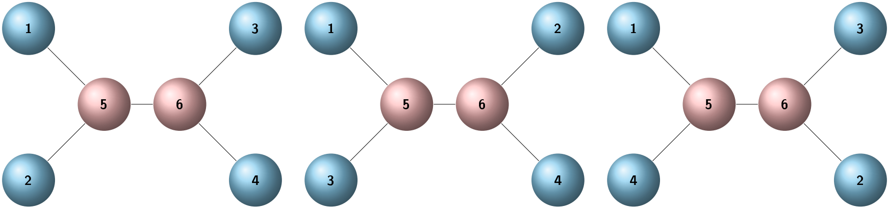


Pruning by isomorphism

Prune nodes where the variables fixed at 1 do not correspond to edges in any representative FST.

Fixing variables

Let the following representative FSTs, be on the set of descendants of a given node



Fix

$$y_{1,5} = 1, \qquad y_{1,6} = 0$$

Tabela: Average results on 10 instances of each dimension

Dimension	Muriqui Branch-and-Bound		SAMBA	
	gap (%)	cpu time (sec)	gap (%)	cpu time (sec)
$n = 3$	84	14400.38	0	240.96
$n = 4$	84	14401.12	0	740.87
$n = 5$	82	14400.34	0	1301.54

- **Muriqui - standard B&B algorithm for convex MINLP**
- **SAMBA - Steiner Adaptations on Muriqui B&B Algorithm**
- **Time limit of 4 hours (3.60 GHz core i7-4790 CPU, 8 MB, 16 GB, Linux)**

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- Capacity of formulating valid inequalities to strengthen the model.
- Solution of non isomorphic subproblems only.

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