Positive semidefinite rank

Hamza Fawzi (MIT, LIDS)

Joint work with João Gouveia (Coimbra), Pablo Parrilo (MIT),
Richard Robinson (Microsoft), James Saunderson (Monash), Rekha Thomas (UW)

DIMACS Workshop on Distance Geometry
July 2016

Euclidean distance matrices

Theorem (Schoenberg, 1935)

M is an Euclidean distance matrix if and only if $\operatorname{diag}(M) = 0$ *and* $[M_{1,i} + M_{1,j} - M_{i,j}]_{2 \leq i,j \leq n}$ *is positive semidefinite.*

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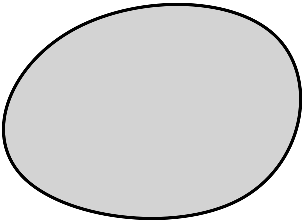
Allows us to express certain distance geometry problems as semidefinite programs

$\rightarrow$ Which convex sets can be "represented" using semidefinite programming?



Semidefinite representation

- Feasible set of a semidefinite program:

$$\begin{cases} X \succeq 0 \text{ (positive semidefinite constraint)} \\ \mathcal{A}(X) = b \text{ (linear constraints)} \end{cases}$$

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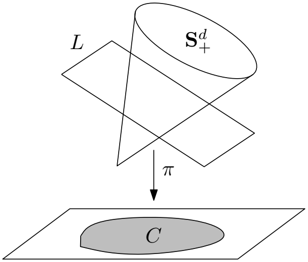
- Convex set C has a **semidefinite representation of size** d if:

$$C = \pi(\mathbf{S}_+^d \cap L)$$

$\mathbf{S}_+^d = d \times d$ positive semidefinite matrices

$L =$ affine subspace

$\pi =$ linear map



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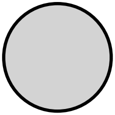
$$x^2 + y^2 \leq 1 \quad \Leftrightarrow \quad \begin{bmatrix} 1 - x & y \\ y & 1 + x \end{bmatrix} \succeq 0$$



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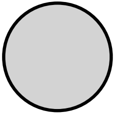
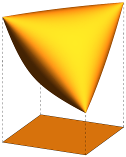


- Square $[-1,1]^2$ has a SDP representation of size 3

$$[-1,1]^2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \exists u \in \mathbb{R} \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & u \\ x_2 & u & 1 \end{bmatrix} \succeq 0 \right\}$$



Existential question vs. complexity question

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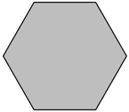
Importance of lifting



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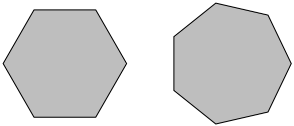

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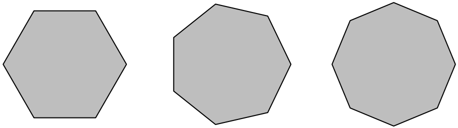

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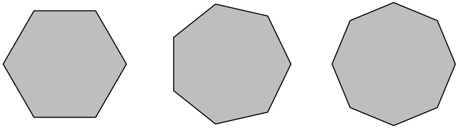

Ben-Tal and Nemirovski: Regular polygon with 2^n sides can be described using only $\approx n$ inequalities!

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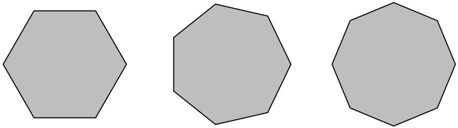

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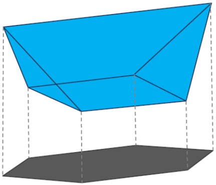


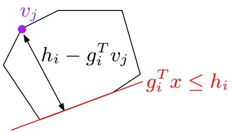
Lift = "inverse" of elimination (cf. Pablo's talk)

Lifts of polytopes and ranks of matrices

P polytope in $\mathbb{R}^d$

Slack matrix of P: Nonnegative matrix M of size $\#\text{facets}(P) \times \#\text{vertices}(P)$:

$$M_{i,j} = h_i - g_i^T v_j$$



where

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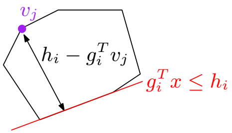

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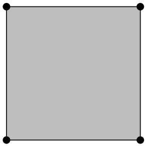

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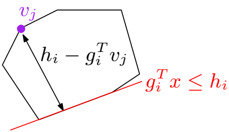

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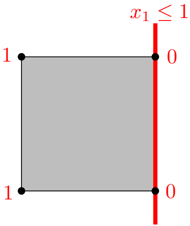

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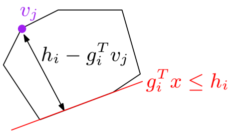

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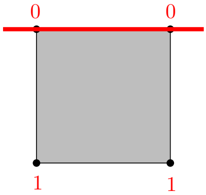
0	1	1	0
0	0	1	1



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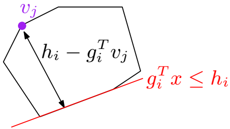

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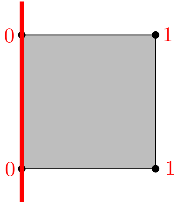

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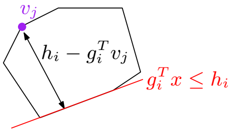

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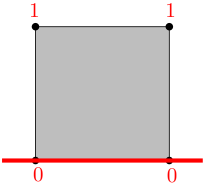


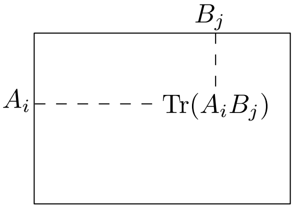
Positive semidefinite rank

$M \in \mathbb{R}^{p \times q}$ with nonnegative entries

- Positive semidefinite factorization:

$$M_{ij} = \mathrm{Tr}(A_i B_j), \quad \text{where} \quad A_i, B_j \in \mathbf{S}_+^k$$

- $\mathrm{rank}_{\mathrm{psd}}(M)$ = size of smallest psd factorization



Example

Consider $M_{ij} = (i-j)^2$ for $1 \leq i, j \leq n$:

$$M = \begin{bmatrix} 0 & 1 & 4 & 9 & 16 \\ 1 & 0 & 1 & 4 & 9 \\ 4 & 1 & 0 & 1 & 4 \\ 9 & 4 & 1 & 0 & 1 \\ 16 & 9 & 4 & 1 & 0 \end{bmatrix}$$

- $\text{rank}_{\text{psd}}(M) = 2$ (independent of n): Let

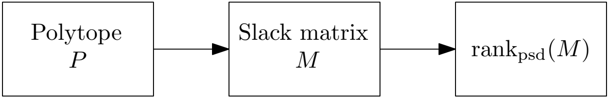
$$A_i = \begin{bmatrix} 1 & i \\ i & i^2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}^T \text{ and } B_j = \begin{bmatrix} j^2 & -j \\ -j & 1 \end{bmatrix} = \begin{bmatrix} -j \\ 1 \end{bmatrix} \begin{bmatrix} -j \\ 1 \end{bmatrix}^T.$$

  One can verify that $M_{ij} = \text{Tr}(A_j B_j)$.

SDP representations and psd rank

Theorem (Gouveia, Parrilo, Thomas, 2011)

Let P be polytope with slack matrix M. The smallest semidefinite representation of P has size exactly $\mathrm{rank}_{\mathrm{psd}}(M)$.



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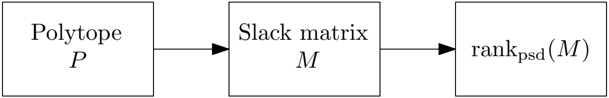

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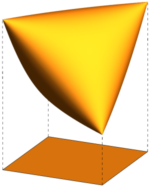
Example:

- Slack matrix of square $[-1, 1]^2$ has positive semidefinite rank 3.



Properties of rank_{psd}

- Satisfies the usual properties one would expect for a rank (invariance under scaling, subadditivity, etc.)

  [Fawzi, Gouveia, Parrilo, Robinson, Thomas, Positive semidefinite rank, Math. Prog., 2015]

- Connection with problems in information theory

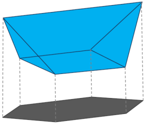
- NP-hard to compute *[Shitov, 2016]*

Linear programming (LP) lifts

- Polytope P has LP lift of size d if it can be written as

$$P = \pi(\mathbb{R}^d_+ \cap L)$$

where L affine subspace and π linear map



- Nonnegative factorization of M of size d:

$$M_{ij} = a_i^T b_j \quad \text{where} \quad a_i, b_j \in \mathbb{R}^d_+$$

$\text{rank}_+(M)$:= size of smallest nonnegative factorization of M

Theorem (Yannakakis, 1991)

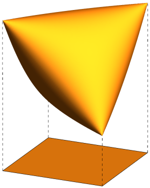
Let P be polytope with slack matrix M. The smallest LP lift of P has size exactly $\text{rank}_+(M)$.

LP lifts vs. SDP lifts

Example The square $P = [-1,1]^2$:

- **SDP lifts**: P has an SDP lift of size 3:

$$[-1,1]^2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \exists u \in \mathbb{R} \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & u \\ x_2 & u & 1 \end{bmatrix} \succeq 0 \right\}$$



SDP lift of size 3.

- **LP lifts**: Can show that any LP lift of $[-1,1]^2$ must have size 4.

Stable set polytope for perfect graphs: SDP lift of linear size (Lovász) but no currently known LP lift of polynomial size

LP lifts vs. SDP lifts

Question: How powerful are SDP lifts compared to LP lifts?

Theorem (Fawzi, Saunderson, Parrilo, 2015)

There is a family of polytopes $P_d \subset \mathbb{R}^{2d}$ such that

$$\frac{\text{rank}_{\text{psd}}(P_d)}{\text{rank}_+(P_d)} \leq O\left(\frac{\log d}{d}\right) \to 0.$$

- P_d = trigonometric cyclic polytope (generalization of regular polygons)

- Construction uses tools from Fourier analysis + *sparse sums of squares*

Conclusion

- Semidefinite representations of convex sets
- Connection with matrix factorization
- Linear programming vs. semidefinite programming lifts for polytopes

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