affine flexes, steven gortler

•joint with bob connelly. and louis theran.

•given a graph G with n vertices and m edges.

•given a framework (G, p) in E^d with full span.

•def: the framework admits an affine flex if there is a d-dimensional affine, but not euclidean, transform A, such that (G, p) is equivalent to $(G, A(p))$

• • here are some examples in 3d and 2d

• • note that when there is an affine flex, there will always be a continuum of them, as in our examples.

•this is obviously a very special situtuation

•given the coordinates of a specific p, one can, in fact check for an affine flex using linear algebra

•the goal of this work is to better understand when this can happen.

one motivating situation

•def: a framework is universally rigid if there is no equivalent framework in any dimension except for congruences.

• • represent frameworks that can be found using SDP

•a stronger property is that of super stability

•def: an equilibrium stress matrix for (G, p) is an n-by-n symmetric matrix. with zero entries on all ij non-edge-pairs. with the all-ones vector in its kernel. with each of the coordinate n-vectors of p in its kernel.

• • so it can have rank at most $n - d - 1$.

•def: a framework is super stable if it has an equilibrium stress matrix that is of rank $n - d - 1$ that is PSD. plus the framework does not have an affine flex.

•t (con): super stability implies universal rigidity.

•so for super stability, we have to explicitly rule out the possibility of an affine flex.

alfakih's thm

•thm: (alf) Suppose (G, p) has has an equilibrium stress matrix that is of rank $n - d - 1$. If each vertex nbhd has a full affine span then the framework does not have an affine flex.

•so we can get super stability with just the stress matrix and local affine span.

quadric stuff

•def: we say that (G, p) has its edge directions on a conic at infinity if there is a non-zero d-by-d symmetric matrix Q such that, for each edge ij, we have $(p_i - p_j)^t Q(p_i - p_j) = 0$.

•thm: (con) a framework has an affine flex iff its edge directions lie on a conic at infinity.

•• in 2D, this means that the edges lie in at most 2 directions.

•def: we say that a framework is ruled by a quadric (or just ruled) if all of the points along all of the edges lie on a quadric.

•• in 2D, this means that the framework lies on 2 lines.

•• note: ruled => conic at infinity, since edge direction is just the intersection of the edge's line with the plane at infinity.

•• here are some examples and non examples in 2d and 3d.

main thm and main cor

•thm: suppose that (G, p) is "NAR" then it has an affine flex iff it is ruled.

• • proof is very simple, and i may get to it.

•it will turn out that a max rank equilibrium stress matrix implies that a framework is NAR, giving us the following corollary.

•cor: suppose (G, p) has has an equilibrium stress matrix that is of rank $n-d-1$. then it has an affine flex iff it is ruled.

•note that a ruled framework cannot have d vertices in general position each with full affine span nbhds.

• • so this corollary is stronger than alfakih's thm.

SAP

•this corrollary is also related to something called the strong arnold property of a matrix.

•indeed, the corollary can also be proven using a different recent theorem by alfakin on SAP together with an older theorem of Godcil on SAP

cone frameworks

•we can use our main cor to study the super stability of cone frameworks.

•def: we denote a cone framework of cone graph (in E^{d+1}) as $p_0 * (G, p)$. G denotes the subgraph induced by removing vetex 0, which is connected to all of the vertices in G. (we assume p_0, the cone vertex position, is not concidient with any of the points in p.)

•note: universal rigidity of a cone framework is the same as the uniqueness of an PSD matrix completion problem with known diagonal entries.

operations

•we can take a framework in E^d and cone it to create a cone framework in E^{d+1}.

•we can take a cone framework and slide it (avoiding p_0)

•we can take a cone framework E^{d+1} and slice it by sliding the vertices of G to lie in a hyperplane and then considering the framwork of G in E^d.

what is known

•if we cone a universally rigid framework, the result is universally rigid

•if we cone a super stable framework, the result is super stable

•if we slide a universally rigid cone framework, the result is universally rigid

•if we slide a super stable cone framework, the result is super stable

•if we slice a universally rigid cone framework, the result might not be universally rigid
• • this happens when a cone framwork does not have its edge directions on a conic at infinity but the slice does have its edges directions on a conic at infinity.

what about super stability under slicing

•lem: if $p_0 * (G, p)$ is super stable, the sliced result must have a max rank PSD equilibrium stress matrix

•main observation: if $p_0 * (G, p)$ is not ruled, then neither is the slice.

•thm: if we slice a super stable cone framework, the result is super stable

projective transforms

•c: if a framework is super stable, then so is the result after any invertible projective transform.

•proof: the projective transform can be modeled using coning, affine transforms in E^{d+1} followed by slicing.

NAR

•def: (G, p) is nbhd affine equivalent to (G, q) if for each vertex, there is an affine transform that maps its nbhd in p to its nbhd in q.

•def: (G, p) is affine congruent to (G, q) if there is a single affine transform that maps p to q.

•def: (G, p) is NAR if for any framework (G, q) to which (G, p) is nae to, we always have that (G, p) is ac to (G, q).

proof of main thm

•we will do the hard direction. if NAR and affine flex with conic Q, then ruled.

perturbation

•suppose that (G, p) has an affine flex, so that its edge directions are on a conic at infinity defined by Q.

•def: let $m(x) := x + [x^t Q x]v$

•lem: (G, p) is nae to $(G, m(p))$.

•the proof is just a two line calculation.

•proof: we have assumed

$$0 = (p_j - p_i)^t Q (p_j - p_i)$$

we get

$$p_j^t Q p_j = -p_i^t Q p_i + 2 p_i^t Q p_j$$

•Treating p_i as a constant, we see that $p_j^t Q p_j$ can be expressed as an affine function of p_j.

•Thus the action of m on the neigborhood of p_i can be modeled with an affine transform.

now add in the NAR assumption

•lem: suppose that (G, p) is NAR and has an affine flex, so that its edge directions are on a conic at infinity defined by Q. then (G, p) is ac to $(G, m(p))$.

what does this congruence imply

•lem: if (G, p) is to $(G, m(p))$ then all of the vertices must lie on a quadric with quadratic terms defined by Q.

•the proof is another 2 line calcuation

•proof: the ac means

$$[p_i^t Q p_i]v = Ap_i + t$$

where A is a d by d matrix.

•multiplying on the left by v^t we get

$$p_i^t Q p_i = \left[v^t A\right] p_i + v^t t$$

•which places p_i on a quadric

three linear points on a quadric

•for each edge, we have its two endpoints on a quadric.

•the edge direction $(p_i - p_j)$ is a point at infinity on this same line.

•and it is on the same quadric.

•since we have 3 colinear points on the conic, the entire line must be on the quadric.

•this gives us a ruled framework.

•QED.