# Polynomial DC decompositions

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# Difference of convex (dc) programming

Problems of the form

$$\min f_0(x)$$
$$s.t.\ f_i(x) \leq 0, i = 1, \ldots, m$$

where $f_i(x) := g_i(x) - h_i(x)$, $g_i, h_i$ convex for $i = 0, \ldots, m$.

where , convex for

**What if such a decomposition is not given?**

regularization) procedures of some kind. However, all the proofs we know are "constructive" in the sense that they indeed yield g and h satisfying (2.3) but could hardly be carried over computational aspects.

Hiriart-Urruty, 1985

stance. For one thing, the d.c. structure of a given problem is not always apparent or easy to disclose, and even when it is known explicitly, there remains for the problem solver the hard task of bringing this structure to a form amenable to computational analysis. However, since the d.c. structure is virtually universal, one can hope that

Tuy, 1995

# Difference of convex (dc) decomposition

- **Difference of convex (dc) decomposition**: given a polynomial $f$, find $g$ and $h$ such that

$$f = g - h,$$

where $g, h$ **convex** polynomials.

- **Questions:**
  - Does such a decomposition always exist?
  - Can I obtain such a decomposition efficiently?
  - Is this decomposition unique?

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# Existence of dc decomposition (1/4)

- A polynomial $p$ is a **sum of squares (sos)** if $\exists q_i$ polynomials, s.t.

$$p(x) = \sum q_i^2(x).$$

- A polynomial $p(x)$ of degree 2d is sos if and only if $\exists Q \succcurlyeq 0$ **such that**

$$p(x) = z(x)^T Q z(x)$$

where $z = [1, x_1, \dots, x_n, x_1 x_2, \dots, x_n^d]^T$ is the vector of monomials up to degree $d$.

- Testing whether a polynomial is sos is a **semidefinite program**.

# Existence of dc decomposition (2/4)

| $f(x)$ convex | $\Leftrightarrow$ | $H_f(x) \succcurlyeq 0, \forall x \in \mathbb{R}^n$ | $\Leftrightarrow$ | $y^T H_f(x) y \geq 0, \forall x, y \in \mathbb{R}^n$ | $\Leftarrow$ | $y^T H_f(x) y$ sos<br>**SOS-convexity** |
|---|---|---|---|---|---|---|

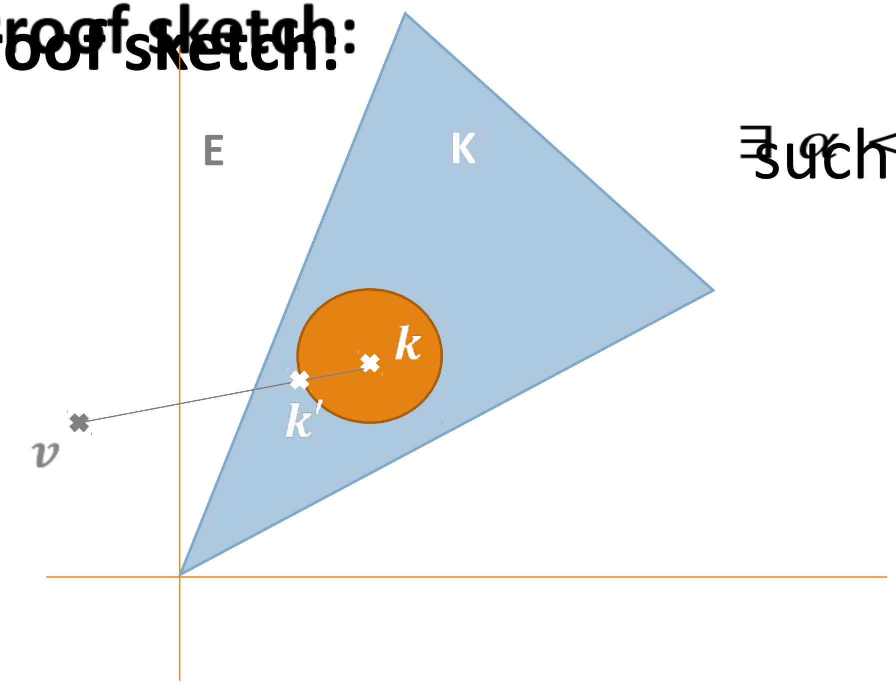
**Theorem:** Any polynomial can be written as the difference of two **sos-convex** polynomials.

**Corollary:** Any polynomial can be written as the difference of two convex polynomials.

# Existence of dc decomposition (3/4)

**Lemma:** Let $K$ be a full dimensional cone in a vector space $E$. Then any $v \in E$ can be written as $v = k_1 - k_2$, with $k_1, k_2 \in K$.

**Proof sketch:**




$\exists\, \alpha < 1$ such that $\overbrace{(1-\alpha)v + \alpha k}^{=:\,k'} \in K$

$$\Leftrightarrow v = \underbrace{\frac{1}{1-\alpha}k'}_{k_1 \in K} - \underbrace{\frac{\alpha}{1-\alpha}k}_{k_2 \in K}$$

# Existence of dc decomposition (4/4)

- Here, $E = \{$polynomials of degree 2d, in n variables$\}$,
$K = \{$sos-convex polynomials of degree 2d and in n variables $\}$.

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- Remains to show that $K$ is full dimensional:

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$\left(\sum_i x_i^2\right)^d$ can be shown to be in the interior of $K$. □

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solving $f = g - h$, $g, h$ sos-convex is an SDP.

sos-convex

- In fact, we show that a decomposition can be found via LP and SOCP (not covered here).

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# Uniqueness of dc decomposition

- **Dc decomposition**: given a polynomial $f$, find $g$ and $h$ convex polynomials such that $\boldsymbol{f = g - h}$.

- **Questions:**
  - Does such a decomposition always exist? ✓ Yes
  - Can I obtain such a decomposition efficiently? ✓ Through sos-convexity
  - Is this decomposition unique?

| Initial decomposition |
| --- |
| $f(x) = g(x) - h(x)$ |

→

| Alternative decompositions |
| --- |
| $f(x) = (g(x) + p(x)) - (h(x) + p(x))$<br>$p(x)$ convex |

**"Best decomposition?"**

# Convex-Concave Procedure (CCP)

- Heuristic for minimizing DC programming problems.

- **Idea:**

**Input**

$k := 0$

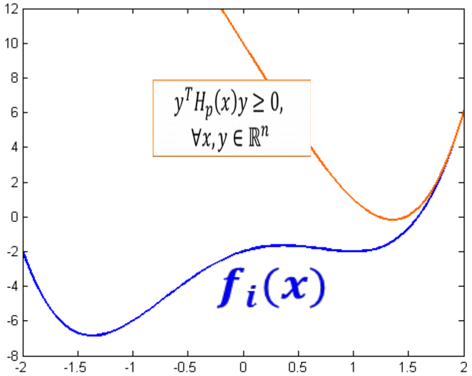
$x_0$, initial point

$f_i = g_i - h_i$,

$i = 0, \dots, m$

**Convexify by linearizing $h$**

$p(x)$ convex




$H_p(x) \succcurlyeq 0, \forall x$

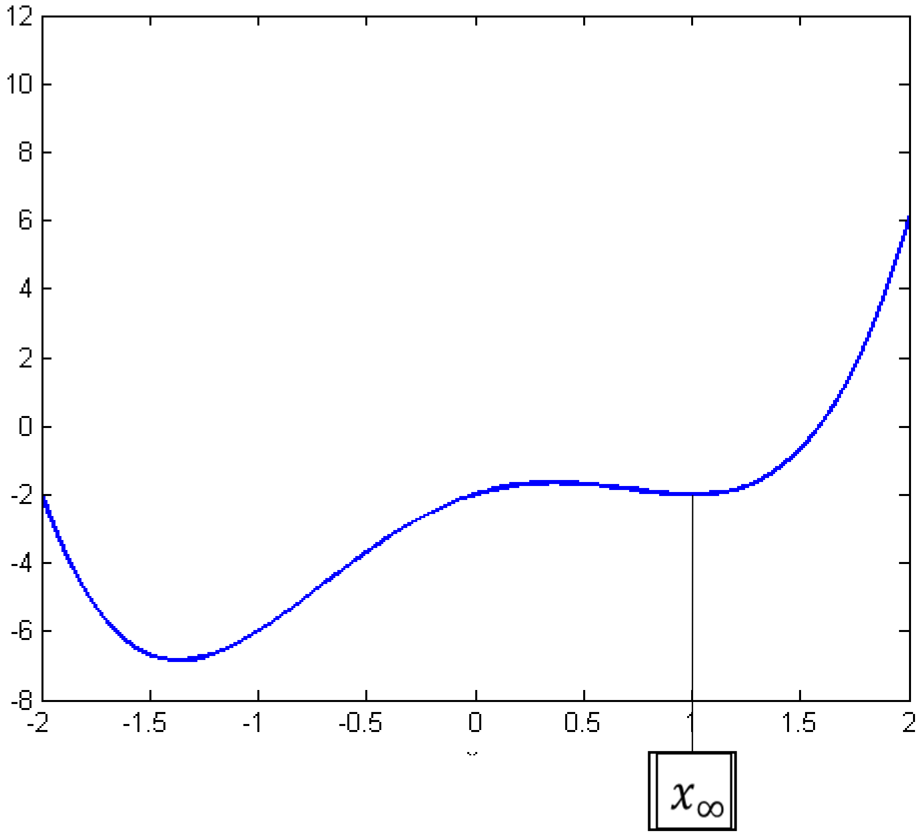
**Solve convex subproblem**

Take  to be the solution of

$\Longleftrightarrow$

# Convex-Concave Procedure (CCP)

- **Toy example:** $\min_{x} f(x)$, where $f(x) = g(x) - h(x)$




Reiterate

# Picking the “best” decomposition for CCP

**Algorithm**

Linearize $\boldsymbol{h(x)}$ around a point $x_k$ to obtain convexified version of $\boldsymbol{f(x)}$

**Idea**

Pick $h(x)$ such that it is as close as possible to affine around $x_k$

**Mathematical translation**

Minimize curvature of $h$ at $x_k$

**Worst-case curvature***

$$\min_{g,h} \lambda_{max}(H_h(x_k))$$
$$\text{s.t. } f = g - h$$
$$g, h \text{ convex}$$

*$\lambda_{max} H_h(x_k) = \max_{y \in S^{n-1}} y^T H_h(x_k) y$

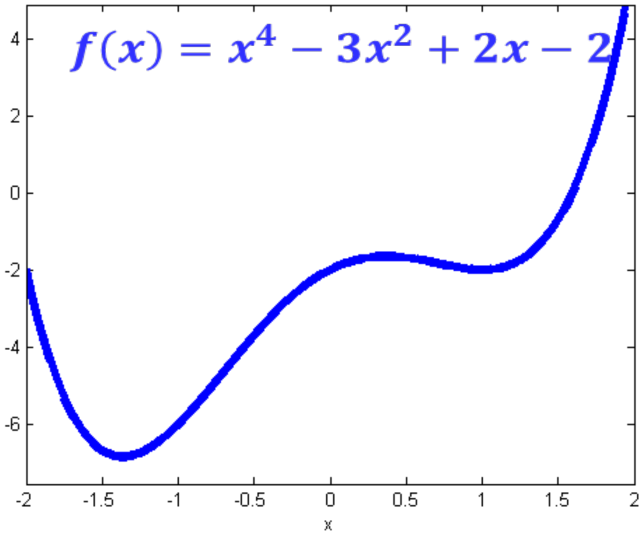
**Average curvature***

$$\min_{g,h} Tr\, H_h(x_k)$$
$$\text{s.t. } f = g - h,$$
$$g, h \text{ convex}$$

* $Tr\, H_h(x_k) = \int_{y \in S^{n-1}} y^T H_h(x_k) y \, d\sigma$

# Undominated decompositions (1/4)

Definition: $(g, h := g - f)$ is an undominated decomposition of $f$ if no other decomposition of $f$ can be obtained by subtracting a (nonaffine) convex function from $g$.




100 instances of an Erdös-Rényi graph with $(n, p) = (20, 0.5)$

Convexify around to get

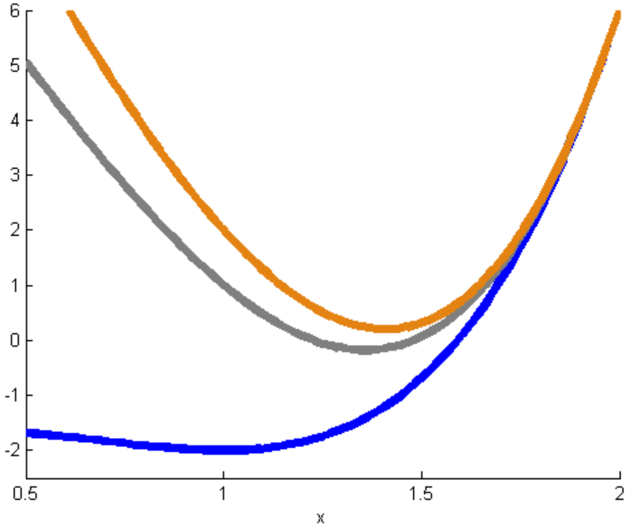
**DOMINATED BY**

$g'(x) = x^4$,

$h'(x) = 3x^2 + 2x - 2$

Convexify around $x_0 = 2$ to get $f^{0'}(x)$

Convexify around to get




If $g'$ dominates $g$ then the next iterate in CCP obtained using $g'$ **always beats** the one obtained using $g$.

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# Undominated decompositions (2/4)

$$D = \begin{pmatrix} 0 & d_{12} & * & d_{14} \\ d_{12} & 0 & d_{23} & * \\ * & d_{23} & 0 & d_{34} \\ d_{14} & * & d_{34} & 0 \end{pmatrix}$$

$$\boldsymbol{x_1} = \begin{pmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \end{pmatrix}, \boldsymbol{x_2} = \begin{pmatrix} x_2^1 \\ x_2^2 \\ x_2^3 \end{pmatrix}, \ldots$$

Incomplete distance matrix ⇒ Recover location of the $m$ points in $\mathbb{R}^d$

Solve:

$$\min_{x_i, x_j \in \mathbb{R}^d} \sum_{i<j} \left( \|x_i - x_j\|_2^2 - d_{ij}^2 \right)^2$$

There is a realization in $\mathbb{R}^d$ iff opt value$= 0$.

# Undominated decompositions (3/4)

$$\min_{x_i, x_j} \sum_{i<j} \underbrace{\left(\|x_i - x_j\|_2^2 - d_{ij}^2\right)^2}$$

$$= \underbrace{\|x_i - x_j\|_2^4 + d_{ij}^4}_{g_{ij}} - \underbrace{2d_{ij}^2 \cdot \|x_i - x_j\|_2^2}_{h_{ij}}$$

$\left(g = \sum_{i<j} g_{ij}, h = \sum_{i<j} h_{ij}\right)$ is an undominated dcd of the objective function.

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# Undominated decompositions (4/4)

- **Theorem:** Given a polynomial $f$, consider

$$(\star) \qquad \min_{g} \; \frac{1}{A_n} \int_{S^{n-1}} Tr\, H_g \, d\sigma, \text{ (where } A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}\text{)}$$
$$\text{s.t. } f = g - h,\; g \text{ convex},\; h \text{ convex}$$

Any optimal solution is an **undominated dcd** of $f$ (and an optimal solution always exists).

**Theorem:** If $f$ has degree 4, it is **strongly NP-hard** to solve $(\star)$.

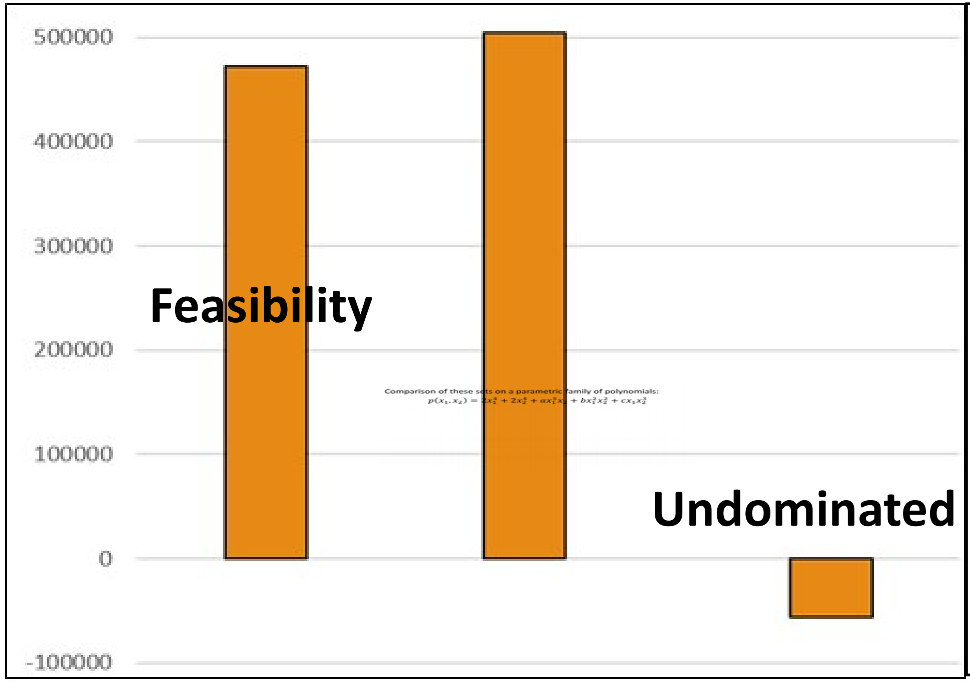
**Idea:** Replace $f = g - h$, $g, h$ convex by $f = g - h$, $g, h$ sos-convex.

# Comparing different decompositions (1/2)

- Solving the problem: $\min_{B=\{x\mid \|x\|\leq R\}} f_0$, where $f_0$ has $n = 8$ and $d = 4$.
- Decompose $f_0$, run CCP for 4 minutes and compare objective value.

| Feasibility | $\lambda_{max}H_h(x_0)$ | Undominated |
|---|---|---|
| $\min_{g,h} 0$ <br> s.t. $f_0 = g - h$ <br> $g, h$ sos-convex | $\min_{g,h} t$ <br> s.t. $f_0 = g - h$ <br> $g, h$ sos-convex <br> $tI - H_h(x_0) \succcurlyeq 0$ | $\min_{g,h} \frac{1}{A_n}\int_{S_{n-1}} Tr\ H_g\, d\sigma$ <br> s.t. $f_0 = g - h$ <br> $g, h$ sos-convex |

# Comparing different decompositions (2/2)




- Average over 30 iterations
- Solver: Mosek
- **Computer:** 8Gb RAM, 2.40GHz processor

**Conclusion:** Rate of convergence of CCP strongly affected by initial decomposition.

# Main messages

- We studied the question of **decomposing a polynomial into the difference of two convex polynomials**.
- This decomposition always **exists** and is **not** unique.
- **Choice of decomposition can** impact convergence rate of the CCP algorithm.
- **Dc decompositions can be efficiently obtained** using the notion of **sos-convexity** (SDP)
- Also possible to use **LP or SOCP-based** relaxations to obtain dc decompositions (not covered here).

# Thank you for listening

## Questions?

Want to learn more? http://scholar.princeton.edu/ghall