Rigidity of Graphs and Frameworks

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Bar-and-Joint Frameworks

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- A d-**dimensional bar-and-joint framework** is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$.
- We consider the framework to be a straight line **realization** of G in $\mathbb{R}^d$ in which the *length* of an edge $uv \in E$ is given by the Euclidean distance $\|p(u) - p(v)\|$ between the points $p(u)$ and $p(v)$.

Rigidity and Global Rigidity

Two d-dimensional frameworks (G, p) and (G, q) are:

- **equivalent** if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for all $uv \in E$;
- **congruent** if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for all $u, v \in V$.

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A framework (G,p) is:

- **globally rigid** if every framework which is equivalent to (G,p) is congruent to (G,p);
- **rigid** if there exists an $\epsilon > 0$ such that every framework (G,q) which is equivalent to (G,p) and satisfies $\|p(v)-q(v)\| < \epsilon$ for all $v \in V$, is congruent to (G,p). (This is equivalent to saying that every continuous motion of the vertices of (G,p) in $\mathbb{R}^d$, which preserves the lengths of all edges of (G,p), also preserves the distances between all pairs of vertices of (G,p).)

Rigidity: Example

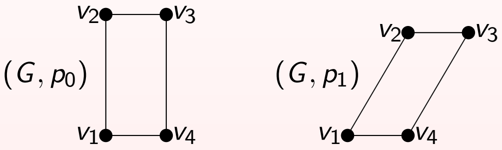

Figure: The framework (G, p_1) can be obtained from (G, p_0) by a continuous motion in $\mathbb{R}^2$ which preserves all edge lengths, but changes the distance between v_1 and v_3. Thus (G, p_0) is not rigid.

Global Rigidity: Example

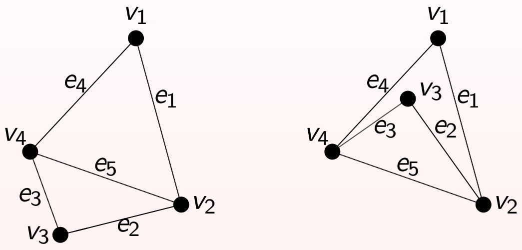

Figure: A rigid 2-dimensional framework which is not globally rigid. Corresponding edges in both frameworks have the same length, but the distances from v_1 to v_3 are different.

Complexity

- It is NP-hard to determine whether a given d-dimensional framework (G, p) is globally rigid for $d \geq 1$ (J. B. Saxe), or rigid for $d \geq 2$ (Abbot).

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- It is NP-hard to determine whether a given d-dimensional framework (G, p) is globally rigid for $d \geq 1$ (J. B. Saxe), or rigid for $d \geq 2$ (Abbot).
- These problems becomes more tractable if we restrict attention to **'generic'** frameworks (those for which the set of coordinates of all points $p(v)$, $v \in V$, is algebraically independent over $\mathbb{Q}$).

The Rigidity Matrix

The **rigidity matrix** $R(G, p)$ of a d-dimensional framework (G, p) is the $|E| \times d|V|$ matrix with rows indexed by E and sequences of d consecutive columns indexed by V.

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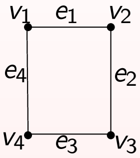
The entries in the row corresponding to an edge $e \in E$ and columns corresponding to a vertex $u \in V$ are given by the vector $p(u) - p(v)$ if $e = uv$ is incident to u and is the zero vector if e is not incident to u.

Rigidity matrix: Example



$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} p(v_1)-p(v_2) & p(v_2)-p(v_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p(v_2)-p(v_3) & p(v_3)-p(v_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p(v_3)-p(v_4) & p(v_4)-p(v_3) \\ p(v_1)-p(v_4) & \mathbf{0} & \mathbf{0} & p(v_4)-p(v_1) \end{pmatrix} \end{array}$$

Infinitesimal Motions

Each vector q in the null space of $R(G,p)$ is an **infinitesimal motion** of (G,p). Taking $q : V \to \mathbb{R}^d$ we have $[q(u) - q(v)] \cdot [p(u) - p(v)]$ for all $e = uv \in E$ so the vectors $q(u)$ are **instantaneous velocities** which preserve lengths of edges.

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$$\text{rank } R(G,p) \leq d|V| - \binom{d+1}{2},$$

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Hence

$$\text{rank } R(G,p) \leq d|V| - \binom{d+1}{2},$$

and (G,p) will be rigid if equality holds.
We say that G is **infinitesimally rigid** if

$$\text{rank } R(G,p) = d|V| - \binom{d+1}{2}.$$

Generic Rigidity and Independence

Theorem [Asimow and Roth, 1979]

Suppose (G, p) is a generic d-dimensional framework with $n \geq d + 1$ vertices. Then (G, p) is rigid if and only if it is infinitesimally rigid.

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We say that $G = (V, E)$ is **independent in** $\mathbb{R}^d$ if the rows of $R(G, p)$ are linearly independent for any generic (G, p). Similarly $F \subseteq E$ is **independent** if the rows of $R(G, p)$ indexed by F are linearly independent

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If we can determine (generic) independence in $\mathbb{R}^d$ then we can determine (generic) rigidity in $\mathbb{R}^d$.

Maxwell's condition and Laman's Theorem

Given a graph G and $X \subseteq V$ let $i(X)$ denote the number of edges of G joining the vertices of X.

Lemma [Maxwell's Condition]

If G is independent in $\mathbb{R}^d$ then $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d+1$.

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It is straightforward to show this condition characterises independence when $d = 1$. Laman showed this also holds when $d = 2$.

Theorem [Laman, 1970]

G is independent in $\mathbb{R}^2$ if and only if $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$.

The Lovász-Yemini Theorem

Laman's Theorem and a classical result of Edmunds from matroid theory characterise generic rigidity when $d = 2$. We need the following concept.

A t-**thin cover** of a graph G is a family $\mathcal{X}$ of subsets of V of size at least two such that each edge of G is induced by at least one set in $\mathcal{X}$ and $|X_i \cap X_j| \leq t$ for all distinct $X_i, X_j \in \mathcal{X}$.

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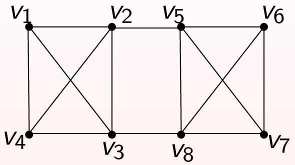
Theorem [Lovász and Yemini, 1982]

G is rigid in $\mathbb{R}^2$ if and only if

$$\sum_{X \in \mathcal{X}} (2|X| - 3) \geq 2|V| - 3$$

for all 1-thin covers $\mathcal{X}$ of G.

Example



Let $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$ where $X_1 = \{v_1, v_2, v_3, v_4\}$, $X_2 = \{v_5, v_6, v_7, v_8\}$, $X_3 = \{v_2, v_5\}$ and $X_4 = \{v_3, v_8\}$. Then

$$\sum_{X \in \mathcal{X}} (2|X| - 3) = 5 + 5 + 1 + 1 = 12 < 2|V| - 3$$

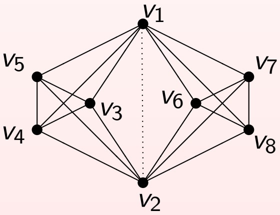
so G is not rigid in $\mathbb{R}^2$.

Independence in $\mathbb{R}^3$

The banana graph shows that Maxwell's condition,

$$i(X) \leq 3|X| - 6 \text{ for all } X \subseteq V \text{ with } |X| \geq 4,$$

does not characterise generic independence in $\mathbb{R}^3$.



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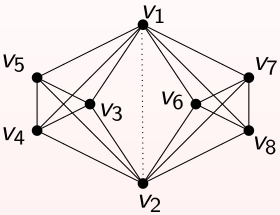

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We can use 2-thin covers to show that G is not independent in $\mathbb{R}^3$. Let $e = v_1v_2$, $X_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $X_2 = \{v_1, v_2, v_6, v_7, v_8\}$. Choose a maximum independent set I of edges of $G + e$ which contains e. Let $I_i = I \cap E(X_i)$. Then $|I_i| \leq 3|X_i| - 6 = 9$ for $i = 1, 2$. Since $e \in I_1 \cap I_2$ we have $I \leq 17$.
Hence rank $R(G + e, p) \leq 17$ for all (generic) p.

The Dress Conjecture

Dress conjectured that 2-thin covers can be used to characterise generic rigidity in $\mathbb{R}^3$. A **hinge** of a 2-thin cover $\mathcal{X}$ is a pair of sets $X_i, X_j \in \mathcal{X}$ such that $|X_i \cap X_j| = 2$. Let $h(\mathcal{X})$ denote the number of hinges of $\mathcal{X}$.

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Conjecture [Dress, 1983]

$G = (V, E)$ is rigid in $\mathbb{R}^3$ if and only if

$$\sum_{X \in \mathcal{X}} f(X) - h(\mathcal{X}) \geq 3|V| - 6 \qquad (1)$$

for all 2-thin covers $\mathcal{X}$ of G, where $f(X) = 1$ if $|X| = 2$ and otherwise $f(X) = 3|X| - 6$.

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Unfortunately this conjecture is FALSE - together with Jordán, we showed in 2005 that the LHS of (1) is *negative* when $|V|$ is large. (The problem is that 2-thin covers can have lots of hinges.)

Modified conjecture

The conjecture may become true if we restrict the number of hinges in the cover. The **hinge graph** of a 2-thin cover $\mathcal{X}$ is the graph $H_{\mathcal{X}}$ with vertex set $\mathcal{X}$, in which two sets $X_i, X_j \in \mathcal{X}$ are adjacent if they intersect in a hinge. The cover $\mathcal{X}$ is k-**degenerate** if $H_{\mathcal{X}}$ can be reduced to the empty graph by recursively deleting vertices of degree at most k.

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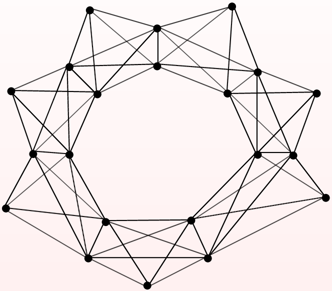
We can at least show that necessity holds i.e. inequality (2) must hold if G is rigid.

Example



Let $\mathcal{X} = \{X_1, X_2, \ldots, X_7\}$ where $G[X_i] = K_5$. Then $H_{\mathcal{X}} = C_7$ so $\mathcal{X}$ is a 2-degenerate, 2-thin cover. We have

$$\sum_{X \in \mathcal{X}} f(X) - h(\mathcal{X}) = 7 \times 9 - 7 = 56 < 3|V| - 6$$

so G is not rigid in $\mathbb{R}^3$.

Global Rigidity - Stress Matrix

A **stress** for a framework (G, p) is a map $w : E \to \mathbb{R}$ such that, for all $v \in V$,

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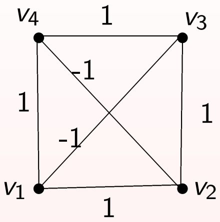
The associated **stress matrix** $S(G, p, w)$ is the $n \times n$ matrix with rows and columns indexed by V in which the entry corresponding to an edge $uv \in E$ is $-w_e$, all other off-diagonal entries are zero, and the diagonal entries are chosen to give zero row and column sums.

Stress Matrix: Example



$$
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$$

Generic Global Rigidity: Stress Matrix Characterisation

Theorem [Connelly (2005); Gortler, Healy and Thurston (2010)]

Let (G, p) be a d-dimensional framework with $n \geq d + 2$ vertices. Then

- rank $S(G, p, w) \leq n - d - 1$ for all stresses w for (G, p).
- When (G, p) is generic, (G, p) is globally rigid if and only if (G, p) has a stress w such that rank $S(G, p, w) = n - d - 1$.

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This result implies that the global rigidity of a generic framework (G, p) *depends only on the graph* G. We say that G is **globally rigid** in $\mathbb{R}^d$ if some (or equivalently, every) generic d-dimensional framework (G, p) is globally rigid.

Generic Global rigidity: Combinatorial Conditions

Theorem [Hendrickson (1992)]

Suppose G is globally rigid in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices, or G is $(d+1)$-connected and redundantly rigid i.e. $G-e$ is rigid in $\mathbb{R}^d$ for all $e \in E$.

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These necessary conditions for global rigidity are sufficient when $d=1$ and when $d=2$ (Connelly, 2005; Jackson and Jordán, 2005). They are not sufficient for $d \geq 3$. However, Tanigawa has shown that a slightly stronger condition is sufficient for all d.

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Theorem [Tanigawa (2015)]

G is globally rigid in $\mathbb{R}^d$ if $G-v$ is rigid in $\mathbb{R}^d$ for all $v \in V$.

Body-Bar Frameworks

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Theorem [Tay (1991); Connelly, Jordán and Whiteley (2013)]

A generic d-dimensional body-bar framework is rigid if and only if its underlying graph G has $\binom{d+1}{2}$ edge-disjoint spanning trees; it is globally rigid if and only if it is redundantly rigid.

Direction-Length Frameworks

A d-dimensional **direction-length framework** is a collection of points in $\mathbb{R}^d$ linked by constraints which fix either the direction or distance between some pairs of points. The underlying graph of this framework is a 'mixed' graph $G = (V, E_D \cup E_L)$.

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Theorem [Servatius and Whiteley (1999)]

A mixed graph G is independent in $\mathbb{R}^2$ if and only if, for all $\emptyset \neq X \subseteq V$, we have $i(X) \leq 2|X| - 2$, $i_D(X) \leq 2|X| - 3$, and $i_L(X) \leq 2|X| - 3$.

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Together with Clinch and Keevash, we have recently characterised mixed graphs with the property that *all* their 2-dimensional generic realisations are globally rigid.

A **point-line framework** is a collection of points and lines in $\mathbb{R}^2$ linked by constraints which fix the distances between some pairs of points and some pairs of points and lines, and the angles between some pairs of lines. Let $G = (V_P \cup V_L, E)$ be the underlying graph of this framework. For $A \subset E$, let ν_P, ν_L denote the numbers of point and line vertices incident with A.

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Theorem[Jackson and Owen (2016)]

A point-line graph G is independent if and only if, for all $\emptyset \neq A \subseteq E$ and all partitions $\{A_1, \ldots, A_s\}$ of A,

$$|A| \leq \sum_{i=1}^{s} (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(A) - 1.$$

Matrix Completability

In this setting, two d-dimensional frameworks (G, p) and (G, q) are:

- **equivalent** if $p(u) \cdot p(v) = q(u) \cdot q(v)$ for all $uv \in E$;
- **congruent** if $p(u) \cdot p(v) = q(u) \cdot q(v)$ for all $u, v \in V$.

Local and **global completability** of (G, p) in $\mathbb{R}^d$ are defined in an analogous way to (local) and global rigidity using the new definitions of equivalence and congruence.

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This terminology comes from the fact that (G, p) is globally completable if and only if the Gram matrix for the points $\{p(v) : v \in V\}$ is uniquely defined by the entries corresponding to the edges of G.

Matrix Completability

Two necessary conditions for the generic independence of G in $\mathbb{R}^d$ are that:

- $i(X) \leq d|X| - \binom{d}{2}$ for all $X \subseteq V$ with $|X| \geq d$;
- $|E(H)| \leq d|V(H)| - d^2$ for all bipartite subgraphs $H \subseteq G$ with at least d vertices on each side of their bipartition.

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The above necessary conditions for generic independence are not sufficient when $d \geq 2$. It seems to be a difficult problem to characterise generic local or global completability even when $d = 2$. (Together with Jordán and Tanigawa (2016), we show that global completability is NOT a generic property when $d = 2$.)