Robust Convex Approximation Methods for TDOA-Based Localization under NLOS Conditions

Anthony Man-Cho So
Department of Systems Engineering & Engineering Management
The Chinese University of Hong Kong (CUHK)

(Joint Work with Gang Wang)

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Source Localization in a Sensor Network

- Basic problem: Localize a signal-emitting source using a number of sensors with *a priori* known locations

- Well-studied problem in signal processing with many applications **[Patwari et al.'05, Sayed-Tarighat-Khajehnouri'05]**:
  - acoustics
  - emergency response
  - target tracking
  - ...

- Typical types of measurements used to perform the positioning:
  - time of arrival (TOA)
  - time-difference of arrival (TDOA)
  - angle of arrival (AOA)
  - received signal strength (RSS)

- Challenge: Measurements are noisy

TDOA-Based Localization in NLOS Environment

- Focus of this talk: TDOA measurements
  - widely applicable
  - better accuracy (over AOA and RSS)
  - less stringent synchronization requirement (over TOA)
- Assuming there are $N+1$ sensors in the network, the TDOA measurements take the form

$$t_i = \frac{1}{c}(\|\boldsymbol{x} - \boldsymbol{s}_i\|_2 - \|\boldsymbol{x} - \boldsymbol{s}_0\|_2 + E_i) \quad \text{for } i = 1, \ldots, N,$$

  where
  - $\boldsymbol{x} \in \mathbb{R}^d$ is the source location to be estimated,
  - $\boldsymbol{s}_i \in \mathbb{R}^d$ is the i-th sensor's given location $(i = 0, 1, \ldots, N)$ with $\boldsymbol{s}_0$ being the reference sensor,
  - $d \geq 1$ is the dimension of the ambient space,
  - c is the signal propagation speed (e.g., speed of light),
  - $\frac{1}{c}E_i$ is the measurement error at the i-th sensor.

TDOA-Based Localization in NLOS Environment

- In this talk, we assume that the measurement error E_i consists of two parts:
  - measurement noise n_i
  - non-line-of-sight (NLOS) error e_i: variable propagation delay of the source signal due to blockage of the direct (or line-of-sight (LOS)) path between the source and the i-th sensor

- Putting $E_i = n_i + e_i$ into the TDOA measurement model, we obtain the following range-difference measurements:

$$d_i = \|\boldsymbol{x} - \boldsymbol{s}_i\|_2 - \|\boldsymbol{x} - \boldsymbol{s}_0\|_2 + n_i + e_i \quad \text{for } i = 1, \ldots, N.$$

Assumptions on the Measurement Error

- Localization accuracy generally depends on the nature of the measurement error.

- The measurement noise n_i is typically modeled as a random variable that is tightly concentrated around zero.

- However, the NLOS error e_i can be environment and time dependent. It is the difference of the NLOS errors incurred at sensors 0 and i. As such, it needs not centered around zero and can be positive or negative/of variable magnitude.

- We shall make the following assumptions:
  - $|n_i| \ll \|\boldsymbol{x} - \boldsymbol{s}_0\|_2$ (measurement noise is almost negligible)
  - $|e_i| \leq \rho_i$ for some given constant $\rho_i \geq 0$ (estimate on the support of the NLOS error is available)

Robust Least Squares Formulation

- Rewrite the range-difference measurements as

$$d_i - \|\boldsymbol{x} - \boldsymbol{s}_i\|_2 - e_i = \|\boldsymbol{x} - \boldsymbol{s}_0\|_2 + n_i.$$

Squaring both sides and using the assumption on n_i, we have

$$\begin{aligned} -2\|\boldsymbol{x} - \boldsymbol{s}_0\|_2 n_i &\approx (d_i - e_i)^2 - 2(d_i - e_i)\|\boldsymbol{x} - \boldsymbol{s}_i\|_2 + \|\boldsymbol{s}_i\|_2^2 - 2\boldsymbol{s}_i^T\boldsymbol{x} \\ &\quad - \|\boldsymbol{s}_0\|_2^2 + 2\boldsymbol{s}_0^T\boldsymbol{x} \\ &= 2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T\boldsymbol{x} - 2d_i\|\boldsymbol{x} - \boldsymbol{s}_i\|_2 - \left(\|\boldsymbol{s}_0\|_2^2 - \|\boldsymbol{s}_i\|_2^2 - d_i^2\right) \\ &\quad + e_i^2 + 2e_i\left(\|\boldsymbol{x} - \boldsymbol{s}_i\|_2 - d_i\right). \end{aligned}$$

- In view of the LHS, we would like the RHS to be small, regardless of what e_i is (provided that $|e_i| \le \rho_i$).

Robust Least Squares Formulation

- This motivates the following robust least squares (RLS) formulation:

$$\begin{array}{ll} \displaystyle\min_{\boldsymbol{x}\in\mathbb{R}^d,\, \boldsymbol{r}\in\mathbb{R}^N} & \displaystyle\max_{-\boldsymbol{\rho}\le\boldsymbol{e}\le\boldsymbol{\rho}} \sum_{i=1}^{N} \left|2(\boldsymbol{s}_0-\boldsymbol{s}_i)^T\boldsymbol{x} - 2d_ir_i - b_i + e_i^2 + 2e_i(r_i-d_i)\right|^2 \\ \text{subject to} & \|\boldsymbol{x}-\boldsymbol{s}_i\|_2 = r_i, \quad i=1,\ldots,N. \end{array}$$

  Here, $b_i = \|\boldsymbol{s}_0\|_2^2 - \|\boldsymbol{s}_i\|_2^2 - d_i^2$ is a known quantity.

- Note that the inner maximization with respect to $\boldsymbol{e}$ is separable. Hence, we can rewrite the objective function as

$$S(\boldsymbol{x},\boldsymbol{r}) = \sum_{i=1}^{N} \left(\underbrace{\max_{-\rho_i\le e_i\le\rho_i} \left|2(\boldsymbol{s}_0-\boldsymbol{s}_i)^T\boldsymbol{x} - 2d_ir_i - b_i + e_i^2 + 2e_i(r_i-d_i)\right|}_{\Gamma_i(\boldsymbol{x},\boldsymbol{r})} \right)^2 .$$

- Note that both objective function and the constraints are non-convex. Moreover, the $\mathcal{S}$-lemma does not apply.

Convex Approximation of the RLS Problem

- By the triangle inequality,

$$\begin{aligned} & \left|2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i + e_i^2 + 2e_i(r_i - d_i)\right| \\ \leq \quad & \left|2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i\right| + \left|e_i^2 + 2e_i(r_i - d_i)\right|. \end{aligned}$$

- It follows that

$$\begin{aligned} \Gamma_i(\boldsymbol{x}, \boldsymbol{r}) \quad &= \quad \max_{-\rho_i \leq e_i \leq \rho_i} \left|2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i + e_i^2 + 2e_i(r_i - d_i)\right| \\ &\leq \quad \left|2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i\right| + \max_{-\rho_i \leq e_i \leq \rho_i} \left|e_i^2 + 2e_i(r_i - d_i)\right|. \end{aligned}$$

- Key Observation:

$$\max_{-\rho_i \leq e_i \leq \rho_i} \left|e_i^2 + 2e_i(r_i - d_i)\right| = \rho_i^2 + 2\rho_i |r_i - d_i|.$$

Convex Approximation of the RLS Problem

- Hence,

$$\Gamma_i(\boldsymbol{x},\boldsymbol{r}) \le \left|2(\boldsymbol{s}_0-\boldsymbol{s}_i)^T\boldsymbol{x} - 2d_ir_i - b_i\right| + \rho_i^2 + 2\rho_i|r_i - d_i|.$$

- Observation: The function Γ_i^+ given by

$$\Gamma_i^+(\boldsymbol{x},\boldsymbol{r}) = \left|2(\boldsymbol{s}_0-\boldsymbol{s}_i)^T\boldsymbol{x} - 2d_ir_i - b_i\right| + \rho_i^2 + 2\rho_i|r_i - d_i|$$

  is non-negative and convex.

- Thus,

$$S^+(\boldsymbol{x},\boldsymbol{r}) = \sum_{i=1}^{N}\left(\Gamma_i^+(\boldsymbol{x},\boldsymbol{r})\right)^2$$

  is a convex majorant of the non-convex objective function of the RLS problem.

Convex Approximation of the RLS Problem

- Using the convex majorant, we have the following approximation of the RLS problem:

$$\begin{array}{ll} \displaystyle\min_{\boldsymbol{x}\in\mathbb{R}^d,\,\boldsymbol{r}\in\mathbb{R}^N} & \displaystyle\sum_{i=1}^{N} \left(\left| 2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i \right| + \rho_i^2 + 2\rho_i |r_i - d_i| \right)^2 \\ \text{subject to} & \|\boldsymbol{x} - \boldsymbol{s}_i\|_2 = r_i, \quad i = 1, \ldots, N. \end{array} \qquad \text{(ARLS)}$$

- This can be relaxed to an SOCP via standard techniques:

$$\begin{array}{ll} \displaystyle\min_{\substack{\boldsymbol{x}\in\mathbb{R}^d,\,\boldsymbol{r}\in\mathbb{R}^N \\ \boldsymbol{\eta}\in\mathbb{R}^N,\,\eta_0\in\mathbb{R}}} & \eta_0 \\ \text{subject to} & \left| 2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i \right| + \rho_i^2 + 2\rho_i |r_i - d_i| \le \eta_i, \ i = 1, \ldots, N, \\ & \|\boldsymbol{x} - \boldsymbol{s}_i\|_2 \le r_i, \quad i = 1, \ldots, N, \\ & \|\boldsymbol{\eta}\|_2^2 \le \eta_0. \end{array} \qquad \text{(SOCP)}$$

Convex Approximation of the RLS Problem

- Alternatively, observe that

$$
\begin{aligned}
& \left(\left|2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i\right| + \rho_i^2 + 2\rho_i |r_i - d_i|\right) \\
= \quad & \max\left\{\pm\left(2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i\right) + \rho_i^2 \pm 2\rho_i (r_i - d_i)\right\}.
\end{aligned}
$$

- Hence, Problem (ARLS) can be written as

$$
\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^d,\, \boldsymbol{r}\in\mathbb{R}^N} \quad & \sum_{i=1}^{N} \tau_i \\
\text{subject to} \quad & \left[\pm\left(2(\boldsymbol{s}_0 - \boldsymbol{s}_i)^T \boldsymbol{x} - 2d_i r_i - b_i\right) + \rho_i^2 \pm 2\rho_i (r_i - d_i)\right]^2 \le \tau_i, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, N, \\
& \|\boldsymbol{x} - \boldsymbol{s}_i\|_2^2 = r_i^2, \quad i = 1, \ldots, N.
\end{aligned}
$$

- The above problem is linear in $\boldsymbol{\tau}$ and $\boldsymbol{Y} = \boldsymbol{y}\boldsymbol{y}^T$, where $\boldsymbol{y} = (\boldsymbol{x}, \boldsymbol{r})$.

Convex Approximation of the RLS Problem

- Hence, we also have the following SDP relaxation of (ARLS):

$$
\begin{array}{cl}
\displaystyle\min_{\substack{\boldsymbol{Y}\in\mathbb{S}^{d+N} \\ \boldsymbol{y}\in\mathbb{R}^{d+N},\,\boldsymbol{\tau}\in\mathbb{R}^N}} & \displaystyle\sum_{i=1}^{N}\tau_i \\
\text{subject to} & \text{some linear constraints in } \boldsymbol{Y},\ \boldsymbol{y}, \text{ and } \boldsymbol{\tau}, \\
& \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{y} \\ \boldsymbol{y}^T & 1 \end{bmatrix} \succeq \mathbf{0}.
\end{array}
\qquad \text{(SDP)}
$$

Theoretical Issues

- When is (ARLS) equivalent to the original RLS problem? In particular, when does the convex majorant $\Gamma_i^+(\boldsymbol{x}, \boldsymbol{r})$ equal the original function $\Gamma_i(\boldsymbol{x}, \boldsymbol{r})$?

- Does (SDP) always yield a tighter relaxation of (ARLS) than (SOCP)?

- Do the relaxations yield a unique solution?

Exactness of Problem (ARLS)

- Consider a fixed $i \in \{1,\ldots,N\}$. Recall

$$\begin{aligned}\Gamma_i(\boldsymbol{x},\boldsymbol{r}) &= \max_{-\rho_i \le e_i \le \rho_i} \left| 2(\boldsymbol{s}_0-\boldsymbol{s}_i)^T\boldsymbol{x} - 2d_ir_i - b_i + e_i^2 + 2e_i(r_i-d_i)\right| \\ \Gamma_i^+(\boldsymbol{x},\boldsymbol{r}) &= \left|2(\boldsymbol{s}_0-\boldsymbol{s}_i)^T\boldsymbol{x} - 2d_ir_i - b_i\right| + \rho_i^2 + 2\rho_i|r_i-d_i|\end{aligned}$$

- Proposition: If $\rho_i = 0$, then $\Gamma_i(\boldsymbol{x},\boldsymbol{r}) = \Gamma_i^+(\boldsymbol{x},\boldsymbol{r})$. Otherwise, $\Gamma_i(\boldsymbol{x},\boldsymbol{r}) = \Gamma_i^+(\boldsymbol{x},\boldsymbol{r})$ iff $2(\boldsymbol{s}_0-\boldsymbol{s}_i)^T\boldsymbol{x} - 2d_ir_i - b_i \ge 0$; i.e. (using the definition of b_i),

$$(n_i+e_i)^2 - 2\|\boldsymbol{x}-\boldsymbol{s}_0\|_2(n_i+e_i) \ge 0. \qquad (1)$$

- Interpretation: Recall that $n_i + e_i$ is the measurement error associated with $\|\boldsymbol{x}^*-\boldsymbol{s}_i\|_2 - \|\boldsymbol{x}^*-\boldsymbol{s}_0\|_2$, where $\boldsymbol{x}^*$ is the true location of the source.
  - Scenario 1: $n_i+e_i \le 0$ or $n_i+e_i \ge 2\|\boldsymbol{x}-\boldsymbol{s}_0\|_2$ (so that (1) holds)
    e.g., $\boldsymbol{x}^* \leftrightarrow \boldsymbol{s}_0$ highly NLOS but $\boldsymbol{x}^* \leftrightarrow \boldsymbol{s}_i$ almost LOS
  - Scenario 2: $0 < n_i+e_i < 2\|\boldsymbol{x}-\boldsymbol{s}_0\|_2$ (so that (1) fails)
    e.g., $\boldsymbol{x}^* \leftrightarrow \boldsymbol{s}_0$ almost LOS but $\boldsymbol{x}^* \leftrightarrow \boldsymbol{s}_i$ mildly NLOS

Relative Tightness of the Approximations

- One may expect that every feasible solution $(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{\eta}, \eta_0)$ to (SOCP) can be used to construct a feasible solution $(\boldsymbol{Y}, \boldsymbol{x}, \boldsymbol{r}, \boldsymbol{\tau})$ to (SDP).

- However, there are instances for which this is not true!

- Reason: Recall that $\boldsymbol{y} = (\boldsymbol{x}, \boldsymbol{r})$. Observe that

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{s}_i\|_2^2 &= \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{s}_i + \|\boldsymbol{s}_i\|_2^2 \\ &\le \sum_{i=1}^{d} Y_{ii} - 2\boldsymbol{x}^T\boldsymbol{s}_i + \|\boldsymbol{s}_i\|_2^2 \qquad (2) \\ &\le Y_{d+i,d+i}, \qquad (3) \end{aligned}$$

where (2) follows from $\boldsymbol{Y} \succeq \boldsymbol{y}\boldsymbol{y}^T$ in (SDP) and (3) is one of the linear constraints in (SDP). Also, $\boldsymbol{Y} \succeq \boldsymbol{y}\boldsymbol{y}^T$ implies that $r_i^2 \le Y_{d+i,d+i}$.

However, we have the tighter constraint $\|\boldsymbol{x} - \boldsymbol{s}_i\|_2 \le r_i$ in (SOCP).

A Refined SDP Approximation

- The above observation suggests that we can tighten (SDP) to

$$
\begin{array}{lll}
\displaystyle\min_{\substack{\boldsymbol{Y} \in \mathbb{S}^{d+N} \\ \boldsymbol{y} \in \mathbb{R}^{d+N},\, \boldsymbol{\tau} \in \mathbb{R}^N}} & \displaystyle\sum_{i=1}^{N} \tau_i & \\
\text{subject to} & \text{some linear constraints in } \boldsymbol{Y},\ \boldsymbol{y}, \text{ and } \boldsymbol{\tau}, & \text{(RSDP)} \\
& \|\boldsymbol{x} - \boldsymbol{s}_i\|_2 \le r_i, \quad i = 1, \ldots, N, & \\
& \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{y} \\ \boldsymbol{y}^T & 1 \end{bmatrix} \succeq \boldsymbol{0}. &
\end{array}
$$

- It is indeed true (and easy to show) that every feasible solution $(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{\eta}, \eta_0)$ to (SOCP) can be used to construct a feasible solution $(\boldsymbol{Y}, \boldsymbol{x}, \boldsymbol{r}, \boldsymbol{\tau})$ to (RSDP).

Solution Uniqueness of the Convex Approximations

- Theorem: Suppose there exists an $i \in \{1, \ldots, N\}$ such that

$$\|\boldsymbol{x}^* - \boldsymbol{s}_i\| = r_i^*$$

holds for all optimal solutions $(\boldsymbol{x}^*, \boldsymbol{r}^*, \boldsymbol{\eta}^*, \eta_0^*)$ (resp. $(\boldsymbol{Y}^*, \boldsymbol{x}^*, \boldsymbol{r}^*, \boldsymbol{\tau}^*)$) to (SOCP) (resp. (RSDP)). Then, both (SOCP) and (RSDP) uniquely localize the source.

- Theorem: Let $\boldsymbol{Y} \in \mathbb{S}^{d+N}$ be decomposed as

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_{11} & \boldsymbol{Y}_{12} \\ \boldsymbol{Y}_{12}^T & \boldsymbol{Y}_{22} \end{bmatrix},$$

where $\boldsymbol{Y}_{11} \in \mathbb{S}^d$, $\boldsymbol{Y}_{12} \in \mathbb{R}^{d \times N}$, and $\boldsymbol{Y}_{22} \in \mathbb{S}^N$. Suppose that every optimal solution $(\boldsymbol{Y}^*, \boldsymbol{x}^*, \boldsymbol{r}^*, \boldsymbol{\tau}^*)$ to (RSDP) satisfies rank$(\boldsymbol{Y}_{11}^*) \leq 1$. Then, (RSDP) uniquely localizes the source.

Numerical Experiments

- Real measurement data from `http://www.eecs.umich.edu/~hero/localize`
- 44 nodes deployed in a room of area 14m×13m; at least 5 and up to 9 from $\mathcal{I} = \{15, 2, 9, 43, 37, 13, 17, 4, 40\}$ are chosen as sensors; node 15 is the reference

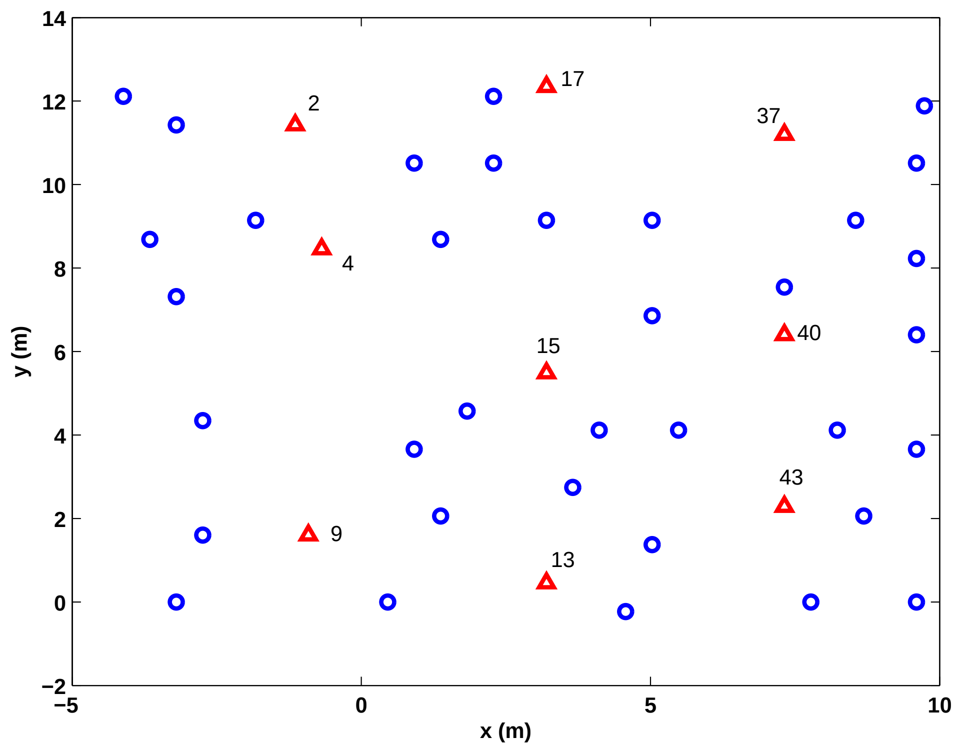

Figure 1: Sensor and source geometry in a real room **[Patwari et al.'03]**: △: sensor, ○: source.

Numerical Experiments

- From the data, we use $\rho = 6.6724$ as an upper bound on the magnitudes of the NLOS errors in the TDOA measurements.

- After fixing the first $N+1$ nodes in $\mathcal{I}$ as sensors, where $N = 4, \ldots, 8$, the remaining $M = 44 - (N+1)$ nodes are regarded as different sources.

- Localization performance is measured by the RMSE criterion:

$$\mathrm{RMSE} = \sqrt{\frac{1}{M}\sum_{i=1}^{M} \|\hat{\boldsymbol{x}}_i - \boldsymbol{x}_i^*\|^2}.$$

  Here, $\hat{\boldsymbol{x}}_i$ and $\boldsymbol{x}_i^*$ are the estimated and true location of the source in the ith run, respectively.

Numerical Experiments

- We compare 5 methods:
  - SDR-Non-Robust: **[Yang-Wang-Luo'09]**
  - WLS-Non-Robust: **[Cheung-So-Ma-Chan'06]**
  - RC-SDR-Non-Robust: **[Xu-Ding-Dasgupta'11]**
  - SOCR-Robust: Formulation (SOCP)
  - SDR-Robust: Formulation (RSDP)

- Simulation environment
  - MATLAB R2012b on a DELL personal computer with a 3.3GHz Intel(R) Core(TM) i5-2500 CPU and 8GB RAM
  - Solver used to solve (SOCP) and (RSDP): SDPT3

Numerical Experiments

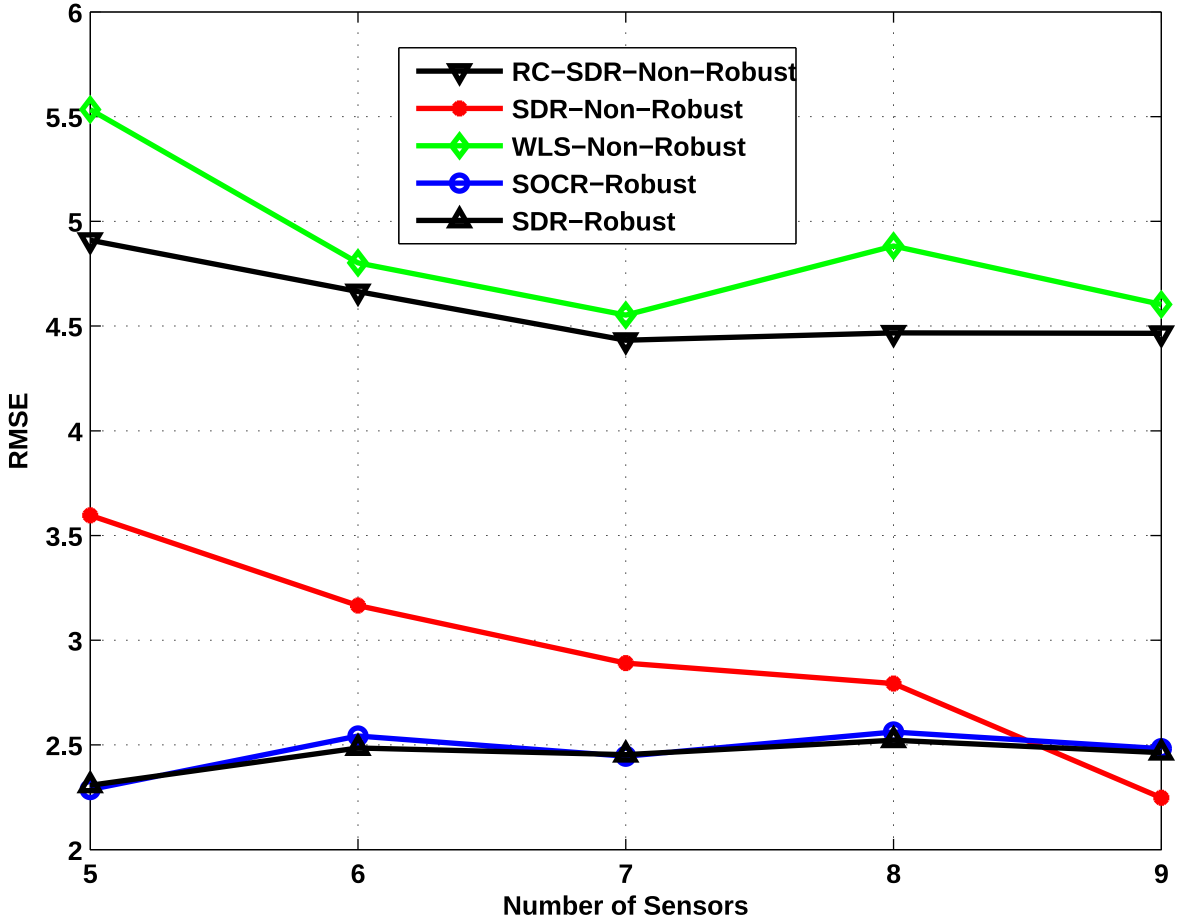

Figure 2: Comparison of RMSE of different methods using real data: $\rho = 6.6724$ and $N = 4, 5, \ldots, 8$.

Numerical Experiments

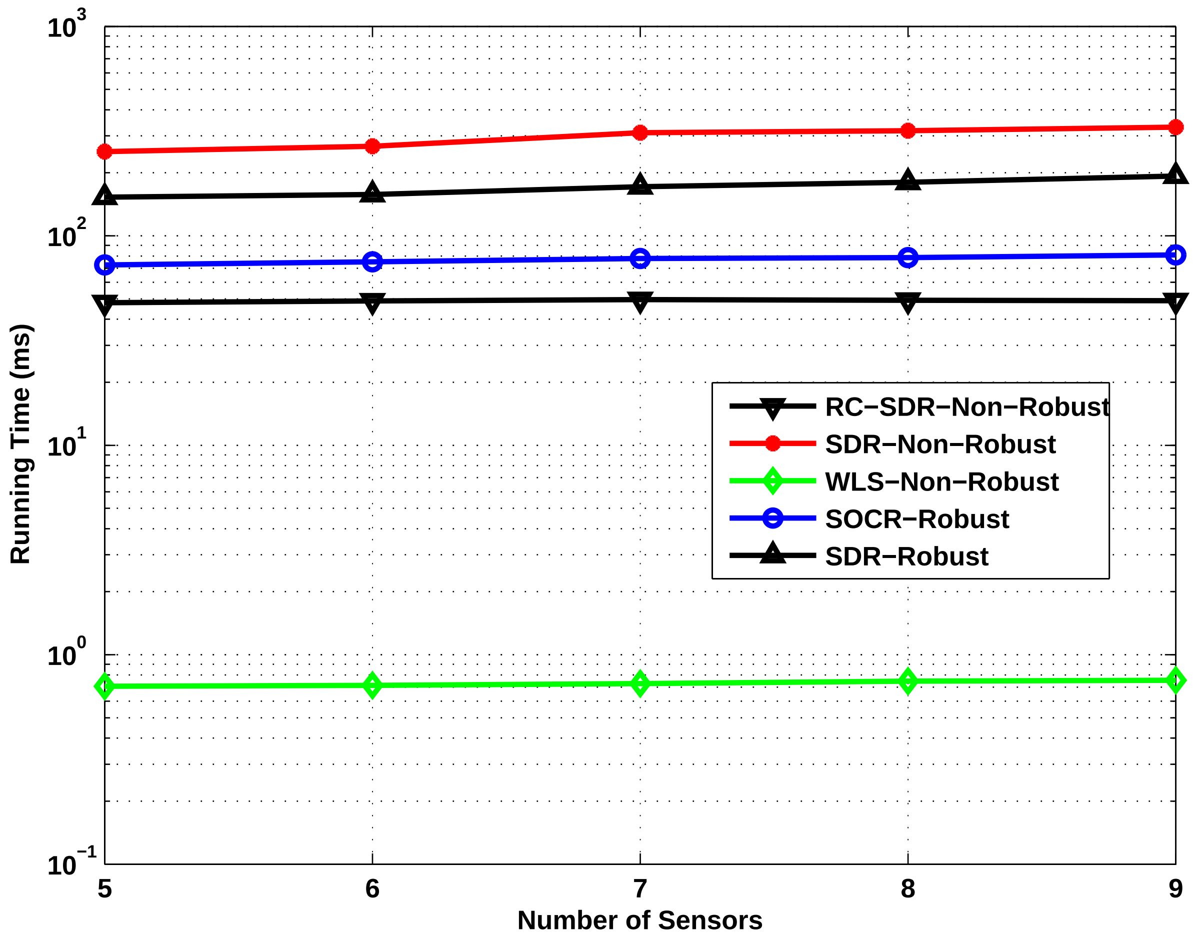

Figure 3: Comparison of average running times of different methods using real data: $\rho = 6.6724$ and $N = 4, 5, \ldots, 8$.

Numerical Experiments

- Further details and more experiments can be found in our paper:

  G. Wang, A. M.-C. So, Y. Li, "Robust Convex Approximation Methods for TDOA-Based Localization under NLOS Conditions", IEEE Transactions on Signal Processing 64(13):3281–3296, 2016.

Thank You!