Singularity Degree of PSD Matrix Completion

Shin-ichi Tanigawa

CWI and Kyoto

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Positive Semidefinite Matrix Completion

PSD completion problem (G, c)

Given $G = (V, E)$ with $V = \{1, \ldots, n\}$ and edge weight $c : E \to [-1, 1]$,

$$\begin{array}{rll} \text{find} & X \in \mathcal{S}^n & \\ \text{s.t.} & X[i,j] = c(ij) & (ij \in E) \\ & X[i,i] = 1 & (i \in V) \\ & X \succeq 0 & \end{array}$$

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$$\begin{array}{ll} \min & \langle \Omega, C \rangle \\ \text{s.t.} & \Omega \in S_+(G) \end{array}$$

where

$$C[i,j] = \begin{cases} c(ij) & (ij \in E) \\ 1 & (i = j) \\ 0 & (otherwise) \end{cases}$$

$$S(G) := \{A \in \mathcal{S}^n : A[i,j] = 0 \ \forall ij \notin V \cup E\}$$

$$S_+(G) := \{A \in S(G) : A \succeq 0\}$$

$$\overline{S}_+(G) := \{A \in S_+(G) : A[i,j] \neq 0 \ \forall ij \in E\}$$

Geometric View

- Given a completion problem (G, c),
- PSD completion $X = PP^\top$ with rank d
  $\Leftrightarrow$ spherical embedding $p : V \to \mathbb{S}^{d-1}$ realizing c, i.e.,

$$p_i \cdot p_j = c(ij) \qquad \forall ij \in E$$

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- dual optimal solution : $\Omega \in S_+(G)$ with $\langle C, \Omega \rangle = 0$

$$\begin{aligned} \langle C, \Omega \rangle = 0 \quad &\Leftrightarrow \quad \langle X, \Omega \rangle = 0 \\ &\Leftrightarrow \quad \Omega P = 0 \\ &\Leftrightarrow \quad \Omega[i,i]p(i) + \sum_{j \sim i} \Omega[i,j]p(j) = 0 \quad (\forall i \in V) \end{aligned} \tag{1}$$

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- Ω is called a stress (matrix) of (G, p) if Ω satisfies (1)
- Given (G, p), $\Omega \in S(G)$ is dual opt iff Ω is a PSD stress of (G, p).

SDP Duality

For any primal and dual optimal pair (X, Ω),

$$\langle X, \Omega \rangle = 0 \quad \Rightarrow \quad \operatorname{rank} X + \operatorname{rank} \Omega \leq n.$$

- high rank dual opt $\Rightarrow$ low rank completion

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Rank maximality certificate

- A completion X for (G, c) attains the maximum rank if $\exists$ dual opt with rank $n - \operatorname{rank} X$.

Parameter ν and Unique Completability

Theorem (Connelly82, Laurent-Varvitsiotis14)

- *A completion X for (G, c) is unique if $\exists$ dual opt Ω with $\operatorname{rank}\Omega = n - \operatorname{rank} X$ and the SAP, i.e.,*

$$\nexists X \in \mathcal{S}^n \setminus \{0\} \text{ with } \Omega X = 0 \text{ and } X[i,j] = 0 \text{ for } ij \in V \cup E$$

- *(G, p) is universally rigid in $\mathbb{S}^{d-1}$ if (G, p) admits a PSD stress Ω with $\operatorname{rank}\Omega = n - d$ and the SAP.*

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Colin de Verdière Parameter ν

$$\nu(G) := \max\{\operatorname{corank}\Omega : \Omega \in \overline{S}_+(G) \text{ has the SAP}\}.$$

$$\nu(G) \leq \max\{d : \exists \text{ universally rigid } (G,p) \text{ in } \mathbb{S}^{d-1}\}$$

Strict Complementarity and Singularity Degree

Strict Complementarity

A primal and dual optimal pair (X, Ω) satisfies a strict complementarity condition if

$$\operatorname{rank} X + \operatorname{rank} \Omega = n$$

- For which problem the strict complementarity can be guaranteed?
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1. $\mathrm{sd}(G) = 1$;
2. The strict complementarity holds for any PSD completion problem with underlying graph G;
3. The projection $\mathcal{E}(G)$ of the elliptope (the set of correlation matrices) onto $\mathbb{R}^E$ is exposed (Druvyatskiy-Pataki-Wolkowicz15).

Facial Reduction (Borwein-Wolkowitcz81)

A sequence $\{\Omega_1, \ldots, \Omega_k\}$ in $\mathcal{S}^n$ is iterated PSD if Ω_i is positive semidefinite on $\mathcal{V}_{i-1}$, where $\mathcal{V}_0 = \mathbb{R}^n$ and

$$\mathcal{V}_i = \{x \in \mathbb{R}^n : \langle xx^\top, \Omega_j \rangle = 0 \ (j = 1, \ldots, i-1)\}.$$

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Theorem (Facial reduction)

For any feasible (G, c), $\exists X$ and $\exists \Omega_1, \ldots, \Omega_k \in S(G)$ s.t.

1. *the sequence is iterated PSD*
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Definition (Sturm 2000)

For a completion problem (G, c), the singularity degree $\mathrm{sd}(G, c)$ is the length of the shortest dual certificate sequence $\{\Omega_1, \ldots, \Omega_k\}$.

Singularity Degree of Graphs

Singularity degree of G

$$\mathrm{sd}(G) = \max_{c} \mathrm{sd}(G, c)$$

Question (Druvyatskiy-Pataki-Wolkowicz15) Characterize G with $\mathrm{sd}(G) = 1$
Question (So15) $\mathrm{sd}(G) = o(n)$?

Main Results

Theorem (T16)

$\mathrm{sd}(G) = 1$ iff G is chordal.

G is chordal if G has no $C_n (n \geq 4)$ as an induced subgraph

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- If G has neither W_n $(n \geq 5)$ nor a proper splitting of W_n $(n \geq 4)$ as an induced subgraph, then $\mathrm{sd}(G) \leq 2$.
- If G has an induced subgraph which is a proper splitting of one of the above forbidden subgraphs, then $\mathrm{sd}(G) > 2$.

If $\mathrm{tw}(G) \leq 2$, then $\mathrm{sd}(G) \leq 2$.

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If $\mathrm{tw}(G) \leq 2$, then $\mathrm{sd}(G) \leq 2$.

Theorem (T16)

For each n there is a graph G with n vertices and $\mathrm{tw}(G) = 3$ whose singularity degree is $\lfloor \frac{n-1}{3} \rfloor$.

Proof of the first theorem

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"$\Leftarrow$" (Druvyatskiy-Pataki-Wolkowicz15)
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Lemma $\mathrm{sd}(C_n) \geq 2$ if $n \geq 4$.

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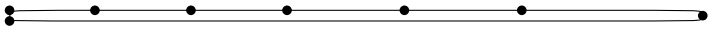

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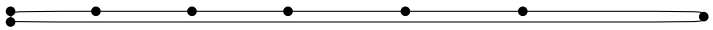

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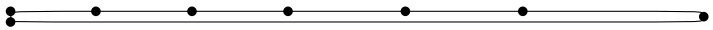

- (G, p) is universally rigid
- there is a unique stress Ω with $\mathrm{rank}\,\Omega = 1 < n - 2$

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Nondegenerate Singularity Degree

A completion problem (G, c) is nondegenerate if $c(ij) \neq \pm 1$ for every $ij \in E(G)$.

Degenerate edges can easily be eliminated.

- Suppose $c(ij) = 1$ for $ij \in E$ ...
- Any solution X of (G, c) satisfies

$$X[i, k] = X[j, k] \quad \text{for every } k$$

- Equivalently, any embedding p realizing c satisfies $p(i) = p(j)$.

The example in the last proof is degenerate...

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Lemma $\mathrm{sd}(G) \leq \mathrm{sd}^*(G) + 1$.

Corollary (T16) $\mathrm{sd}(G) \leq 2$ if G has no forbidden induced subgraph listed above.

Example of Large Singularity Degree

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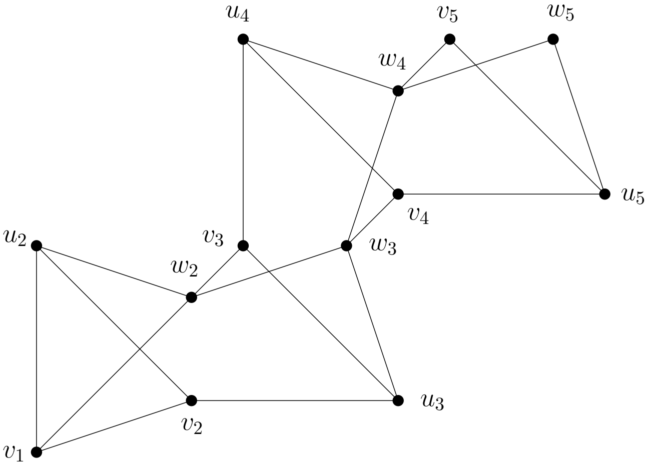

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