#### Graph Cores via Universal Completability

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#### Outline



Universal completable frameworks



Sufficient condition for UC



Least eigenvalue frameworks



Sufficient condition for showing a graph is a core

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ex: a 
$$C_{2k+1}$$
- framework  
 $p_x = \left(\cos\left(\frac{2\pi kx}{2k+1}\right), \sin\left(\frac{2\pi kx}{2k+1}\right)\right), \ 0 \le x \le 2k,$ 

Def: A framework  $G(\mathbf{p})$  is called UC if  $\forall q_1, \dots, q_n$  $\langle p_i, p_j \rangle = \langle q_i, q_j \rangle$  for  $[i = j \& ij \in E]$ implies that  $\langle p_i, p_j \rangle = \langle q_i, q_j \rangle$  for  $i, j \in [n]$ 

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the G-partial matrix corresponding to  $\mathrm{G}(\mathbf{p})$  has a unique PSD completion

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find G-partial PSD matrix



all its PSD completions have rank > k

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#### Lem: Let $G(\mathbf{p}) \subseteq \mathbb{R}^d$ with $\operatorname{span}(p_1, \ldots, p_n) = \mathbb{R}^d$ . The framework $G(\mathbf{p})$ is UC if $\exists Z \in S^n$ satisfying:

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 $\operatorname{corank}(Z) = d$ 



 $\implies \operatorname{span}(p_i p_j^T + p_j p_i^T : i, j \in V \cup E)$ 

#### How to use this?

#### Nullspace representations

Def: Let  $M \in S^n$  with  $\operatorname{corank}(M) = d$  and  $\{u_i\}_{i=1}^d$ an orth. basis for  $\operatorname{Ker}(M)$ . Set  $P = [u_1, \dots, u_d]$ and let  $\{p_i\}_{i=1}^n \subseteq \mathbb{R}^d$  be the rows of P. The map  $[n] \mapsto \mathbb{R}^d$ , where  $i \mapsto p_i$ is called a nullspace representation of M

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e.g. [Lovász, Schrijver'99, '06]

Constructing outerplanar embeddings

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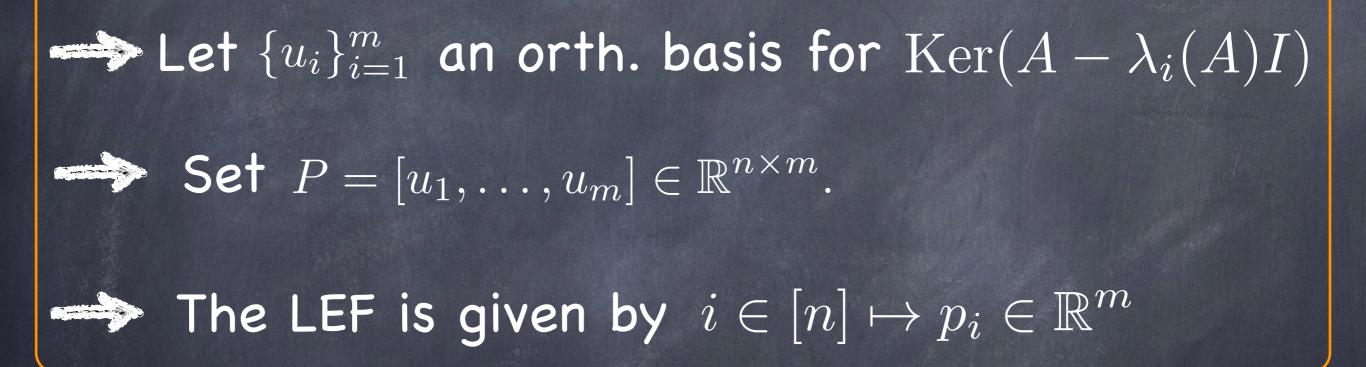
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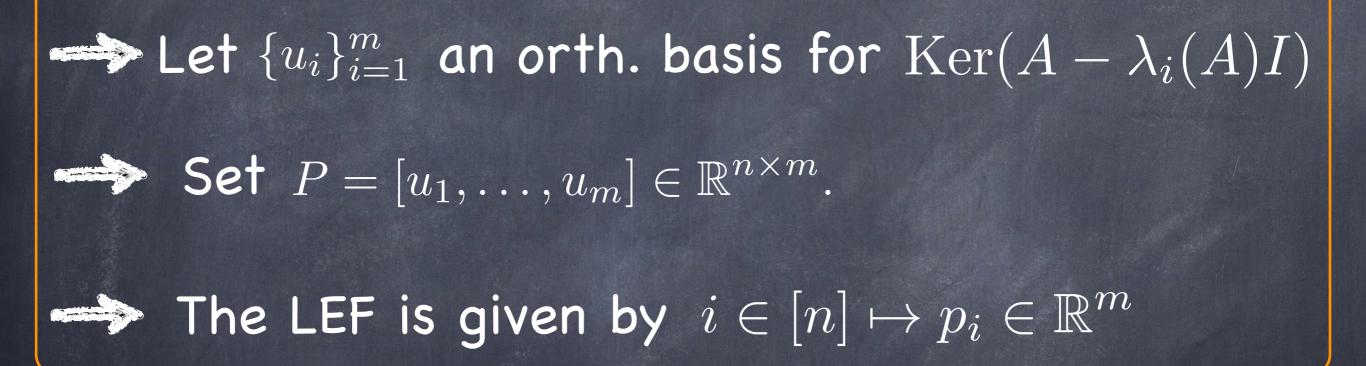
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Brouwer & Haemers (Euclidean reps.)

Thm: A least eig. framework  $G(p) \subseteq \mathbb{R}^m$  is UC iff  $\forall R \in S^m : p_i^{\mathsf{T}} R p_j = 0 \text{ for } i = j \text{ and } (i, j) \in E \Longrightarrow R = 0.$ 

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Pf:  $\Leftarrow$  Use the sufficient condition for UC with  $Z = A - \lambda_{min}(A)I$ 

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Pf:  $\Leftarrow$  Use the sufficient condition for UC with  $Z = A - \lambda_{min}(A)I$   $\implies$  Scale so  $\lambda_{min}(R) = -1$  and note that  $P(I+R)P^T \neq PP^T$ 

# Graph cores

#### **Def:** A map $f: V(G) \to V(G)$ is an endomorphism if: $u \sim v \Longrightarrow f(u) \sim f(v)$



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Bipartite graphs are not cores



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Thm [Nešetřil]: Every locally injective endomorphism is an automorphism

Thm: Consider a framework  $G(\mathbf{p})$  with

 $d(i,j) = 2 \implies p_i \neq p_j$   $||p_i|| \text{ constant for } i \in [n]$   $\langle p_i, p_j \rangle \text{ constant for } ij \in E$ 

 $\rightarrow G(\mathbf{p})$  is UC

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Thm: Consider a framework  $G(\mathbf{p})$  with  $d(i,j) = 2 \Longrightarrow p_i \neq p_j$  $||p_i||$  constant for  $i \in [n]$  $\langle p_i, p_j \rangle$  constant for  $ij \in E$  $\rightarrow G(\mathbf{p})$  is UC Then G is a core.

Pf: Show that any end. f is locally injective

**Pf:** Say  $\exists u, v$  such that d(u, v) = 2 and f(u) = f(v) $\rightarrow$  Set  $p'_i = p_{f(i)}, \forall i$  $G(\mathbf{p})$  and  $G(\mathbf{p}')$  are equivalent  $\exists$  isometry U such that  $p'_i = Up_i, \forall i$ So  $\langle p_u, p_v \rangle = \langle p'_u, p'_v \rangle = \langle p_{f(u)}, p_{f(v)} \rangle$  which implies  $p_u = p_v$  contradiction

#### Thm: If there exists a framework $G(\mathbf{p})$ such that:



 $d(i,j) = 2 \Longrightarrow p_i \neq p_j$ 





 $\langle p_i, p_j \rangle$  constant for  $ij \in E$ 



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 $d(i,j) = 2 \Longrightarrow p_i \neq p_j$  $||p_i||$  constant for  $i \in [n]$  $\langle p_i, p_j \rangle$  constant for  $ij \in E$  $\rightarrow G(\mathbf{p}) \text{ is UC }$  Use LEF



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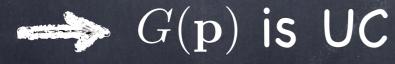


 $d(i,j) = 2 \Longrightarrow p_i \neq p_j$ 

 $||p_i||$  constant for  $i \in [n]$ 

 $\langle p_i, p_j \rangle$  constant for  $ij \in E$ 

What about these?



#### then G is a core

#### **Def:** G is 1WR if $\exists a_k, b_k \in \mathbb{N}$ such that:

$$A^k \circ I = a_k I$$

$$\implies A^k \circ A = b_k A$$

for all  $k \in \mathbb{N}$ 

Recall  $(X \circ Y)_{ij} = X_{ij}Y_{ij}$ 

#### Thm: Let G be 1WR and $G(\mathbf{p}) \subseteq \mathbb{R}^m$ a LEF. Then:

$$\implies ||p_i||^2 = \frac{m}{n} \quad \text{for all } i \in [n]$$

$$\implies \langle p_i, p_j \rangle = \frac{\tau m}{na_2} \quad \text{for all } ij \in E$$

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 $\rightarrow E_{\tau}$  is a polynomial in A

 $\exists a, b \in \mathbb{R}$  such that

 $E_{\tau} \circ I = aI$  and  $E_{\tau} \circ A = bA$ 

Pf: Let  $E_{\tau}$  the projector on the least eigenspace

- $\rightarrow E_{\tau}$  is a polynomial in A
- $\exists a, b \in \mathbb{R}$  such that

 $E_{\tau} \circ I = aI$  and  $E_{\tau} \circ A = bA$ 

 $F_{\tau} = \operatorname{Gram}(p_1, \dots, p_n)$ 

## Putting everything together

# Thm: Let G be 1WR and $G(\mathbf{p}) \subseteq \mathbb{R}^m$ a LEF. Say that $d(i,j) = 2 \Longrightarrow p_i \neq p_j$ $span(p_i p_j^T + p_j p_i^T : i, j \in V \cup E)$ Then G is a core.

# Applications

Def: The Kneser graph  $K_{n:r}$  has: Vertices: Subsets of [n] of size rEdges:  $X \sim Y$  if  $X \cap Y = \emptyset$ 

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The proof uses EKR theorem

To show  $K_{n:r}$  is a core using our sufficient condition:

#### Find LEF and show that

$$p_i \neq p_j, \ \forall i, j$$

 $\Rightarrow \operatorname{span}(p_i p_j^T + p_j p_i^T : i, j \in V \cup E)$ 

#### References

Universal completability, least eigenvalue frameworks and vector colorings, arXiv:1512.04972

Vector colorings and graph homomorphisms, coming soon

# Thank you!