Graph Cores via Universal Completability

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Outline

- Universal completable frameworks
- Sufficient condition for UC
- Least eigenvalue frameworks
- Sufficient condition for showing a graph is a core

Universal completability

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ex: a C_{2k+1}- framework

$$p_x = \left(\cos\left(\frac{2\pi kx}{2k+1}\right), \sin\left(\frac{2\pi kx}{2k+1}\right)\right), \; 0 \leq x \leq 2k,$$

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→ the G-partial matrix corresponding to $\mathrm{G}(\mathbf{p})$ has a unique PSD completion

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- $\operatorname{corank}(Z) = d$
- $\operatorname{span}(p_i p_j^T + p_j p_i^T : i, j \in V \cup E)$

How to use this?

Nullspace representations

Def: Let $M \in \mathcal{S}^n$ with $\mathrm{corank}(M) = d$ and $\{u_i\}_{i=1}^d$ an orth. basis for $\mathrm{Ker}(M)$. Set $P = [u_1, \ldots, u_d]$ and let $\{p_i\}_{i=1}^n \subseteq \mathbb{R}^d$ be the rows of P. The map

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e.g. [Lovász, Schrijver'99, '06]

→ Constructing outerplanar embeddings

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- Chan, Godsil (eigenpolytopes)
- Brouwer & Haemers (Euclidean reps.)

Least eigenvalue framework

Thm: A least eig. framework $G(p) \subseteq \mathbb{R}^m$ is UC iff

$$\forall R \in \mathcal{S}^m : \; p_i^\mathsf{T} R p_j = 0 \text{ for } i = j \text{ and } (i,j) \in E \implies R = 0.$$

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Pf: $\Longleftarrow$ Use the sufficient condition for UC with

$$Z = A - \lambda_{min}(A) I$$

$\Longrightarrow$ Scale so $\lambda_{min}(R) = -1$ and note that

$$P(I + R)P^T \neq PP^T$$

Graph cores

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Def: A graph is a core if all its endomorphisms are automorphisms

- Complete graphs, odd cycles are cores
- Bipartite graphs are not cores

Cores

Def: An endomorphism f is **locally injective** if

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Thm [Nešetřil]: Every locally injective endomorphism is an automorphism

Graph cores via UC

Thm: Consider a framework $G(\mathbf{p})$ with

- $d(i,j) = 2 \implies p_i \neq p_j$
- $\|p_i\|$ constant for $i \in [n]$
- $\langle p_i, p_j \rangle$ constant for $ij \in E$
- $G(\mathbf{p})$ is UC

Then G is a core.

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Pf: Show that any end. f is locally injective

Graph cores via UC

Pf: Say $\exists u, v$ such that $d(u,v) = 2$ and $f(u) = f(v)$

→ Set $p'_i = p_{f(i)}, \ \forall i$

→ $G(\mathbf{p})$ and $G(\mathbf{p}')$ are equivalent

→ $\exists$ isometry U such that $p'_i = Up_i, \forall i$

So $\langle p_u, p_v \rangle = \langle p'_u, p'_v \rangle = \langle p_{f(u)}, p_{f(v)} \rangle$ which implies

$p_u = p_v$ contradiction

Graph cores via UC

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Use LEF

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What about these?

1-walk-regular graphs

Def: G is **1WR** if $\exists a_k, b_k \in \mathbb{N}$ such that:

⟹ $A^k \circ I = a_k I$

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for all $k \in \mathbb{N}$

Recall $(X \circ Y)_{ij} = X_{ij} Y_{ij}$

1-walk-regular graphs

Thm: Let G be 1WR and $G(\mathbf{p}) \subseteq \mathbb{R}^m$ a LEF. Then:

→ $\|p_i\|^2 = \dfrac{m}{n}$ for all $i \in [n]$

→ $\langle p_i, p_j \rangle = \dfrac{\tau m}{n a_2}$ for all $ij \in E$

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→ $\exists a, b \in \mathbb{R}$ such that

$$E_\tau \circ I = aI \quad \text{and} \quad E_\tau \circ A = bA$$

1-walk-regular graphs

Pf: Let E_τ the projector on the least eigenspace

→ E_τ is a polynomial in A

→ $\exists a, b \in \mathbb{R}$ such that

$E_\tau \circ I = aI$ and $E_\tau \circ A = bA$

→ $E_\tau = \mathrm{Gram}(p_1, \ldots, p_n)$

Putting everything together

Thm: Let G be 1WR and $G(\mathbf{p}) \subseteq \mathbb{R}^m$ a LEF. Say that

- $d(i,j) = 2 \implies p_i \neq p_j$
- $\text{span}(p_i p_j^T + p_j p_i^T : i,j \in V \cup E)$

Then G is a core.

Applications

Kneser graphs

Def: The Kneser graph $K_{n:r}$ has:

Vertices: Subsets of $[n]$ of size r

Edges: $X \sim Y$ if $X \cap Y = \emptyset$

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Thm: [Hahn, Tardif'97]

For $n \geq 2r+1$ the graph $K_{n:r}$ is a core

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Thm: [Hahn, Tardif'97]

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→ The proof uses EKR theorem

Kneser graphs

To show $K_{n:r}$ is a core using our sufficient condition:

- Find LEF and show that
  - $p_i \neq p_j, \ \forall i, j$
  - $\text{span}(p_i p_j^T + p_j p_i^T : i, j \in V \cup E)$

References

- Universal completability, least eigenvalue frameworks and vector colorings, arXiv:1512.04972

- Vector colorings and graph homomorphisms, coming soon

Thank you!