Graph Cores via Universal Completableability

Antonios Varvitsiotis

Nanyang Technological University & Centre for Quantum Technologies, Singapore

Joint work with:
C. Godsil, D. Roberson, B. Rooney, R. Šámal
Outline

- Universal completable frameworks
- Sufficient condition for UC
- Least eigenvalue frameworks
- Sufficient condition for showing a graph is a core
Universal completability

Def: A framework for $G$ is assignment of real vectors $p_1, \ldots, p_n$ to its vertices.
Universal completability

**Def:** A **framework** for $G$ is assignment of real vectors $p_1, \ldots, p_n$ to its vertices

**Notation:** $G(p) \subseteq \mathbb{R}^d$ where $p := \{p_1, \ldots, p_n\}$
Universal completability

**Def:** A framework for $G$ is assignment of real vectors $p_1, \ldots, p_n$ to its vertices

**Notation:** $G(p) \subseteq \mathbb{R}^d$ where $p := \{p_1, \ldots, p_n\}$

**ex:** a $C_{2k+1}$-framework

$$p_x = \left( \cos\left(\frac{2\pi k x}{2k + 1}\right), \sin\left(\frac{2\pi k x}{2k + 1}\right) \right), \ 0 \leq x \leq 2k,$$
Universal completability

**Def:** A framework $G(p)$ is called **UC** if $\forall q_1, \ldots, q_n$

$$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \quad \text{for} \quad [i = j \& ij \in E]$$

implies that

$$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \quad \text{for} \quad i, j \in [n]$$
Universal completability

**Def:** A framework $G(p)$ is called **UC** if \( \forall q_1, \ldots, q_n \)

\[
\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \quad \text{for} \quad [i = j & ij \in E]
\]

implies that

\[
\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \quad \text{for} \quad i, j \in [n]
\]

→ the $G$-partial matrix corresponding to $G(p)$ has a **unique** PSD completion
Why is it interesting?

Def [LV’14]: The Gram dimension of $G$ is the smallest $k$ for which every $G$-partial PSD matrix has a PSD completion of rank $\leq k$. 


Why is it interesting?

Def [LV’14]: The Gram dimension of $G$ is the smallest $k$ for which every $G$-partial PSD matrix has a PSD completion of rank $\leq k$.

→ the Gram dimension is minor-monotone.
Def [LV’14]: The Gram dimension of $G$ is the smallest $k$ for which every $G$-partial PSD matrix has a PSD completion of rank $\leq k$.

→ the Gram dimension is minor-monotone

To show $G$ is forbidden for $\text{gd}(G) \leq k$:

→ find $G$-partial PSD matrix

→ all its PSD completions have rank $> k$
Why is it interesting?

Def [LV’14]: The \textbf{Gram dimension} of \( G \) is the smallest \( k \) for which every \( G \)-partial PSD matrix has a PSD completion of rank \( \leq k \).

\[ \text{the Gram dimension is minor-monotone} \]

To show \( G \) is \textbf{forbidden} for \( \text{gd}(G) \leq k \):

\[ \text{find } G\text{-partial PSD matrix} \]

\[ \text{has unique PSD completion of rank } > k \]
Constructing UC frameworks

**Lem:** Let $G(p) \subseteq \mathbb{R}^d$ with $\text{span}(p_1, \ldots, p_n) = \mathbb{R}^d$. The framework $G(p)$ is UC if $\exists Z \in S^n$ satisfying:
Lem: Let $G(p) \subseteq \mathbb{R}^d$ with $\text{span}(p_1, \ldots, p_n) = \mathbb{R}^d$. The framework $G(p)$ is UC if $\exists Z \in S^n$ satisfying:

$Z$ is positive semidefinite
Constructing UC frameworks

**Lem:** Let $G(p) \subseteq \mathbb{R}^d$ with $\text{span}(p_1, \ldots, p_n) = \mathbb{R}^d$. The framework $G(p)$ is UC if $\exists Z \in S^n$ satisfying:

- $Z$ is positive semidefinite
- $Z_{ij} = 0$ when $ij \notin E$
Lem: Let $G(p) \subseteq \mathbb{R}^d$ with $\text{span}(p_1, \ldots, p_n) = \mathbb{R}^d$. The framework $G(p)$ is UC if $\exists Z \in \mathbb{S}^n$ satisfying:

- $Z$ is positive semidefinite
- $Z_{ij} = 0$ when $ij \notin E$
- $\sum_{j\in[n]} Z_{ij} p_j = 0$ for $i \in [n]$
Constructing UC frameworks

**Lem:** Let $G(p) \subseteq \mathbb{R}^d$ with $\text{span}(p_1, \ldots, p_n) = \mathbb{R}^d$.

The framework $G(p)$ is UC if \( \exists Z \in S^n \) satisfying:

1. $Z$ is positive semidefinite
2. $Z_{ij} = 0$ when $ij \notin E$
3. $\sum_{j \in [n]} Z_{ij} p_j = 0$ for $i \in [n]$
4. $\text{corank}(Z) = d$
Constructing UC frameworks

Lem: Let $G(p) \subseteq \mathbb{R}^d$ with $\text{span}(p_1, \ldots, p_n) = \mathbb{R}^d$. The framework $G(p)$ is UC if there exists $Z \in S^n$ satisfying:

1. $Z$ is positive semidefinite
2. $Z_{ij} = 0$ when $ij \notin E$
3. $\sum_{j \in [n]} Z_{ij} p_j = 0$ for $i \in [n]$
4. $\text{corank}(Z) = d$
5. $\text{span}(p_ip_j^T + p_jp_i^T : i, j \in V \cup E)$
How to use this?
**Def:** Let $M \in S^n$ with $\text{corank}(M) = d$ and $\{u_i\}_{i=1}^d$ an orth. basis for $\text{Ker}(M)$. Set $P = [u_1, \ldots, u_d]$ and let $\{p_i\}_{i=1}^n \subseteq \mathbb{R}^d$ be the rows of $P$. The map

$$[n] \mapsto \mathbb{R}^d, \text{ where } i \mapsto p_i$$

is called a **nullspace representation** of $M$. 
Nullspace representations

**Def:** Let $M \in S^n$ with $\operatorname{corank}(M) = d$ and $\{u_i\}_{i=1}^d$ an orth. basis for $\ker(M)$. Set $P = [u_1, \ldots, u_d]$ and let $\{p_i\}_{i=1}^n \subseteq \mathbb{R}^d$ be the rows of $P$. The map

$$[n] \mapsto \mathbb{R}^d, \quad \text{where } i \mapsto p_i$$

is called a nullspace representation of $M$

e.g. [Lovász, Schrijver’99, ’06]

Constructing outerplanar embeddings
Def [GRRSV’15]: A least eigenvalue framework (LEF) for G is a nullspace representation of $A - \lambda_{min}(A)I$. 
Least eigenvalue framework

Def [GRRSV’15]: A least eigenvalue framework (LEF) for $G$ is a nullspace representation of $A - \lambda_{\text{min}}(A)I$.

Let $\{u_i\}_{i=1}^{m}$ an orth. basis for $\text{Ker}(A - \lambda_{\text{min}}(A)I)$.
Least eigenvalue framework

Def [GRRSV'15]: A least eigenvalue framework (LEF) for G is a nullspace representation of $A - \lambda_{\text{min}}(A)I$.

Let $\{u_i\}_{i=1}^m$ an orth. basis for $\text{Ker}(A - \lambda_{\text{min}}(A)I)$

Set $P = [u_1, \ldots, u_m] \in \mathbb{R}^{n \times m}$. 
Least eigenvalue framework

Definition [GRRSV'15]: A least eigenvalue framework (LEF) for $G$ is a nullspace representation of $A - \lambda_{\min}(A)I$.

1. Let $\{u_i\}_{i=1}^m$ be an orthonormal basis for $\text{Ker}(A - \lambda_{\min}(A)I)$.
2. Set $P = [u_1, \ldots, u_m] \in \mathbb{R}^{n \times m}$.
3. The LEF is given by $i \in [n] \mapsto p_i \in \mathbb{R}^m$. 
Let \( \{u_i\}_{i=1}^m \) an orth. basis for \( \text{Ker}(A - \lambda_i(A)I) \)

Set \( P = [u_1, \ldots, u_m] \in \mathbb{R}^{n \times m} \).

The LEF is given by \( i \in [n] \mapsto p_i \in \mathbb{R}^m \).
Let \( \{u_i\}_{i=1}^m \) an orth. basis for \( \text{Ker}(A - \lambda_i(A)I) \)

Set \( P = [u_1, \ldots, u_m] \in \mathbb{R}^{n \times m} \).

The LEF is given by \( i \in [n] \mapsto p_i \in \mathbb{R}^m \)

e.g. see
- Chan, Godsil (eigenpolytopes)
General eigenvalue frameworks

Let \( \{u_i\}_{i=1}^m \) an orth. basis for \( \text{Ker}(A - \lambda_i(A)I) \)

Set \( P = [u_1, \ldots, u_m] \in \mathbb{R}^{n \times m} \).

The LEF is given by \( i \in [n] \mapsto p_i \in \mathbb{R}^m \)

e.g. see

- Chan, Godsil (eigenpolytopes)
- Brouwer & Haemers (Euclidean reps.)
Least eigenvalue framework

**Thm:** A least eig. framework $G(p) \subseteq \mathbb{R}^m$ is UC iff

$$\forall R \in S^m : \ p_i^T R p_j = 0 \text{ for } i = j \text{ and } (i, j) \in E \implies R = 0.$$
Least eigenvalue framework

**Thm:** A least eig. framework $G(p) \subseteq \mathbb{R}^m$ is UC iff

$$\forall R \in S^m : p_i^T R p_j = 0 \text{ for } i = j \text{ and } (i, j) \in E \implies R = 0.$$ 

**Pf:** $\iff$ Use the sufficient condition for UC with

$$Z = A - \lambda_{\min}(A)I$$
Least eigenvalue framework

**Thm:** A least eig. framework $G(p) \subseteq \mathbb{R}^m$ is UC iff

$$\forall R \in S^m : \quad p_i^T R p_j = 0 \text{ for } i = j \text{ and } (i, j) \in E \implies R = 0.$$ 

**Pf:** $\iff$ Use the sufficient condition for UC with

$$Z = A - \lambda_{\text{min}}(A)I$$

$\implies$ Scale so $\lambda_{\text{min}}(R) = -1$ and note that

$$P(I + R)P^T \neq P P^T$$
Graph cores
Def: A map $f : V(G) \to V(G)$ is an **endomorphism** if:

$$u \sim v \implies f(u) \sim f(v)$$
Cores

**Def:** A map \( f : V(G) \rightarrow V(G) \) is an endomorphism if:

\[
u \sim v \implies f(u) \sim f(v)
\]

**Def:** A graph is a core if all its endomorphisms are automorphisms.
Cores

**Def:** A map $f : V(G) \rightarrow V(G)$ is an **endomorphism** if:

$$u \sim v \implies f(u) \sim f(v)$$

**Def:** A graph is a **core** if all its endomorphisms are automorphisms

→ Complete graphs, odd cycles are cores
Cores

Def: A map $f : V(G) \rightarrow V(G)$ is an endomorphism if:

$$u \sim v \implies f(u) \sim f(v)$$

Def: A graph is a core if all its endomorphisms are automorphisms

→ Complete graphs, odd cycles are cores

→ Bipartite graphs are not cores
Cores

**Def:** An endomorphism $f$ is *locally injective* if

$$d(i, j) = 2 \implies f(i) \neq f(j)$$
Cores

**Def:** An endomorphism $f$ is **locally injective** if

$$d(i, j) = 2 \implies f(i) \neq f(j)$$

**Thm [Nešetřil]:** Every locally injective endomorphism is an automorphism
Graph cores via UC

Thm: Consider a framework $G(p)$ with

- $d(i, j) = 2 \implies p_i \neq p_j$
- $\|p_i\|$ constant for $i \in [n]$
- $\langle p_i, p_j \rangle$ constant for $ij \in E$
- $G(p)$ is UC

Then $G$ is a core.
Graph cores via UC

**Thm:** Consider a framework $G(p)$ with

- $d(i, j) = 2 \implies p_i \neq p_j$
- $\lVert p_i \rVert$ constant for $i \in [n]$
- $\langle p_i, p_j \rangle$ constant for $ij \in E$
- $G(p)$ is UC

Then $G$ is a core.

**Pf:** Show that any end. $f$ is locally injective
**Graph cores via UC**

**Pf:** Say $\exists u, v$ such that $d(u, v) = 2$ and $f(u) = f(v)$

$\implies$ Set $p'_i = p_{f(i)}$, $\forall i$

$\implies$ $G(p)$ and $G(p')$ are equivalent

$\implies$ $\exists$ isometry $U$ such that $p'_i = Up_i$, $\forall i$

So $\langle p_u, p_v \rangle = \langle p'_u, p'_v \rangle = \langle p_{f(u)}, p_{f(v)} \rangle$ which implies $p_u = p_v$ contradiction
Graph cores via UC

**Thm:** If there exists a framework $G(p)$ such that:

- $d(i, j) = 2 \implies p_i \neq p_j$
- $\|p_i\|$ constant for $i \in [n]$
- $\langle p_i, p_j \rangle$ constant for $ij \in E$
- $G(p)$ is UC

then $G$ is a core
Graph cores via UC

Thm: If there exists a framework $G(p)$ such that:

1. $d(i, j) = 2 \implies p_i \neq p_j$
2. $\|p_i\|$ constant for $i \in [n]$
3. $\langle p_i, p_j \rangle$ constant for $ij \in E$

Then $G(p)$ is UC

Use LEF

$G$ is a core
Graph cores via UC

**Thm:** If there exists a framework $G(p)$ such that:

- $d(i, j) = 2 \implies p_i \neq p_j$
- $\|p_i\|$ constant for $i \in [n]$
- $\langle p_i, p_j \rangle$ constant for $ij \in E$
- $G(p)$ is UC

then $G$ is a core
**1-walk-regular graphs**

**Def:** $G$ is 1WR if $\exists a_k, b_k \in \mathbb{N}$ such that:

\[ A^k \circ I = a_k I \]
\[ A^k \circ A = b_k A \]

for all $k \in \mathbb{N}$

Recall $(X \circ Y)_{ij} = X_{ij}Y_{ij}$
1-walk-regular graphs

Thm: Let $G$ be 1WR and $G(p) \subseteq \mathbb{R}^m$ a LEF. Then:

\[ \|p_i\|^2 = \frac{m}{n} \quad \text{for all } i \in [n] \]

\[ \langle p_i, p_j \rangle = \frac{\tau m}{na_2} \quad \text{for all } ij \in E \]
1-walk-regular graphs

**Pf:** Let $E_\tau$ the projector on the least eigenspace
1-walk-regular graphs

**Pf:** Let $E_τ$ the projector on the least eigenspace

$E_τ$ is a polynomial in $A$
1-walk-regular graphs

**Pf:** Let $E_\tau$ the projector on the least eigenspace

$E_\tau$ is a polynomial in $A$

$\exists a, b \in \mathbb{R}$ such that

$E_\tau \circ I = aI$ \text{ and } $E_\tau \circ A = bA$
1-walk-regular graphs

**Pf:** Let $E_{\tau}$ the projector on the least eigenspace

$\Rightarrow$ $E_{\tau}$ is a polynomial in $A$

$\Rightarrow$ $\exists a, b \in \mathbb{R}$ such that

$E_{\tau} \circ I = aI \text{ and } E_{\tau} \circ A = bA$

$\Rightarrow$ $E_{\tau} = \text{Gram}(p_1, \ldots, p_n)$
Putting everything together

**Thm:** Let $G$ be 1WR and $G(p) \subseteq \mathbb{R}^m$ a LEF. Say that

- $d(i, j) = 2 \implies p_i \neq p_j$
- $\text{span}(p_ip_j^T + p_jp_i^T : i, j \in V \cup E)$

Then $G$ is a core.
Applications
Kneser graphs

Def: The Kneser graph $K_{n:r}$ has:

Vertices: Subsets of $[n]$ of size $r$

Edges: $X \sim Y$ if $X \cap Y = \emptyset$
**Kneser graphs**

**Def:** The **Kneser graph** $K_{n:r}$ has:

- **Vertices:** Subsets of $[n]$ of size $r$
- **Edges:** $X \sim Y$ if $X \cap Y = \emptyset$

**Thm:** [Hahn, Tardif’97]

For $n \geq 2r + 1$ the graph $K_{n:r}$ is a core
**Def:** The Kneser graph $K_{n:r}$ has:

- Vertices: Subsets of $[n]$ of size $r$
- Edges: $X \sim Y$ if $X \cap Y = \emptyset$

**Thm:** [Hahn, Tardif’97]

For $n \geq 2r + 1$ the graph $K_{n:r}$ is a core

The proof uses EKR theorem
To show $K_{n:r}$ is a core using our sufficient condition:

Find LEF and show that

$p_i \neq p_j, \forall i, j$

$\text{span}(p_ip_j^T + p_jp_i^T : i, j \in V \cup E)$
References

Universal completability, least eigenvalue frameworks and vector colorings, arXiv:1512.04972

Vector colorings and graph homomorphisms, coming soon

Thank you!