Co-degree Density of Hypergraphs

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Extremal (Hyper)graph Problems

Study the max/min value of a function over a class of (hyper)graphs

- r-graph: r-uniform (hyper)graph.
- extremal graph: realizing the extreme value.
- $\mathcal{F}$-free: containing no member of $\mathcal{F}$ as a subgraph.

	Turán problem	codegree problem
function	size	min codegree
class	$\mathcal{F}$-free	$\mathcal{F}$-free
max	ex$(n, \mathcal{F})$	co-ex$(n, \mathcal{F})$

Graphs ($r = 2$)

Turán Theorem

ex(n, K_r) *is attained only by balanced* $(r-1)$*-partite graphs, so ex*(n, K_r) *is (about)* $(1 - \frac{1}{r-1})\binom{n}{2}$

Erdős-Simonovits-Stone theorem (ESS)

Fundamental theorem of (extremal) graph theory

ex(n, F) *is* $(1 + o(1))(1 - \frac{1}{\chi(F)-1})\binom{n}{2}$.

χ: chromatic number

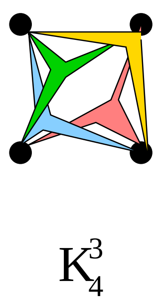
The only unknown case is bipartite graphs.

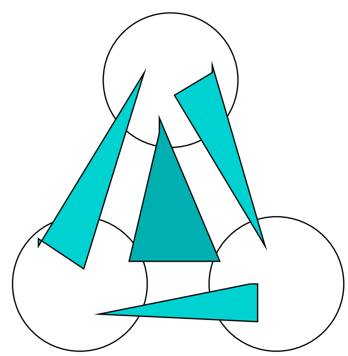
Hypergraphs

No Turán or **ESS** theorems: ex(n, F) is not even known for the complete 3-graph on 4 vertices.

Turán's Conjecture

ex(n, K_4^3) *is attained by*





(Erdős \$500) lim ex$(n, K_4^3)/\binom{n}{3} = \frac{5}{9}$.

Definition (Turán density)

$\pi(\mathcal{F}) = \lim_{n\to\infty}$ ex$(n, \mathcal{F})/\binom{n}{r}$

For graphs, $\pi(\mathcal{F}) = \min_{F\in\mathcal{F}} 1 - \frac{1}{\chi(F)-1}$ (**ESS**).

Degree problem = Turán problem

$x \in V(G)$, deg(x) = # edges containing x.
$\delta(G) = \min_{x \in V(G)} \deg(x)$.

Facts:
Let G be an n-vertex r-graph.

1. If $\delta(G) \geq c\binom{n-1}{r-1}$, then $e(G) \geq c\binom{n}{r}$.

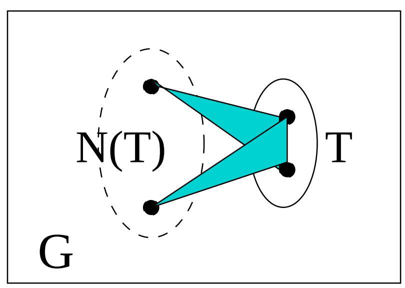
2. If $e(G) \geq (c+\varepsilon)\binom{n}{r}$, then G contains a subgraph G' on $m \geq \varepsilon^{1/r} n$ vertices with $\delta(G') \geq c\binom{m}{r-1}$.

ESS: *Every graph G_n with $\delta(G_n) \geq (1+\varepsilon)(1 - \frac{1}{\chi(F)-1})n$ contains a copy of F.*

Co-degree

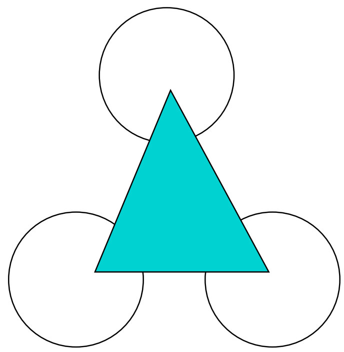
In r-graph G_n, $T \subset V(G)$ with $|T| = r - 1$,

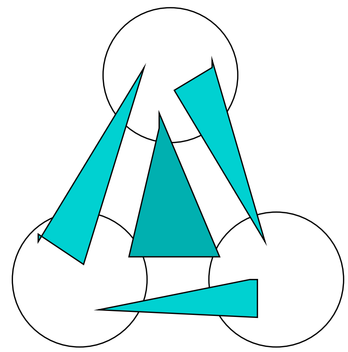
$N(T) = \{v \in V(G) : T \cup \{v\} \in E(G)\}$.



Co-degree $\text{codeg}(T) = |N(T)|$.

Let $\mathcal{C}(G) = \min_{T \subset V, |T| = r-1} \{\text{codeg}(T)\}$ and $c(G) = \mathcal{C}(G)/n$.





$C(K_3^3(t)) = 0$ $\qquad$ $C(T^3(n)) = \frac{n}{3}$

$e(K_3^3(t)) = \frac{2}{9}n$ $\qquad$ $e(T^3(n)) = \frac{5}{9}n$

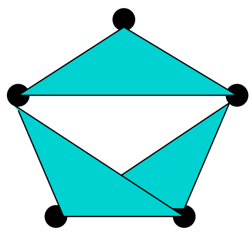
Definition:
The co-degree Turán number co-ex$(n, \mathcal{F})$ *of* $\mathcal{F}$ *is the maximum of* $\mathcal{C}(G_n)$ *over all* $\mathcal{F}$*-free* r*-graphs* G_n*. The co-degree density of* $\mathcal{F}$ *is*

$$\gamma(\mathcal{F}) := \limsup_{n\to\infty} \frac{\text{co-ex}(n,\mathcal{F})}{n}.$$

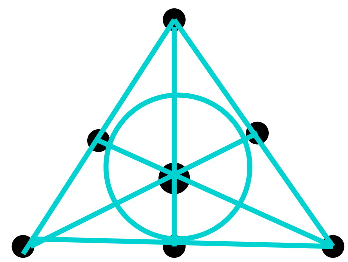
Fact 1: $\gamma(\mathcal{F}) \le \pi(\mathcal{F})$ (averaging).

Fact 2: $\gamma(\mathcal{F}) = \pi(\mathcal{F})$ when $r = 2$ (co-degree = degree)

Examples:



$\mathcal{D}3$



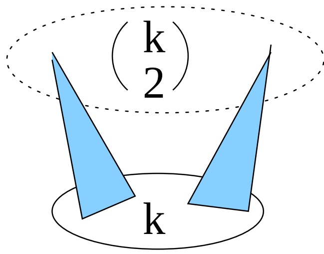
F

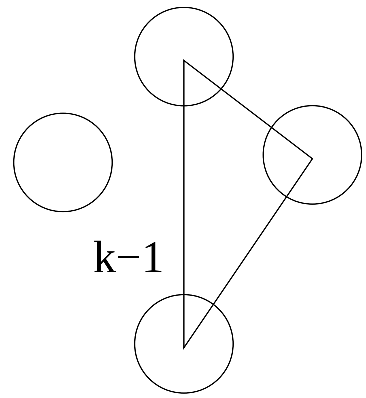
$\gamma(\mathcal{D}_3) = 0$ trivial; $\pi(\mathcal{D}_3) = 2/9$ (Frankl-Füredi)

$\gamma(\mathbf{F}) = 1/2$ (Mubayi); $\pi(\mathbf{F}) = 3/4$ (de Caen-Füredi)

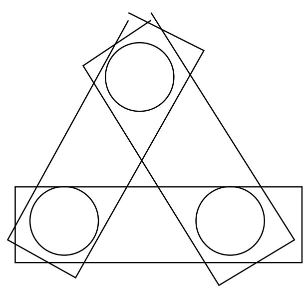
Conjectures: $\gamma(K_4^3) = 1/2$ (Nagle-Czygrinow), $\pi(K_4^3) = 5/9$ (Turán)

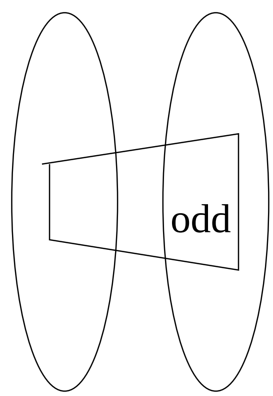
Example of $\gamma = 0$, $\pi \to 1$.





Example of $0 < \gamma = \pi$ (even r only).





(Frankl) $\pi(C_3^{2k}) = 1/2$.

Because of the symmetry of the extremal graph, this implies that $\gamma(C_3^{2k}) = 1/2$.

Fundamental questions on γ:

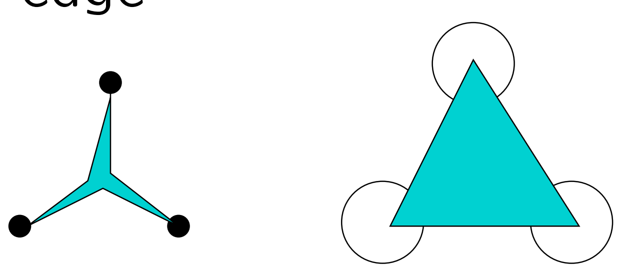
1. supersaturation
2. jumps
3. principality

Supersaturation

Theorem (Erdős, Simonovits)
Fix f-vertex F. For every $\varepsilon > 0$, there exists $\delta > 0$, s.t. every r-graph G_n (n sufficiently large) of size $\geq (\pi(F) + \varepsilon)\binom{n}{r}$ contains $\geq \delta\binom{n}{f}$ copies of F.

Corollary:
$\pi(F) = \pi(F(t))$, where $F(t)$ is a blow-up of F.

blow-up an edge



Theorem (Mubayi-Z)
Supersaturation holds for γ, and
$\gamma(F) = \gamma(F(t))$.

Jumps

Let $\Pi_r = \{\pi(\mathcal{F}) : \mathcal{F}$ is a family of r-graphs$\}$.

$\Pi_2 = \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \}$ (**ESS**).

Much less known when $r \geq 3$:

Proposition: $\pi(\mathcal{F}) \notin (0, r!/r^r)$ for any $\mathcal{F}$.

Definition (Jump).
Given a function f and $r \geq 2$. A real number $0 \leq \alpha < 1$ is called a jump for r in terms of f if $\exists \delta > 0$, such that no family $\mathcal{G}$ of r-graphs satisfies $f(\mathcal{G}) \in (\alpha, \alpha + \delta)$.

In terms of π
every $0 \leq \alpha < 1$ is a jump for $r = 2$,
every $0 \leq \alpha < r!/r^r$ is a jump for $r \geq 3$ (Proposition).

Conjecture (Erdős 1977):
every $c \in [0, 1)$ *is a jump for* $r \geq 3$.

Theorem (Frankl-Rödl 1984):
$1 - 1/\ell^{r-1}$ *is not a jump for* $r \geq 3$ *and* $\ell > 2r$.

Problem: *is* $r!/r^r$ *a jump for* $r \geq 3$?

Theorem (Mubayi-Z):
For each $r \geq 3$, *no* $\alpha \in [0, 1)$ *is a jump for* γ.

Corollary: *For each* $r \geq 3$,

$$\Gamma_r = \{\gamma(\mathcal{F}) : \mathcal{F} \text{ is family of } r\text{-graphs}\}$$

is dense in $[0, 1)$.

Principality

Clearly $\pi(\mathcal{F}) \leq \pi(F)$, for all $F \in \mathcal{F}$.

Definition:
π is principal for r if $\pi(\mathcal{F}) = \min_{F\in\mathcal{F}} \pi(F)$ for every finite family $\mathcal{F}$ of r-graphs.

$r = 2$, principal (**ESS**)

$r \geq 3$, non-principal (Balogh, Mubayi-Pikhurko)

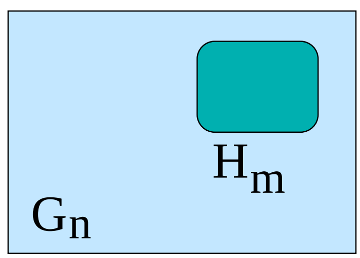
Theorem(Mubayi-Z):
γ is not principal for each $r \geq 3$, i.e., *there exists a finite family $\mathcal{F}$ of r-graphs s.t.* $0 < \gamma(\mathcal{F}) < \min_{F\in\mathcal{F}} \gamma(F)$.

Comparing γ and π

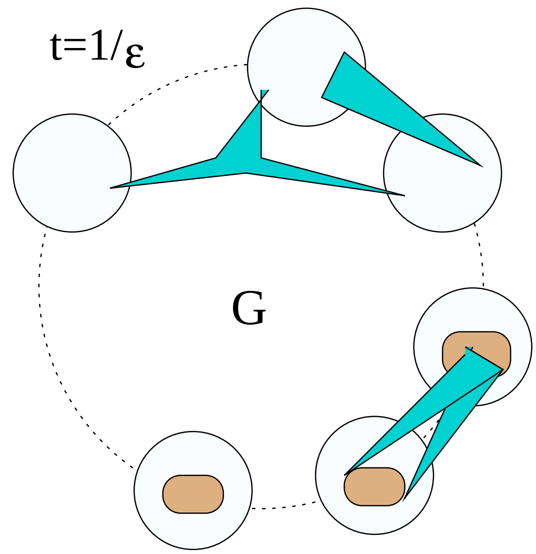
	graphs	(h) π	(h) γ
supersaturation	$\surd$	$\surd$	$\surd$
Jumps	$\surd$	$\surd, \times$	$\times$
principality	$\surd$	$\times$	$\times$

An equivalent definition for jumps

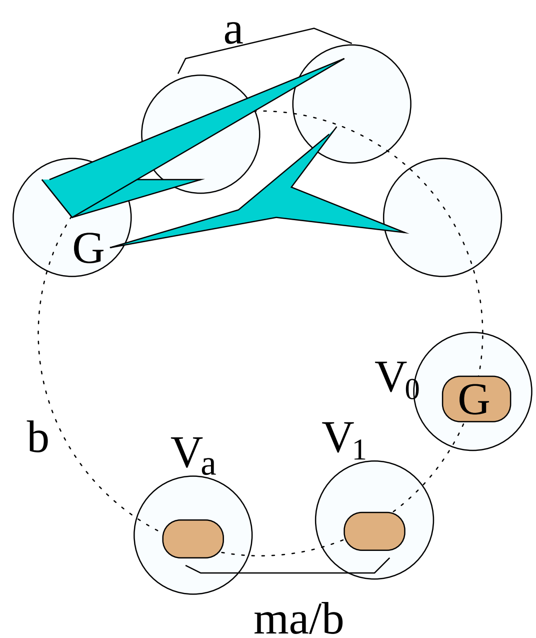
(Definition) α *is a jump if* $\exists\,\delta > 0$ *s.t.* $\forall \varepsilon > 0$, *every large* $\{G_n\}$ *with* $c(G_n) \geq \alpha + \varepsilon$ *contains a subgraph* $H_m \subseteq G_n$ *for which* $m \to \infty$ *as* $n \to \infty$ *and* $c(H_m) \geq \alpha + \delta$.



Proof that 0 is not a jump.



Proof that $\alpha = \frac{a}{b}$ is not a jump.



Open problems:

• What if replacing $\mathcal{F}$ be F in the definition of jump (for π or γ)? Harder to prove no jumps

• $\Gamma_r = [0, 1)$?
where $\Gamma_r = \{\gamma(\mathcal{F}) : \mathcal{F}$ is a family of r-graphs$\}$.

• Find two 3-graphs F_1, F_2 with
$0 < \gamma(F_1, F_2) < \min\{\gamma(F_1), \gamma(F_2)\}$.