DC power flow in rectangular coordinates

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Abstract

"DC power flow" is an analogy between approximations to the real power components of the power flow equations and a direct current resistive circuit. It can also be interpreted as linearization of the real power components expressed in terms of phasor voltage magnitude and phase, linearized about a "flat start." The accuracy of DC power flow for estimating real power flow is surprisingly good in many cases, although it has large errors in some cases.

We explore linearization of both the real and reactive power equations expressed in terms of real and imaginary parts of the voltage phasor. We focus on linearization about a flat start, which in rectangular coordinates has the voltage phasors each with real part one per unit and imaginary part zero.

The resulting approximation has relatively good performance for real power. Because of the analog with linearization in terms of polar voltage representation, we call this approximation "DC power flow in rectangular coordinates." We also exhibit an exact solution to the power flow equations for the particular case of a lossless network.

Keywords DC power flow, linearization.

1 Introduction

- Exact or approximate solution of the power flow equations is essential to the planning, operation, and control of power systems.
- DC power flow [1, §4.1.4][2] is an analogy between an approximation to the real power components of the AC power flow equations and a related direct current circuit:
  - can be interpreted as linearization of the real power components in terms of phasor voltage magnitude and phase, that is, "polar coordinates,"
  - linearization is about a "flat start," where the voltage phasors each have magnitude one per unit and angle zero [3].
- Most day-ahead locational electricity markets use DC power flow [4] or linearization.

1.1 Accuracy of DC power flow using polar coordinates

- DC power flow is accurate for real power flow, except when:
  - angle differences across lines are large, or
  - voltage magnitudes deviate significantly from one per unit [3, 5, 6].
- Precisely the most important conditions for evaluating limits, particularly post-contingency limits!
- Linearization for reactive power in terms of polar coordinates is poor.

1.2 Rectangular coordinates

- Recent significant progress in power flow and optimal power flow (OPF) has used phasor voltage real and imaginary parts:
  - "rectangular coordinates" [7, 8, 9].
- Semi-definite programming formulations of the OPF problem using rectangular coordinates have provided provably optimal solutions [10]:
  - may become the dominant approach to OPF.
- Linearization still standard currently in online applications.
- Consider linearization in rectangular coordinates.

2 Formulation

- $v \in \mathbb{C}^n$ is the vector of complex phasor voltages at all n buses in the transmission system,
- $i \in \mathbb{C}^n$ is the vector of complex current injections into the transmission system,
- $s \in \mathbb{C}^n$ is the vector of complex power injections into the transmission system.
- Represent v in rectangular coordinates by writing:

$$v = \mathbf{1} + e + jf,$$

where:

$\mathbf{1}$ is the vector of all ones,
$e, f, \in \mathbb{R}^n$, and
j is the square root of minus 1.

- Define a "flat start" to be $v = \mathbf{1}$, corresponding to $e = f = \mathbf{0}$.

Formulation, continued

- $Y = G + jB$ is the bus admittance matrix for the system:
  - Off-diagonal entries are equal to minus the admittance of the corresponding *series* elements joining corresponding buses,
  - Diagonal entries are the sum of the admittances joined to the corresponding buses, due to both series and *shunt* elements.

$$\begin{aligned} G &= G^{\text{series}} + G^{\text{shunt}}, \\ B &= B^{\text{series}} + B^{\text{shunt}}, \end{aligned}$$

  where:

  G^{shunt} and B^{shunt} are diagonal,

  G^{series} and B^{series} are symmetric (putting aside cases such as where transformers have off-nominal turns ratios), and

  $G^{\text{series}}\mathbf{1} = B^{\text{series}}\mathbf{1} = \mathbf{0}$, where $\mathbf{0}$ is the vector (or depending on interpretation, matrix) of all zeros.
  - Assume $G^{\text{shunt}} = \mathbf{0}$.

Formulation, continued

- Kirchhoff's current law:

$$
\begin{aligned}
i &= Yv, \\
&= (G + jB)(\mathbf{1} + e + jf), \\
&= (G^{\text{series}} + G^{\text{shunt}} + j(B^{\text{series}} + B^{\text{shunt}}))(\mathbf{1} + e + jf), \\
&= (G^{\text{series}} + j(B^{\text{series}} + B^{\text{shunt}}))(\mathbf{1} + e + jf), \text{ since } G^{\text{shunt}} = \mathbf{0}, \\
&= G^{\text{series}}\mathbf{1} + G^{\text{series}}e - (B^{\text{series}} + B^{\text{shunt}})f \\
&\quad + j(G^{\text{series}}f + B^{\text{series}}\mathbf{1} + B^{\text{series}}e + B^{\text{shunt}}\mathbf{1} + B^{\text{shunt}}e), \\
&= G^{\text{series}}e - (B^{\text{series}} + B^{\text{shunt}})f + j(G^{\text{series}}f + B^{\text{series}}e + B^{\text{shunt}}\mathbf{1} + B^{\text{shunt}}e),
\end{aligned}
$$

- since $G^{\text{series}}\mathbf{1} = B^{\text{series}}\mathbf{1} = \mathbf{0}$.
- Define superscript $\dagger$ to mean transpose, superscript $\ddagger$ to mean Hermitian transpose (that is, transpose of complex conjugate), and $\text{diag}(\bullet)$ to be a vector consisting of the diagonal elements of its argument.
- For any vector x and diagonal matrix D, $\text{diag}\left(\mathbf{1}x^{\dagger}\right) = x$ and $\text{diag}\left(x(D\mathbf{1})^{\dagger}\right) = Dx$.

Formulation, continued

- Then:

$$
\begin{aligned}
s &= \operatorname{diag}\left(v i^{\ddagger}\right), \\
&= \operatorname{diag}\left((\mathbf{1}+e+jf)\begin{pmatrix} G^{\text{series}} e - (B^{\text{series}} + B^{\text{shunt}}) f \\ -j\left(G^{\text{series}} f + B^{\text{series}} e + B^{\text{shunt}}\mathbf{1} + B^{\text{shunt}} e\right) \end{pmatrix}^{\dagger}\right).
\end{aligned}
$$

- Linearization of s about a flat start of $v = \mathbf{1}$ preserves the affine terms and discards the purely quadratic terms:

$$
\begin{aligned}
s &\approx \operatorname{diag}\left(\mathbf{1}\begin{pmatrix} G^{\text{series}} e - (B^{\text{series}} + B^{\text{shunt}}) f \\ -j\left(G^{\text{series}} f + B^{\text{series}} e + B^{\text{shunt}}\mathbf{1} + B^{\text{shunt}} e\right) \end{pmatrix}^{\dagger}\right) \\
&\quad + \operatorname{diag}\left((e+jf)\left(-jB^{\text{shunt}}\mathbf{1}\right)^{\dagger}\right), \\
&= G^{\text{series}} e - (B^{\text{series}} + B^{\text{shunt}}) f - j(G^{\text{series}} f + B^{\text{series}} e + B^{\text{shunt}}\mathbf{1} + B^{\text{shunt}} e) \\
&\quad - jB^{\text{shunt}}(e+jf), \text{ since } B^{\text{shunt}} \text{ is diagonal}, \\
&= G^{\text{series}} e - B^{\text{series}} f + j(-(B^{\text{series}} + 2B^{\text{shunt}})e - G^{\text{series}} f - jB^{\text{shunt}}\mathbf{1}).
\end{aligned}
$$

Formulation, continued

- Separating s into real and reactive power injections:

$$\begin{bmatrix} p \\ q \end{bmatrix} \approx \begin{bmatrix} G^{\text{series}} & -B^{\text{series}} \\ -(B^{\text{series}} + 2B^{\text{shunt}}) & -G^{\text{series}} \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ B^{\text{shunt}}\mathbf{1} \end{bmatrix}. \qquad (1)$$

- Similar form to case in polar coordinates.

3 Special case

- Purely quadratic terms in s are:

$$\mathrm{diag}\left((e+jf)\left(G^{\text{series}}e-(B^{\text{series}}+B^{\text{shunt}})f-j\left(G^{\text{series}}f+B^{\text{series}}e+B^{\text{shunt}}e\right)\right)^{\dagger}\right),$$

- which has real part:

$$\mathrm{diag}\left(e\left(G^{\text{series}}e-(B^{\text{series}}+B^{\text{shunt}})f\right)^{\dagger}+f\left(G^{\text{series}}f+B^{\text{series}}e+B^{\text{shunt}}e\right)^{\dagger}\right),$$

- Following Zhang and Tse [9, Appendix], if $e=\mathbf{0}$ and $G^{\text{series}}=\mathbf{0}$ then the real part of the quadratic term equals zero.
- For given p, if we solve $p=-B^{\text{series}}f$ for f then $v=1+jf$ is an exact solution to the power flow equations.
- Corresponding reactive power injections are:

$$\begin{aligned} q &= \mathrm{diag}\left(-\mathbf{1}\left(G^{\text{series}}f+B^{\text{shunt}}\mathbf{1}\right)^{\dagger}-f\left((B^{\text{series}}+B^{\text{shunt}})f\right)^{\dagger}\right), \\ &= -G^{\text{series}}f-B^{\text{shunt}}\mathbf{1}-\mathrm{diag}\left(f\left((B^{\text{series}}+B^{\text{shunt}})f\right)^{\dagger}\right) \end{aligned}$$

Special case, continued

- Analogous to the DC approximation in polar coordinates where the relationship between power and angle is derived under the assumption that the voltage magnitudes are fixed, with the values of reactive injection implicitly determined:
  - however, DC approximation in polar coordinates is not exact even if $G^{\text{series}} = \mathbf{0}$.
- In contrast,

$$\begin{aligned} -B^{\text{series}} f &= p, \quad (2)\\ q &= -G^{\text{series}} f - B^{\text{shunt}} \mathbf{1} - \operatorname{diag}\left(f \left(\left(B^{\text{series}} + B^{\text{shunt}} \right) f \right)^{\dagger} \right), \quad (3) \end{aligned}$$

- are an *exact* solution to the power flow equations in the lossless case.

4 Numerical example

- Consider a two bus system with:
  - a single line joining the two buses having series admittance $1 - j10$ and no shunt admittance,
  - one of the buses having phasor voltage held at $1 + j0 \in \mathbb{C}$, and
  - the other bus having phasor voltage $1 + e + jf \in \mathbb{C}$.
- We consider the exact and linearized approximations for the real and reactive power injected at the other bus as a function of the bus phasor voltage.
- Consider DC approximations in terms of both polar and rectangular coordinates.

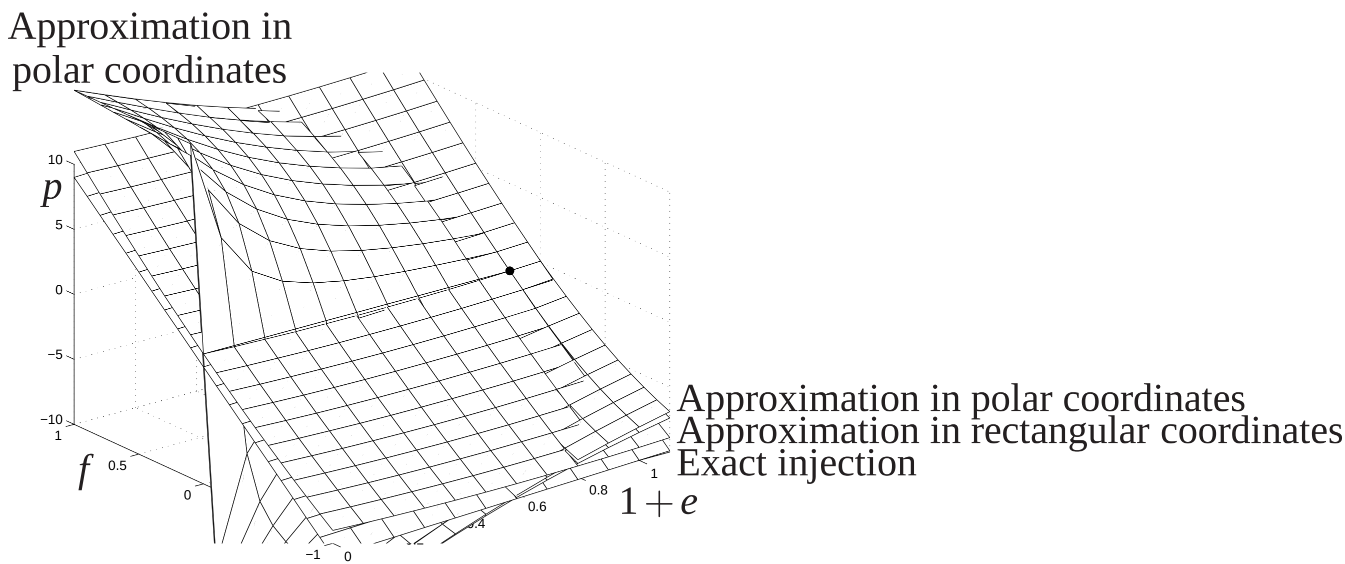

Fig. 1. Real power injection p versus $1+e+jf$.

Real power, continued

- Exact real power injection and the approximation in terms of rectangular coordinates are qualitatively close together throughout the graphed range:
  - approximate sensitivities also close to the actual sensitivities.
- DC approximation of real power in terms of polar coordinates deviates significantly from the exact value for some values of $1+e+jf$.
- In the half-annular region where voltage magnitudes are within 10% of 1 per unit:
  - for small values of voltage phasor angle, smaller than $\pi/4$ say, and voltage magnitudes close to one per unit, DC approximation of real power in polar coordinates is highly accurate.
  - for large angles or for voltage magnitudes deviating significantly from one per unit, the approximation can be quite poor.
- Sensitivities of real power injection to changes in the magnitude and angle are relatively far from the actual sensitivities.
- Approximation of real power in rectangular coordinates is qualitatively better match to actual injection than approximation in polar coordinates.

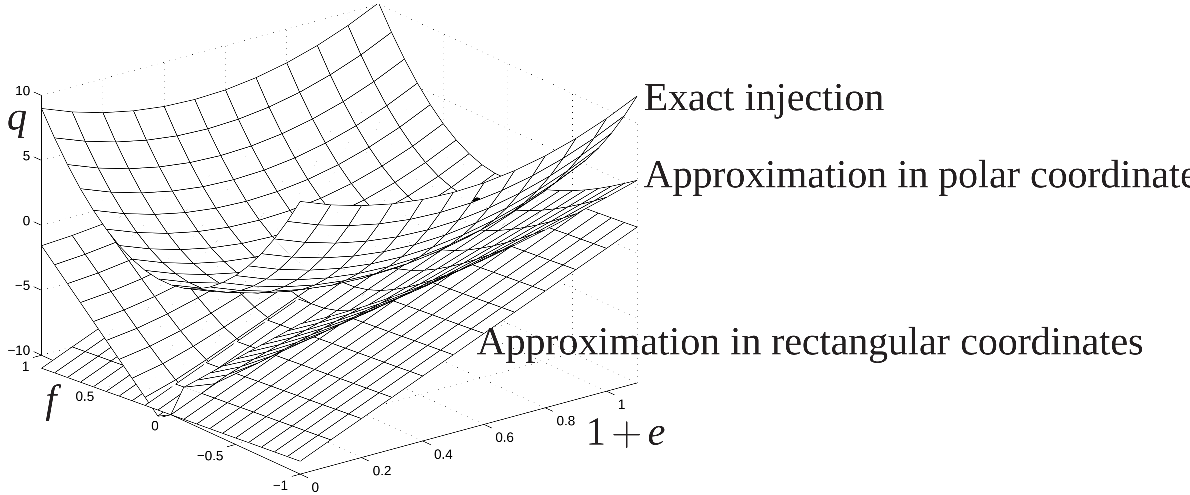

Fig. 2. Reactive power injection q versus $1+e+jf$.

Reactive power, continued

- Approximate reactive power injections in terms of rectangular coordinates is poorer than the conventional DC approximation in terms of polar coordinates.
- Functional dependence on e and f is highly non-linear.

5 Improved approximation for reactive power

- Following Coffrin and Van Hentenryck [11], propose piecewise linearizing reactive power while maintaining the linear approximation of real power:

$$
\begin{aligned}
p &= \begin{bmatrix} G^{\text{series}} & -B^{\text{series}} \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}, \\
\begin{bmatrix} q \\ \vdots \\ q \end{bmatrix} &\geq \begin{bmatrix} -(B^{\text{series}} + 2B^{\text{shunt}}) & -G^{\text{series}} \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} - \begin{bmatrix} B^{\text{shunt}}\mathbf{1} \\ \cdots \\ \cdots \end{bmatrix}.
\end{aligned}
$$

- where the terms $\cdots$ would be calculated from linearization about other operating points besides $v = \mathbf{1}$:
  - could be calculated off-line.

6 Extensions and conclusions

- Developed linearization of power flow in rectangular coordinates.
- Accurate for real power, but not for reactive power.
- In the lossless case without explicit representation of reactive power, non-linear power flow can be solved exactly through the solution of a linear equation and substitution to evaluate reactive power injections.
- Piecewise linearization may be an effective approach where reactive power is to be explicitly represented.
- Analogous approximations can be developed for current and complex power flow as a function of the rectangular representation [3].

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