QUASIRANDOMNESS

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Van der Waerden		1927	
Erdős–Turán conjecture		1936	
Roth	$k=3$	1952	
Szemerédi	$k=4$	1969	
Szemerédi	$\forall k$	1975	
Szemerédi	reg. lemma	1978	! (75)
Fürstenberg		1977	
		⋮	

Tao (06)

"There are many diffent proofs of this deep theorem, but they are all based on a fundamental dichotomy between structure and randomness, which in turn leads to a decomposition of any object into a structured component and a random component"

THEOREM (SZEMEREDI, 1975)

All subsets of the integers of positive upper density contain arbitrarily long arithmetic progressions.

Szemerédi	combinatorial
Fürstenberg, ..	ergodic th.
Gowers	Fourier anal.
Gowers	hypergr.
Rödl, Nagle, Schacht, Skokan	hypergr

Green – Tao: (200?)

All subsets of the primes of positive relative upper density contain arbitrarily long arithmetic progressions.

Szemerédi regularity lemma

counting lemma

removal lemma

(6,3) problem - Ruzsa-Szemerédi

?(7,4)?

regularity lemma for

hypergraphs → numberth. — Frankl-Rödl, Gowers, Nagel, Rödl, Schacht, Skokan, Tao

abelian groups → number th, extr. groupth — Green

Permutations — Cooper

Matrices — Frieze-Kannan,..

Analysis — Lovász - B. Szegedy

Probability, inf. — Tao

Szemerédi regularity lemma

~ approximation of dense graphs
by piecewise randomlike graphs

? random-like ?
pseudo-random
quasirandom

graphs	Chung + Graham, Wilson
hypergraphs	" "
integers	" "
abelian groups	" "
set systems	" "
permutation matrices	Cooper
groups	Gowers

Chung, Graham, Wilson

Haviland, Thomason

Kohayakawa, Rödl, Skokan

Gowers

Babai, Nisan, M. Szegedi

Chung, Tetali

Krivelevich, Sudakov

Simonovits - S.

NUMBER THEORY $\longleftrightarrow$ GRAPH THEORY

$A \subseteq [N]$ $\longleftrightarrow$

$x+y \in E(G) \Leftrightarrow x+y \in A$

$x-y$

exp $a+b=c+d \longleftrightarrow C_4$ in G

solution of $M\underline{x}=0 \longleftrightarrow \exists$ some small subgraph

{C. Borgs, J. Chayes, L. Lovász, V.T. Sós,
K. Vesztergombi}

physics

graphs

{M. Freedman, L. Lovász, A. Schrijver}

algebraic

{L. Lovász, B. Szegedy}

analysis

R. Wilson ~ 1981

"Let H be a Hadamard matrix of order n and M an $s \times t$ matrix of $+1$'s and -1's.

Let N denote the number of ordered choices $(i_1, \dots, i_s; j_1, \dots, j_t)$ of s distinct rows and t distinct columns such that the corresponding submatrix of H coincides with M, i.e $H(i_\alpha, j_\beta) = M(\alpha, \beta)$.

Then

$$\left| N - \frac{n(n-1)\dots(n-s+1)\, n(n-1)\dots(n-t+1)}{2^{st}} \right| < c\, n^{s+t-\frac{1}{2}}."$$

Contractors and connectors of graph algebras*

László Lovász and Balázs Szegedy
Microsoft Research
One Microsoft Way

April 2005

Contents

*AMS Subject Classification: Primary 05C99, Secondary 16S99

Thomason (87) (Pseudo-random gr.)

A graph G_n is <u>(p,α)-jumbled</u>

if for every induced subgraph $G_n(X)$

$$\left| e(G_n(X)) - p\binom{|X|}{2} \right| < \alpha |X|. \quad (*)$$

" (p,α)-jumbled graphs behave in many ways like a random graph with edge-probability p."

Erdős - Spencer (72')

$$\alpha > (pn)^{1/2} ;$$

the discrepancy in $(*)$ cannot be too small!

QUASIRANDOM GRAPH SEQUENCES

F, G simple graphs, $|V(G_n)| = n$

$\hom(F,G) =$ # homomorphism from F to G

↓

adjacency preserving maps

A graph sequence (G_n) is

<u>p-quasirandom</u>, if for

$\forall F_k$

$$\underbrace{t(F_k, G_n)}_{\text{F-density of } G_n} := \frac{\hom(F, G_n)}{n^{k}} \longrightarrow \underbrace{p^{|E(F_i)|}}_{\text{density in p-random gr.}}$$

Theorem (Chung-Graham-Wilson 89')

$\forall\ (G_n)$ the following are equivalent:

① for some fixed $\boxed{\nu \geq 4}$, $\forall\ F_\nu$

$$t(F_\nu, G_n) \to p^{|E(F_\nu)|}$$

$\Updownarrow$

② $\begin{cases} t(\,|\,, G_n) \to p \\ t(\square, G_n) \to p^4 \end{cases}$

$\Updownarrow$

③ $\begin{cases} t(\,|\,, G_n) \to p \\ \dfrac{\lambda_1(G_n)}{n} \to p \\ \dfrac{\lambda_2(G_n)}{n} \to 0 \end{cases}$

$\Updownarrow$

④ $\forall\ X \subset V(G_n)$, $|X| = \frac{n}{2}$

$$e(G_n(X)) \sim p^2 \frac{n^2}{4}$$

$\Updownarrow$

⑤ $$\sum_{x,y \in V} |n(x,y) - p^2 n| = o(n^3)$$

$n(x,y)$: # common neighbors

• let $n = |V(G_n)|$

$\Updownarrow$

6 α fixed, $\boxed{\alpha \neq \frac{1}{2}}$, $0 < \alpha < 1$

$\forall \ X \subseteq V(G_n)$, $|X| = \alpha n$

$$e(X, V-X) = \frac{1}{2}\alpha(1-\alpha)n^2 + o(n^2)$$

$\Updownarrow$

7 (Simonovits - S.)

~ in Szemerédi - regular partitions

"a. all" the densities are $p + o(1)$.

Constructions

① Paley-graphs $\qquad p \equiv 1 \ (4)$

$V = \{1, \dots, p\}$

$E = \{(i,j) : |i-j| \text{ quadr. res. mod } p\}$

② $V = 2^{[n]}$

$(A_i, A_j) \in E \iff |A_i \cap A_j| \equiv 0 \ (2)$

③ α irrat $\qquad 0 < \delta < 1$

$V = \{1, \dots, n\}$

$(i,j) \in E \iff \{(i-j)^2 \alpha\} < \delta$

Erdős – Bollobás

⋮

Krivelevich – Sudakov

strongly regular graphs

Hereditarily extended properties

$$\forall X \subseteq V(G_n) \quad e(X) = p\binom{|X|}{2} + o(n^2) \iff (G_n) \text{ p-quasi-random}$$

 edges are uniformly distributed

$$\Downarrow$$

(a) $\forall H_\nu$ induced copies of H_ν are uniformly distributed

(b) $\forall H_\nu$ not necessarily induced copies of H_ν are uniformly distributed.

¿ Is the same true for arbitrary F instead of  ?

Simonovits-S.:

(b) YES

(a) ¿ NO for 

YES for some F

F, G simple graphs

$\hom(F,G)$ = # adjacency preserving
$\varphi : V(F) \to V(G)$

homomorphisms density:

$$t(F,G) = \frac{1}{|V(G)|^{|V(F)|}} \hom(F,G)$$

i.e. the probability that a random $\varphi : V(F) \to V(G)$ is a homomorphism

CONVERGENCE OF GRAPHS

(G_n), F, G_n simple

$$t(F,G_n) = \frac{\hom(F,G_n)}{|V_{G_n}|^{|V_F|}}$$

Def (G_n) is convergent, if

$\forall F$ $\quad t(F,G_n)$ is convergent

Exp $(G(n;p))$ p-random

$$t(F,G_n) \longrightarrow p^{|E_F|}$$ with prob

$(G(n;p))$ p-quasirandom

$$t(F,G_m) \longrightarrow p^{|E_F|}$$

random,

Exp (G_n) generalized H-quasiranda

$$t(F,G_n) \longrightarrow t(F,H)$$

F simple, unweighted, H weighted:

H : $\alpha_i > 0 \quad \forall i \in V(H)$

$\beta_{ij} \quad (i,j) \in E(H)$

$$\hom(F,H) = \sum_{\varphi : V_F \to V_H} \prod_{i \in V_F} \alpha_{\varphi(i)} \prod_{ij \in E_F} \beta_{\varphi(i)\varphi(j)}$$

$$t(F,H) = \frac{\hom(F,H)}{(\sum \alpha_i)^{|V_F|}}$$

H unweighted ~

all vertex-weights, edge-weights are 1 (no edge: 0)

FACT.

$\forall F \quad t(F, G(n;H)) \longrightarrow t(F,H)$

with prob. 1

GENERALIZED RANDOM GRAPHS

H weighted graph

$V(H) = \{1, \dots, q\}$,

vertex-weights : $\alpha_i > 0 \quad 1 \le i \le q$

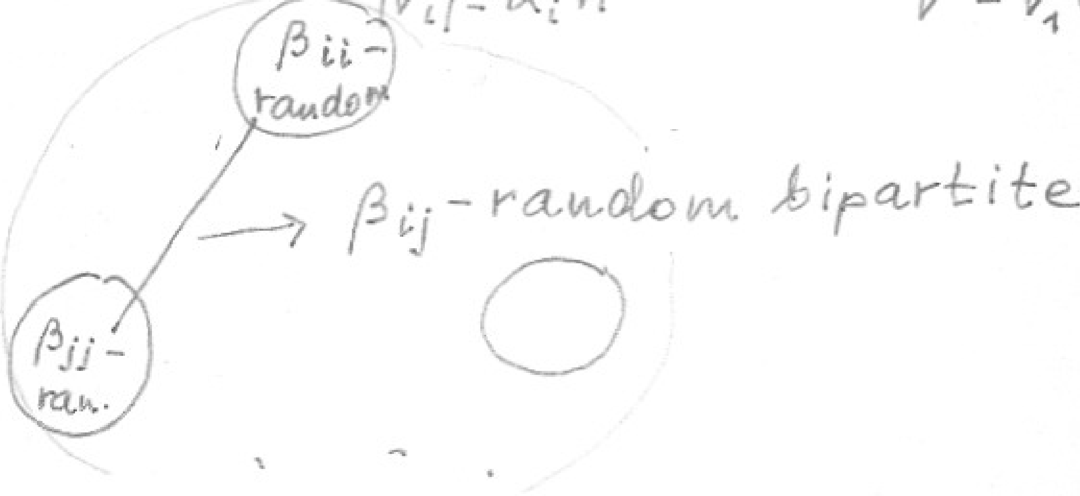
edge-weights : $0 \le \beta_{ij} \le 1$

We may assume: H is complete, with loops

<u>Generalized random graph</u>

<u>with model H</u> : $G(n;H)$

$|V_i| = \alpha_i n \qquad V = V_1 \cup \dots \cup V_q$



<u>Def</u> (G_n) is an <u>H-quasirandom sequence</u>, if

$$\forall F \qquad t(F, G_n) \longrightarrow t(F, H) \quad (*)$$

<u>Exp</u> $G(n;H)$ H-random

$$t(F, G(n;H)) \longrightarrow t(F, H)$$

with prob. 1

Questions

? Is the structure of G_n similar to $G(n, H)$?

? Is it enough to require $(*)$ for a finite set $\{F_1, \dots, F_\nu\}$ (depending on H)?

Chung-Graham-Wilson

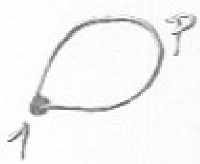
$$\left.\begin{array}{l} t(\bullet\!\!-\!\!\bullet, G_n) \to p \\ t(\square, G_n) \to p^4 \end{array}\right\} \Longrightarrow (G_n) \text{ is } p\text{-quasirandom}$$

? WHAT IS THE LIMIT OBJECT ?

(G_n) p-random

p-quasirandom

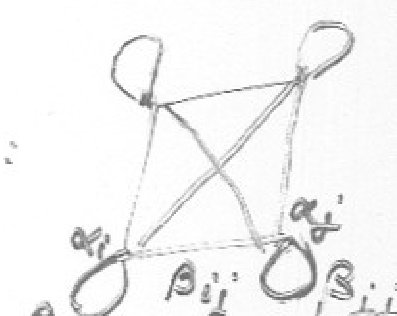
$$\forall F \qquad t(F, G_n) \longrightarrow p^{\epsilon(F)} = t(F, H)$$

H : 

(G_n) H-random

H-quasirandom

$$\forall F \qquad t(F, G_n) \longrightarrow t(F, H)$$

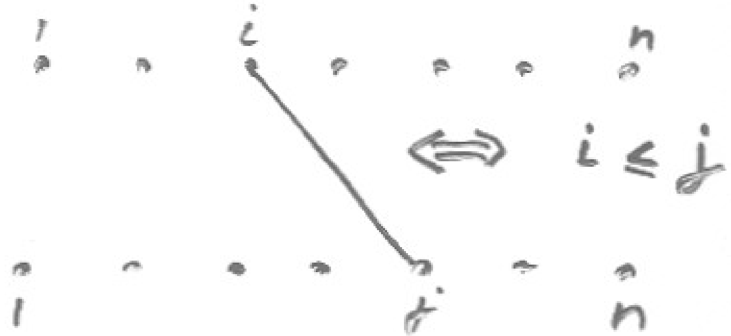
H : 

? is H the limit ?

? (G_n) convergent, but not H-quasirandom ?

(G_n) convergent, but not generalized quasirandom

<u>Exp</u> (G_{2n}) half-graphs



p - quasirandom

$t(F, G_n) \to p^{e(F)}$

$H =$ 

$\to t(F,$  $)$

$t(F, G_n) \to t(F, H)$

t H

Lovász – B. Szegedy

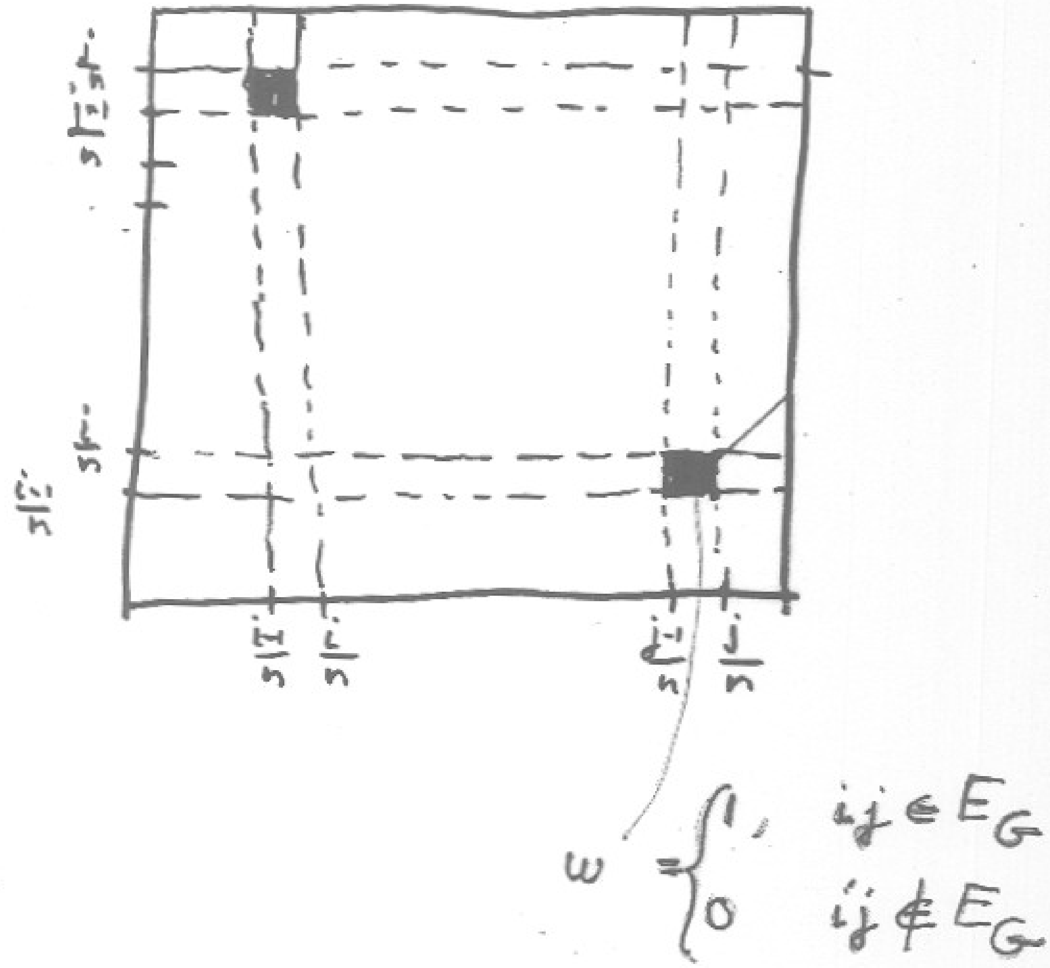
Let

$$\mathcal{W} = \{ w : [0,1]^2 \to [0,1], \text{ meas., symm.} \}$$

- $$t(F,w) := \int_{[0,1]^k} \prod_{ij \in E_F} w(x_i, x_j)\, dx$$

- $k = |E_F|$

$G_n \longleftrightarrow \omega_{G_n}$



$$t(F, G) = t(F, \omega_G)$$

Theorem (Lovász - B. Szegedy)

- For every convergent graph sequence (G_n) there is a $w \in \mathcal{W}$ such that $\forall F$

$$\lim_{n\to\infty} t(F, G_n) = t(F, w)$$

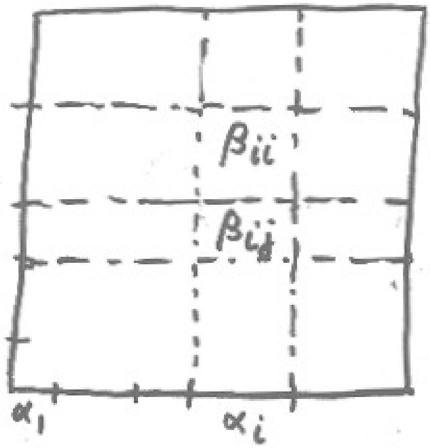
- $\forall\, w \in \mathcal{W}$ arises as a limit of some sequence (G_n)

Borgs - Chayes - Lovász
w determined upto meas. pres. trans

$$\boxed{\begin{array}{l} \forall F \quad \lim t(F, G_n) = t(F, w) \\ \quad\Updownarrow \\ \qquad\qquad \lim G_n = w \end{array}}$$

Let H be a weighted graph

$w_H \in W$ $\qquad \sum_1^k \alpha_i = 1$



step-function

$\left.\begin{array}{l} G(n,p), \\ (G_n)\ p\text{-quasirandom} \end{array}\right\} \longrightarrow \lim = w_p = p$ const.

$\left.\begin{array}{l} G(n;H)\ \ H\text{-random} \\ (G_n)\ \ H\text{-qusirandom} \end{array}\right\} \longrightarrow \lim = w_H$ step f.

Szemerédi-lemma ~

For every weighted graph H there is a step function $w_H \in \mathcal{W}_0$ with

$$t(F, H) = t(F, w_H) \quad \forall F .$$

(G_n) convergent and $\exists H$ such that

$$\lim t(F, G_n) = t(F, H) \Longleftrightarrow W((G_n)) \text{ step f.}$$

(G_n) generalized quasirandom

$\Updownarrow$

(G_n) convergent and the limit is a step function

(G_n) convergent $\nearrow$ $\exists$ finite, weighted H $t(F, G_n) \to t(F, H)$ (G_n) quasirandom

(G_n) convergent $\searrow$ $\nexists$ finite H, $\exists\, W \in \mathcal{W}_0$ measurable $t(F, G_n) \to t(F, W)$

17-9

Theorem (Lovász - S.)

Let (G_n) be H-quasirandom. Then

(a) structure of G_n ~ structure of $G(n;H)$:

$\forall n \ \exists \ V(G_n) = \bigcup_{i=1}^{q} V_i$ such that

- $\frac{|V_i|}{|V|} \to \alpha_i \qquad 1 \le i \le q$
- $G_n(V_i)$ is β_{ii}-quasirandom
- $G_n(V_i, V_j)$ is β_{ij}-qu.r. bipartite

(b) $\exists$ finite test-class:

(G_n) is H-quasirandom iff

$$t(F, G_n) \to t(F, H)$$

for $\forall F$ with $|V(F)| < (10q)^q$

$q = |V(H)|$

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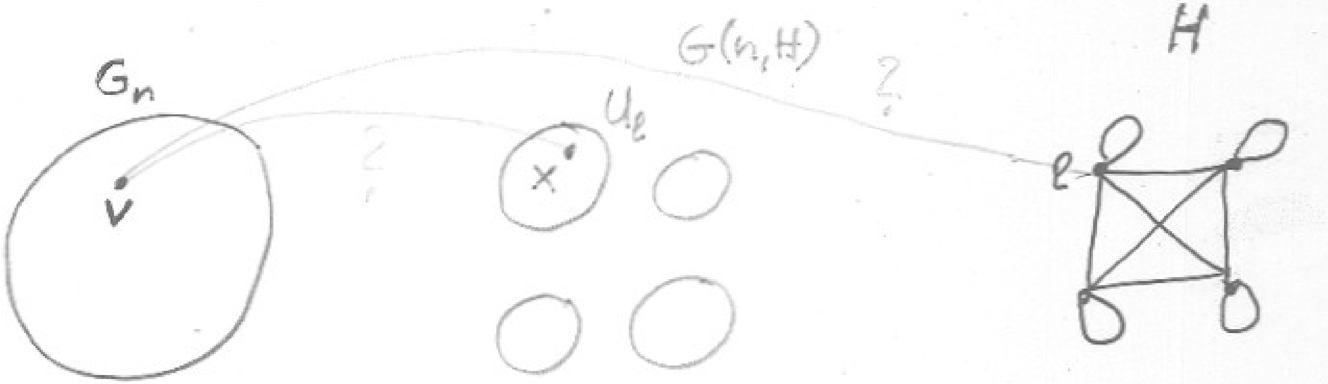


¿ v similar to x ~ ℓ ?

$$\hom_{1\to v}(F, G_n) = \#\varphi \quad \text{with } \varphi(1) = v$$

$$v \sim x: \ \hom_{1\to x}(F, G(n,H)) \sim \hom_{1\to v}(F, G(n,H))$$

$$\sim \hom_{\ell}(F,H) \cdot n^{k-1} =$$

$$= \prod_{i\neq \ell} \alpha(i) \prod_{(i,j)\in E(F)} \beta_{\varphi(i),\varphi(j)} \cdot n^{k-1}$$

$$h_F = (\hom_1(F,H), \dots, \hom_q(F,H)) \in R^q$$

Suppose $h_{F_1}, \dots, h_{F_q}$ is a basis for R^q.

Let $e_\ell = (0, 0, , 1, , 0) \in R^q$

$$e_\ell = \sum_{i=1}^{q} \lambda_i h_{F_i}$$

Let

$$g_F = \left(\frac{\hom_1(F, G_n)}{n^{k-1}}, \dots, \frac{\hom_n(F, G_n)}{n^{k-1}} \right) \in R^n$$

$$s = \sum_1^q \lambda_i g_{F_i} = (s_1, \dots, s_n) \in R^n$$

$v \in G_n$ "similar" to $x \in U_\ell$; $\ell \in V(H)$

$\updownarrow$

$s_v \sim 1$

not "similar" $\quad s_u \sim 0$

In $\underline{s}$ $\quad s_i = 1$ for $\sim \alpha_\ell n$ $\quad i \in [n]$

Problem: twin vertices in H
automorphismes of H

GRAPH ALGEBRAS

Lovász, Freedman, Schrijver

Quantum graph

finite linear combination of graphs

$$\mathcal{G}_0 = \{ \sum \lambda_i F_i \ , \ \lambda_i \in R \}$$

k - labeled graphs

k nodes of F labeled 1,..,k

product of k - labeled graphs

$F_1 F_2$: $F_1 \cup F_2$ labeled nodes identified

$\mathcal{G}_k$: algebra of k - labeled quantum graphs

L. Lovász – B. Szegedy

(G_n) convergent

- $\exists$ finite, weighted H
  $t(F, G_n) \longrightarrow t(F, H)$
  (G_n) is H – qu. r

- $\nexists$ finite H,
  $\exists$ W meas. in $[0,1]^2$
  $t(F, G_n) \longrightarrow t(F, W)$

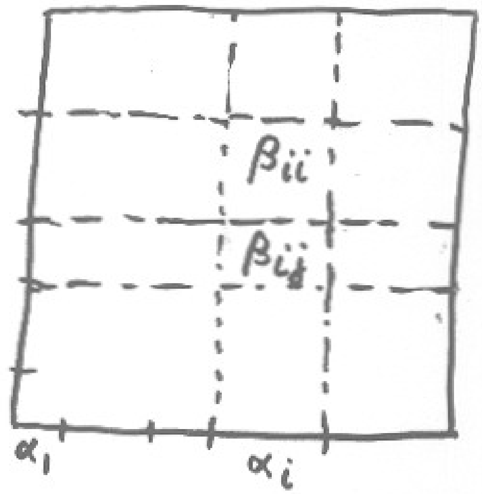
(G_n) generalized quasirandom

$$\Updownarrow$$

(G_n) convergent. and the limit is a step-function

Let H be a weighted graph

$w_H \in W$ $\qquad \sum_1^k \alpha_i =$



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Szemerédi-lemma $\sim$

approximation by step-function

¿ minimal finite family $\mathcal{F}$
structure,
$|\mathcal{F}|$?

Spencer: $|\mathcal{F}| < q^{c}$

(2) Is there a characterization of generalized quasirandom graph seqs. in terms of the eigenvalues of (G_n)?

(for $q = 1$: $\lambda_1 \sim pn$, $\lambda_2 \sim o(n)$)

$q = 2$?

(We can formulate some other characterizations, in terms of Szemerédi partitions, cuts etc.)