Topology, Lattices, and Logic Programming

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DIMACS Lattice Workshop, July 8-10, 2003

Background

- How to assign meanings to a logic program P, e.g.
```
odd(s(0)),
odd(x) -> odd(ss(x)).
```
- Herbrand universe: $U_P = \{\mathtt{s}^i\mathtt{0} \mid i \geq 0\}$
- Herbrand base: all ground atomic formulas formed using terms from U_P and predicates in P.
- ground(P): the set of ground instances of P.
```
odd(s0),
odd(0) -> odd(ss0),
odd(s0) -> odd(sss0)), ...
```
- The meaning of logic programs reduces to the interpretation of a set of "implications" of the form

$$X \rightarrow a \qquad \text{or} \qquad X \rightarrow Y$$

Motivation

- Fixed-point semantics has a lot to do with topology. It is the topological property of the immediate consequence operator T_P that determines the property of the semantics. (E.g. existence and uniqueness)

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- Topology has a lot to do with lattices. It is the lattice of opens collectively that gives the topological space its identity. (E.g. T_0, Hausdorff, sobriety)
- This leads to frames/locales, the abstract notion of topology which takes open sets as the starting point.
- Paradigm: open sets as propositions, points as models. Open sets first, points secondary.

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  - Make available tools from many areas
- Outline of this talk: coverage relations, definite programs, logic programs with negation, disjunctive logic programs, other issues.

Definite logic programs

- Given $(A, \vdash)$, where $\vdash$ is a set of implications $X \vdash a$, with X a finite subset of A and a a member of A.
- Interpretation: think of each element of A as an open set, and each implication as containment $\bigcap X \subseteq a$.
- Question: which topological space?
- The “topological space" consists of all finite meets $\wedge$ and arbitrary joins $\bigvee$ generated from A, subject to the interpretation of constraints $\vdash$ given above.
- How to generate a frame $\mathsf{Frm}(A)$ from $(A, \vdash)$?

Frames and coverage relations

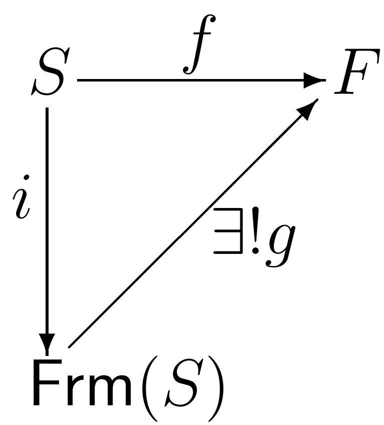
- A *frame* (locale) is a poset with finite meets and arbitrary joins which satisfies the infinite distributive law
$$x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}.$$
- A frame morphism is a function $f : F \to G$ that preserves finite meets and arbitrary joins.
- Let $(S, \wedge, \leq)$ be a meet-semi-lattice. A *coverage* on S is a relation $\succ \subseteq 2^S \times S$ satisfying
  - if $Y \succ a$ then $Y \subseteq \downarrow a$.
  - if $Y \succ a$ then for any $b \leq a$, $\{y \wedge b \mid y \in Y\} \succ b$.
- A coverage relation (or coverage) $\succ$ is *compact* if

$$Y \succ a \text{ implies } X \succ a \text{ for some finite } X \subseteq^{\text{fin}} Y.$$

Semilattice generated frames

A meet-semi-lattice S with a coverage $\succ$ is called a *site*.
A frame $\mathsf{Frm}(S)$ is *generated* from $(S, \succ)$ if there exists i s.t.

- $i : S \to \mathsf{Frm}(S)$ preserves finite meets,
- i transforms covers to joins: $Y \succ a \Rightarrow i(a) = \bigvee i(Y)$, and
- $\mathsf{Frm}(S), i$ is universal, i.e., for any frame F and any meet-preserving and cover-to-join transforming function $f : S \to F$, there exists a unique frame morphism $g : \mathsf{Frm}(S) \to F$ s.t. the following diagram commutes:



Ideals and filters

- An *ideal* of a poset is a lower closed, directed subset of the poset.
  In a lattice, an ideal is a $\vee$-closed, lower set.
- A *filter* of a frame F is a subset $u \subseteq F$ which is $\wedge$-closed, upper set. Ideals always contain the bottom element and filters always contain the top element.
  A filter u of a frame F is *completely prime* if $\bigvee P \in u \Rightarrow P \cap u \neq \emptyset$ for any $P \subseteq u$.

$\succ$-ideals

- Given a site $(S, \succ)$, a $\succ$-ideal is a subset I of S which is
  - lower-closed: $a \in I \;\&\; b \leq a \Rightarrow b \in I$,
  - covered: $U \succ a \;\&\; U \subseteq I \Rightarrow a \in I$.
- Example. Let D be a distributive lattice. Let the coverage be defined as $U \succ a$ if
  - $U \subseteq {\downarrow}a$ and
  - $\exists X \subseteq^{\text{fin}} U$, $a = \vee X$
- A $\succ$-ideal is then exactly an *ideal* of D in this case.
- Definition. A frame (locale) is said to be *spectral* if it is isomorphic to the ideal completion of a distributive lattice.

Coverage theorem

- Theorem (Johnstone82). The collection of $\succ$-ideals under inclusion is the frame generated from a site $(S, \succ)$.
- Definition. A *point* of a frame is a completely prime filter.
- Fact. If H is generated from $(S, \succ)$ (with i) then *points* are exactly *filters* F of S such that

$$i(a) \in F \;\&\; Y \succ a \Rightarrow (\exists b \in Y)\, i(b) \in F$$

- Definition. frame H is *spatial* if for any $a, b \in H$,

$$a \leq b \;\text{ iff }\; \forall\text{point } F,\; a \in F \Rightarrow b \in F.$$

- Fact. Spectral frames are spatial.

Compact coverages and spectral frames

- Recall: a coverage relation (or coverage) $\succ$ is called *compact* if for every $X \subseteq S$ and every $a \in S$,
$$X \succ a \text{ implies } Y \succ a \text{ for some finite } Y \subseteq^{\text{fin}} X.$$
- Lemma. If $(S, \succ)$ is a site for which the coverage relation $\succ$ is compact, then for any directed set F of $\succ$-ideals, $\bigvee F = \bigcup F$.
- Lemma. Suppose $(S, \succ)$ is a site and $\succ$ is compact. Then a $\succ$-ideal is a compact element in the generated frame if and only if it is generated by a finite subset of S.
- Compact Coverage Theorem (Z.03). A frame is spectral iff it can be generated from a compact coverage relation.

Information systems (without Con)

- Definition. An information system is a pair $(A, \vdash)$ such that the relation $\vdash \subseteq \mathrm{Fin}(A) \times A$ is reflexive and transitive
- Definition. An ideal element of A is a subset $x \subseteq A$ such that $X \subseteq x \;\&\; X \vdash a \Rightarrow a \in x$.
- Theorem (Scott82) For any information system $(A, \vdash)$, the set of ideal elements under inclusion $(|A|, \subseteq)$ is a complete algebraic lattice. Conversely, any complete algebraic lattice is order-isomorphic to one from some information system.

Semantics of definite logic programs

- A definite logic program $(A, \vdash)$ gives rise to a site $(A^\wedge, \succ)$, with $A^\wedge$ the freely generated meet-semi-lattice from A, and $\{a \wedge (\wedge X)\} \succ \wedge X$ iff $X \vdash a$.
(Note $\bigcap X \subseteq a$ iff $\bigcap X = a \cap \bigcap X$)
- Proposition. This compact coverage relation generates a spectral frame $\mathsf{Frm}(A)$. The "points" of the frame are in 1-1 correspondence with ideal elements of $|A|$.
- The Compact Coverage Theorem implies that $\leq$ is sound and complete with respect to these models. In particular, $X \vdash a$ if for each point x, $x \models X \Rightarrow x \models a$.
- Moreover, since $\vdash$ is "embedded in" $\leq$, we obtain the "derived rules", e.g. reflexivity and transitivity.

What kind of topology?

- Scott topology.
- A set is *Scott open* if it is upwards closed and inaccessible by lubs of directed sets. Sets of the from $[\![X]\!] := \{x \mid X \subseteq x \;\&\; x \in |A|\}$ form a *basis* of the Scott topology over $(|A|, \subseteq)$.
- From $\succ$-ideals u to Scott opens: $u \longmapsto \bigcup\{[\![X]\!] \mid \wedge X \in u\}$
- From Scott opens O to $\succ$-ideals: $O \longmapsto \{\wedge X \mid [\![X]\!] \subseteq O\}$
- Consistent with Fitting85, Fitting87, Seda-Hitzler95, 99, Batarekh-Subrahmanian89, Rounds-Z.01, Z.-Rounds01

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- Coverage relation: for each rule $X, \neg Y \to a$, for each x s.t. $x \wedge y \leq 0$ for all $y \in Y$, put $\{a \wedge x \wedge z \wedge (\wedge X) \mid \forall y \in Y\ (z \wedge y \leq 0)\} \succ x \wedge (\wedge X)$

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- By the Coverage Theorem, we have (in the frame)

$$\forall x(.) x \wedge (\wedge X) = \bigvee \{a \wedge x \wedge z \wedge (\wedge X) \mid \forall y \in Y\ (z \wedge y \leq 0)\}$$

iff $\bigvee \{x \wedge (\wedge X) \mid x \wedge (\vee Y) \leq 0\}$ (Simplifying the right $\bigvee \bigvee$ to $\bigvee$)

$$= \bigvee \{a \wedge z \wedge (\wedge X) \mid z \wedge (\vee Y) \leq 0)\}$$

iff $(\wedge X) \wedge \bigvee \{x \mid x \wedge (\vee Y) \leq 0\}$

$$= a \wedge (\wedge X) \wedge \bigvee \{z \mid z \wedge (\vee Y) \leq 0)\}$$

iff $(\wedge X) \wedge \wedge \{\neg b \mid b \in Y\} \leq a$

Note that $\neg b := b \to 0 := \bigvee \{x \mid x \wedge b \leq 0\}$

Strong negation

- Parameter: the underlying semi-lattice, corresponding intuitively to a “basis” of the topology we want to generate.
- $(\mathsf{Fin}(A \cup \bar{A}), \cup, \supseteq)$ subject to $a \wedge \bar{a} \leq 0$ for all $a \in A$.
- Coverage relation: for each rule $X, \neg Y \to a$, for each x s.t. $x \wedge y \leq 0$ for all $y \in Y$, put $\{a \wedge x \wedge z \wedge (\wedge X) \mid \forall y \in Y\ (z \wedge y \leq 0)\} \succ x \wedge (\wedge X)$
- But now there is a unique largest element $\wedge\bar{Y}$ for which $\wedge\bar{Y} \wedge y \leq 0$ for each $y \in Y$. So the coverage relation reduces to $\{a \wedge \wedge\bar{Y} \wedge (\wedge X)\} \succ \wedge\bar{Y} \wedge (\wedge X)$ for each rule $X, \neg Y \to a$.
- By the Coverage Theorem, we have $(\wedge X) \wedge \wedge\bar{Y} \leq a$ for each $X, \neg Y \to a$, in the generated frame.

Weak negation (patch/Lawson topology)

- $(\mathsf{Fin}(A \cup \bar{A}), \cup, \supseteq)$ subject to $a \vee \bar{a} = 1$ and $a \wedge \bar{a} = 0$ for all $a \in A$.
- Coverage relation: for each rule $X, \neg Y \to a$, for each x s.t. $x \wedge y \leq 0$ for all $y \in Y$, put $\{a \wedge x \wedge z \wedge (\wedge X) \mid \forall y \in Y \ (z \wedge y \leq 0)\} \succ x \wedge (\wedge X)$
- $\wedge \bar{Y}$ the largest element for which $\wedge \bar{Y} \wedge y \leq 0$ for each $y \in Y$. So the coverage relation reduces to $\{a \wedge \wedge \bar{Y} \wedge (\wedge X)\} \succ \wedge \bar{Y} \wedge (\wedge X)$ for each rule $X, \neg Y \to a$.
- By the Coverage Theorem, we have $(\wedge X) \wedge \wedge \bar{Y} \leq a$ for each $X, \neg Y \to a$, in the generated frame.

Disjunctive logic programs

- Definition. A sequent structure $(A, \vdash)$ is a set of implications $X \vdash Y$, with X, Y finite subsets of A.
- Interpretation: think of each element of A as an open set, and each implication as containment $\bigcap X \subseteq \bigcup Y$.
- Coverage: $Y \succ \wedge X$?
- Not quite. Here is the fix: $\{b \wedge (\wedge X) \mid b \in Y\} \succ \wedge X$.
- $\succ$-ideals are subsets $U \subseteq \mathrm{Fin}(A)$ such that
  - if $X \in U$ and $Y \supseteq X$, then $Y \in U$;
  - if $\{a_1 \wedge (\wedge X), \ldots,\ a_n \wedge (\wedge X)\} \subseteq U$ and $X \vdash a_1, \ldots, a_n$, then $\wedge X \in U$.

Generated frame (Coquand-Z.00)

- For $U \subseteq \mathrm{Fin}(A)$, write c$U$ for the $\succ$-ideal generated by U.
- Corollary. For any sequent structure $(A, \vdash)$, the set of its $\succ$-ideals H_0 is a frame under inclusion. Moreover, the interpretation $m_0 : A \to H_0$ mapping a to c$\{a\}$ is universal. Furthermore we have $X \vdash Y$ if and only if $\wedge m_0(X) \leq \vee m_0(Y)$ for all finite subsets X, Y of A.
- Definition. $x \subseteq A$ is an *ideal element* if for each instance $X \vdash Y$ of $\vdash$, $X \subseteq x$ implies $x \cap Y \neq \emptyset$.
- Ideal elements corresponds to completely prime filters in H_0. Therefore, if for each ideal element x, $X \subseteq x$ implies $x \cap Y \neq \emptyset$, then $X \vdash Y$.

Clausal logic (Rounds-Z.01)

- With respect to a sequent structure $(A, \vdash)$, a *clause* is a finite subset of A. A *clause set* is a collection of clauses.
- An ideal element x is a *model of a clause* u if $x \cap u \neq \emptyset$. x is a *model of a clause set* W if it is a model of every clause in W.
- Define
  - $W \models u$ if any model of W is a model of u
  - $W \vdash^*_{hr} u$ if either $\emptyset \in W$, or u can be deduced from W by the HR rule $\dfrac{a_1, X \quad \ldots \quad a_n, X}{X}$ (if $a_1, \ldots, a_n \vdash X$)
  - $\{X_1, \ldots, X_n\} \dashrightarrow u$ if for any choice $a_1 \in X_1, \ldots, a_n \in X_n$, $\{a_i \mid 1 \ldots n\} \vdash u$.
- Theorem. $\models \; = \; \vdash^*_{hr} \; = \; \dashrightarrow u$

Generating $\models$-closed clause sets

- Given sequent structure $(A, \vdash)$, consider the freely generated distributive lattice from A as the underlying meet-semi-lattice.
- Coverage relation:
  - $X \succ \vee X$,
  - $X \succ (a_1 \vee (\vee X)) \wedge \cdots \wedge (a_n \vee (\vee X))$ if $a_1, \ldots, a_n \vdash X$.
- Theorem (Coquand-Z.01). u is a $\succ$-ideal iff u is a $\models$-closed clause set.

Concluding remarks

- A general topological approach to construct semantic models.
- Completeness ensured by spatiality.
- Inference rules derived equationally.
- Treated definite logic programs, disjunctive logic programs, and negation.
- Stable model semantics and other semantics?
- Metrics?