Convergence Rates in Decentralized Optimization

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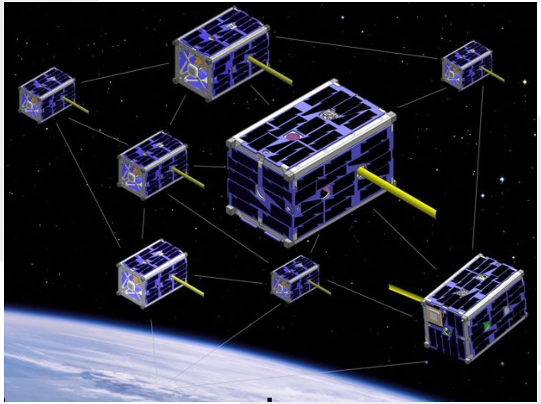
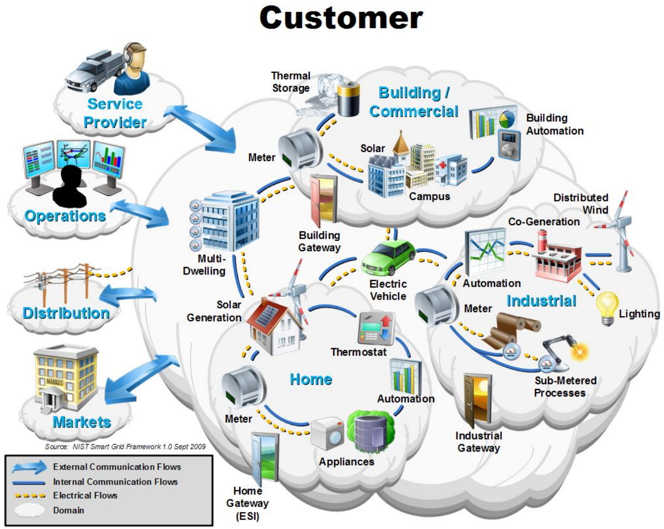
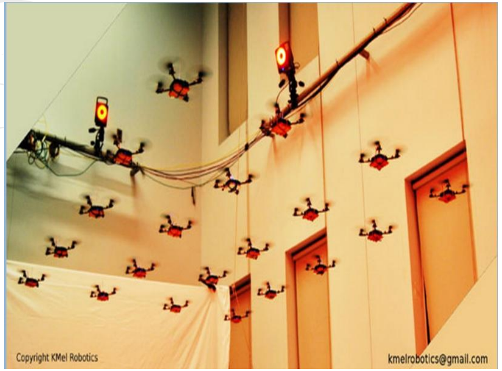
Distributed and Multi-agent Control



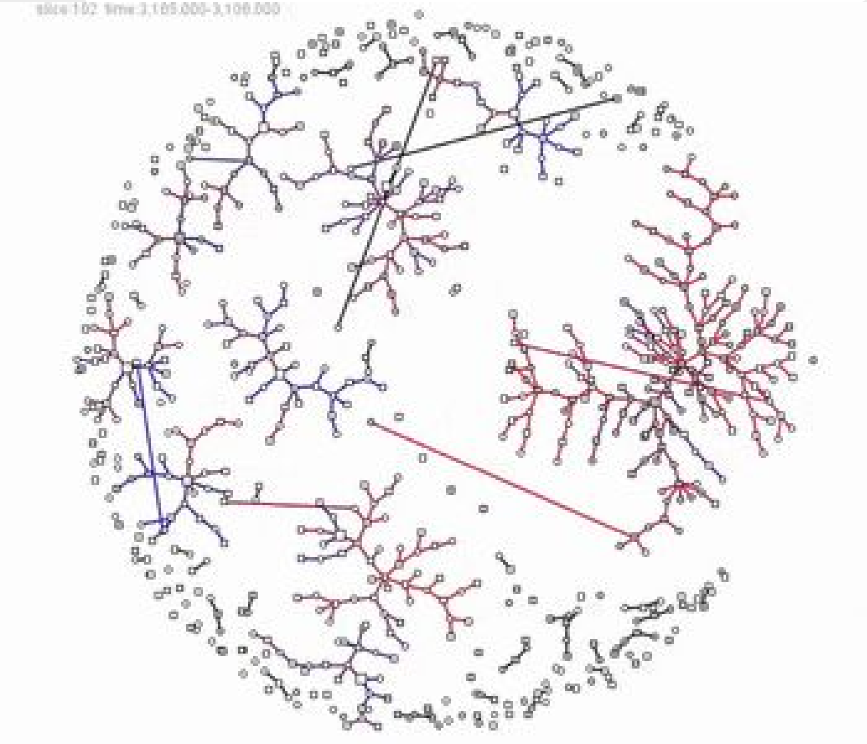




- Strong need for protocols to coordinate multiple agents.
- Such protocols need to be distributed in the sense of involving only local interactions among agents.

Image credit: CubeSat, TCLabs, Kmel Robotics

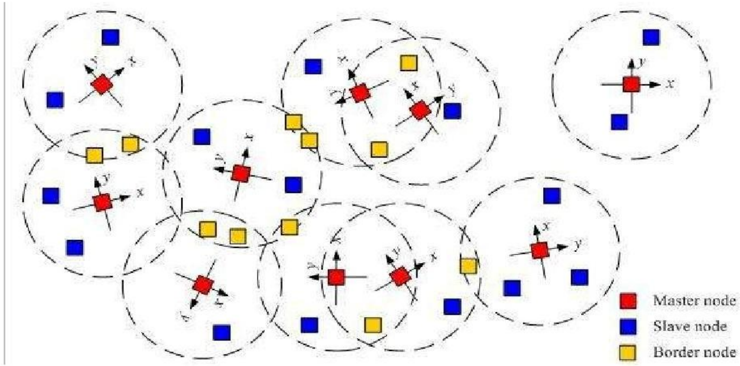
Challenges



- Decentralized methods.
- Unreliable links.
- Node failures.
- Too much data.
- Too much local information.
- Malicious nodes.
- Fast & scalable performance.
- Interaction of cyber & physical components.

Image credit: UW Center for Demography

Problems of Interest



- Formation control
- Target Localization
- Cooperative Estimation
- Distributed Learning
- Leader-following
- Coverage control

- Load balancing
- Clock synchronization in sensor networks
- Resource allocation
- Dynamics in social networks
- Distributed Optimization

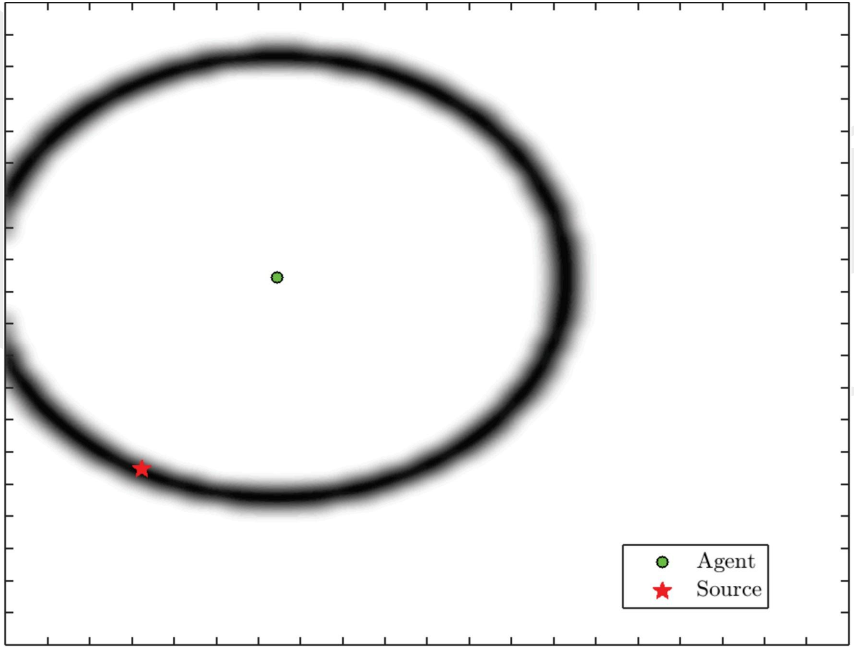
This presentation

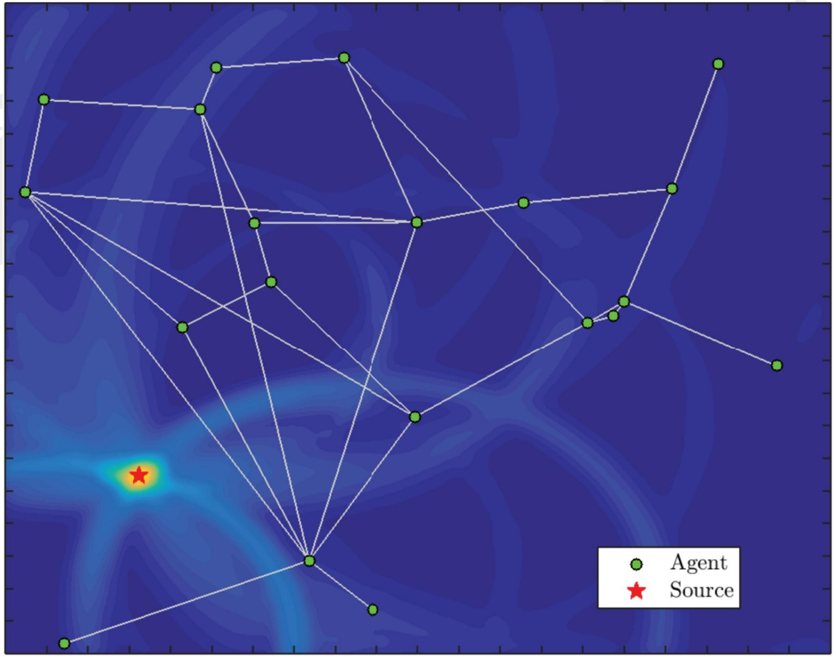
1. Major concerns in multi-agent control (3 slides)
2. **Three problems (4 slides)**
   a) Distributed learning
   b) Localization from distance measurements
   c) Distributed optimization
3. A common theme: average consensus protocols (10 slides)
   a) Introduction
   b) Main result
   c) Intuition
4. Revisiting the three problems from part 2 (21 slides)
5. Conclusion (1 slide)

Distributed learning

- There is a true state of the world θ^* that belongs to a finite set of hypotheses Θ.
- At time t, agent i receives i.i.d. random variables $s_i(t)$, lying in some finite set. These measurements have distributions $P_i(.|\theta)$, which are known to node i.
- Want to cooperate and identify the true state of the world. Can only interact with neighbors in some graph(s).
- A variation: no true state of the world, some hypotheses just explain things better than others.
- Will focus on source localization as a particular example.

Distributed learning -- example





Each agent (imprecisely) measures distance to source; these give rise to beliefs, which need to be fused in order to decide a hypotheses on the location of the source.

Decentralized optimization

- There are n agents. Only agent i knows the convex function $f_i(x)$.
- Agents want to cooperate to compute a minimizer of

$$F(x) = (1/n) \sum_i f_i(x)$$

- As always, agents can only interact with neighbors in an undirected graph -- or a time-varying sequence of graphs.
- Too expensive to share all the functions with everyone.
- But: everyone can compute their own function values and (sub)gradients.

Distributed regression -- an example

- Users with feature vectors a_i are shown an ad.
- y_i is a binary variable measuring whether they ``liked it.''
- One usually looks for vectors z corresponding to predictors $sign(z'a_i + b)$
- Some relaxations considered in the literature:

$$\Sigma_i\ 1 - y_i(z'a_i + b) + \lambda\ ||z||_1$$
$$\Sigma_i\ max(0,1 - y_i(z'a_i + b)) + \lambda\ ||z||_1$$
$$\Sigma_i\ \log\ (1 + e^{-y\_i(z'a\_i + b)}) + \lambda\ ||z||_1$$

Want to find z & b that minimize the above.

- If the k'th cluster has data $(y_i, a_i, i\ in\ S_k)$, then setting

$$f_k(z,b) = \Sigma_{i \in Sk}\ 1 - y_i(z'a_i + b) + \lambda'\ ||z||_1$$

recovers the problem of finding a minimizer of $\Sigma_k f_k$

This presentation

1. Major concerns in multi-agent control (3 slides)
2. Three problems (4 slides)
   a) Distributed learning
   b) Localization from distance measurements
   c) Distributed optimization & distributed regression
3. **Average consensus protocols (10 slides)**
   a) Introduction
   b) Main result
   c) Intuition
4. Revisiting the three problems from part 2 (15 slides)
5. Conclusion (2 slides)

The Consensus Problem - I

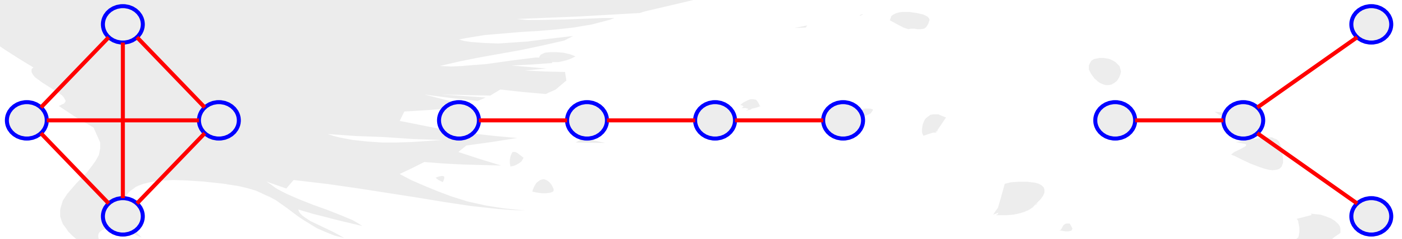
- There are n agents, which we will label $1, \ldots, n$
- Agent i begins with a real number $x_i(0)$ stored in memory
- Goal is to compute the average

$$(1/n) \sum_i x_i(0)$$

- Nodes are limited to interacting with neighbors in an **undirected** graph or a sequence of undirected graphs.

The Consensus Problem - II

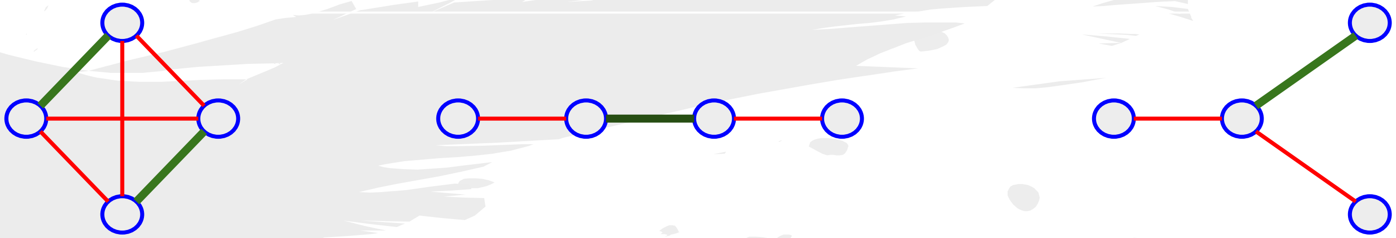
- Protocols need to be fully distributed, based only on local information and interaction between neighbors. Some kind of connectivity assumption will be needed.



- Want protocols inherently robust to failing links, failing or malicious nodes, don't suffer from a ``data curse'' by storing everything.
- Want to avoid protocols based on flooding or leader election.
- Preview: this seems like a toy problem, but plays a key role in all the problems previously described.

Consensus Algorithms: Gossip

Nodes break up into a matching



...and update as

$$x_i(t+1),\ x_j(t+1) = \tfrac{1}{2}\left(x_i(t) + x_j(t)\right)$$

First studied by [Cybenko, 1989] in the context of load balancing (processors want to equalize work along a network).

Consensus Algorithms: Equal-neighbor

$$x_i(t+1) = x_i(t) + c \sum_{j \text{ in } N(i,t)} x_j(t)-x_i(t)$$

- Here $N(i,t)$ is the set of neighbors of node i at time t.
- Works if c is small enough (on a fixed graph, c should be smaller than the inverse of the largest degree)
- First proposed by [Mehyar, Spanos, Pongsajapan, Low, Murray, 2007].

Consensus Algorithms: Metropolis

$$x_i(t+1) = x_i(t) + \sum_{j \in N(i,t)} w_{ij}(t)\,(x_j(t)-x_i(t))$$

- First proposed in this context by [Xiao, Boyd, 2004].
- Here $w_{ij}(t)$ are the Metropolis weights

$$w_{ij}(t) = \min(\,1+d_i(t), 1 + d_j(t)\,)^{-1}$$

where $d_i(t)$ is the degree of node i at time t.

- Avoids the hassle of choosing the constant c before.

Consensus Algorithms: others

- All of the above protocols are linear:

$$x(t+1) = A(t)\, x(t)$$

where $A(t)=[a_{ij}(t)]$ is a stochastic matrix. Note that $A(t)$ is always compatible with the graph is the sense of $a_{ij}(t)=0$ whenever there is no edge between i and j.

- Can design nonlinear protocols [Chapman and Mesbahi, 2012], [Krause 2000],[Hui and Haddad, 2008], [Srivastava, Moehlis, Bullo, 2011], many others….
- Most prominent is the so-called *push-sum* protocol [Dobra, Kempe, Gehrke 2003]which takes the ratio of two linear updates.

Our Focus: Designing Good Protocols

- **Our goal**: simple and robust protocols that work quickly...even in the worst case.
- What does ``worst-case'' mean?
- Look at time until the measure of disagreement $S(t) = \max_i x_i(t) - \min_i x_i(t)$ is shrunk by a factor of $\boldsymbol{\varepsilon}$. Call this $T(n,\varepsilon)$.
- We can take worst-case over either all fixed connected graphs or all time-varying graph sequence (satisfying some long-term connectivity conditions).

Previous Work and Our Result

Authors	Bound for $T(n,\varepsilon)$	Worst-case over
[Tsitsiklis, Bertsekas, Athans, 1986]	$O(n^n \log (1/\varepsilon))$	Time-varying directed graphs
[Jadbabaie, Lin, Morse, 2003]	$O(n^n \log (1/\varepsilon))$	Time-varying directed graphs
[**O.**,Tsitsiklis, 2009]	$O(n^3 \log (n/\varepsilon))$	Time-varying undirected graphs
[Nedic, **O.**, Ozdaglar, Tsitsiklis, 2011]	$O(n^2 \log (n/\varepsilon))$	Time-varying undirected graphs
[**O.**, 2015] , **this presentation**	$O(n \log (n/\varepsilon))$	Fixed undirected graphs

The Accelerated Metropolis Protocol - I

$$y_i(t+1) = \Sigma_j\, a_{ij}\, x_j(t)$$

$$x_i(t+1) = y_i(t+1) + (1-(9n)^{-1})\,(y_i(t+1) - y_i(t))$$

- Here a_{ij} is *half* of the Metropolis weight whenever i,j are neighbors. $A(t)=[a_{ij}]$ is a stochastic matrix.
- Must be initialized as $x(0)=y(0)$.
- **Theorem** [O., 2015]: If each node of an undirected connected graph uses the AM method, then each $x_i(t)$ converges to the average of the initial values. Furthermore, $S(t) \leq \varepsilon S(0)$ after $O(n \log (n/\varepsilon))$ updates.

The Accelerated Metropolis Protocol - II

$$y_i(t+1) = \Sigma_j \, a_{ij} \, x_j(t)$$

$$x_i(t+1) = y_i(t+1) + \left(1-(9n)^{-1}\right) (y_i(t+1) - y_i(t))$$

- The idea that iterative methods for linear systems can benefit from extrapolation is very old (~1950s). Used in consensus by [Cao, Spielman, Yeh 2006], [Johansson, Johansson 2008], [Kokiopoulou, Frossard, 2009], [Oreshkin, Coates, Rabbat 2010], [Chen, Tron, Terzis, Vidal 2011], [Liu, Anderson, Cao, Morse 2013], ...
- As written, requires knowledge of the number of nodes by each node. This can be relaxed: each node only needs to know an upper bound correct within a constant factor.

Proof idea

- The natural update $x(t+1) = A\,x(t)$ with stochastic A corresponds to asking about the speed at which a Markov chain converges to a stationary distribution.
- Main insight 1: Metropolis chain mixes well because it decreases the centrality of high-degree vertices.
- In particular: whereas the ordinary random walk takes $O(n^3)$ to mix, the Metropolis walk takes $O(n^2)$
- Main insight 2: can think of Markov chain mixing as gradient descent, and use Nesterov acceleration to take square root of running time.
- This argument can give $O(diameter)$ convergence (up to log factors) on geometric random graphs or 2D grids.

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3. A common theme: consensus protocols (10 slides)
   a) Introduction
   b) Main result
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4. **Revisiting the three problems from part 2 (15 slides)**
5. Conclusion (2 slides)

Back to Decentralized Optimization

- There are n agents. Agent i knows the convex function $f_i(x)$.
- Agents want to cooperate to compute a minimizer of

$$F(x) = (1/n) \sum_i f_i(x)$$

  This contains the consensus problem as a special case.
- In the centralized setup, assuming each $f_i(x)$ has subgradient bounded by L, the subgradient method on the function $F(x)$ results in $F(x_a(t))-F(x^*) = O(1/\sqrt{t})$
  This means that the time until the objective is within epsilon of the optimal value is $O(1/\epsilon^2)$

Previous work

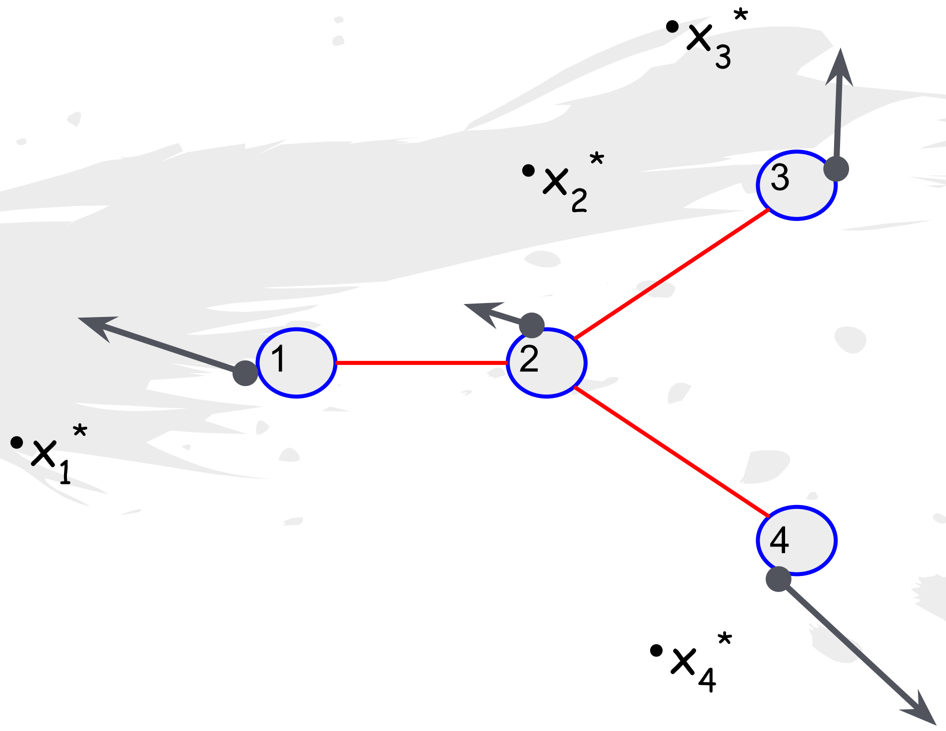
- [Nedic, Ozdaglar 2009] proposed that node i maintain the variable $x_i(t)$ which is updated as

$$x_i(t+1) = \sum_j a_{ij}(t)\, x_j(t) - \alpha\, g_i(t)$$

where $g_i(t)$ is the subgradient of $f_i(x)$ at $x_i(t)$ and $[a_{ij}(t)]$ is any of the consensus matrices above.

- [Nedic, Ozdaglar, 2009] showed that each averaged $x_i(t)$ converges to a small neighborhood of the same minimizer of $F(\cdot)$

Intuition



Linear Time Decentralized Optimization - I

There is a natural algorithm inspired by the AM Method:

$$y_i(t+1) = \Sigma_j\, a_{ij}\, x_j(t) - \alpha\, g_i(t)$$

$$z_i(t+1) = y_i(t) - \alpha\, g_i(t)$$

$$x_i(t+1) = y_i(t+1) + (1-1/(9n))\,(y_i(t+1) - z_i(t+1))$$

...where $g_i(t)$ is the subgradient of f_i at $x_i(t)$, L is an upper bound on the norm of $g_i(t)$, $\alpha=1/(L\sqrt{n}\sqrt{T})$, and a_{ij} are half-Metropolis weights.

Main idea: this interleaves gradient descent with an averaging scheme.

Linear Time Decentralized Optimization - II

- **Theorem** [**O.**, 2015]: on any undirected connected graph, we have that all $x_i(t)$ approach the same minimizer of F and $F(x_a(t))-F(x^*) < \epsilon$ after $O(n/\epsilon^2)$ iterations.
- Initial paper [Nedic, Ozdaglar 2009] had a bound of $O(n^{2n}/\epsilon^2)$ to get within ϵ
- Later improved by [Ram, Nedic, Veeravalli 2011] to $O(n^4/\epsilon^2)$ time to get within ϵ
- In simulations, the linear convergence time still holds on time-varying graphs.

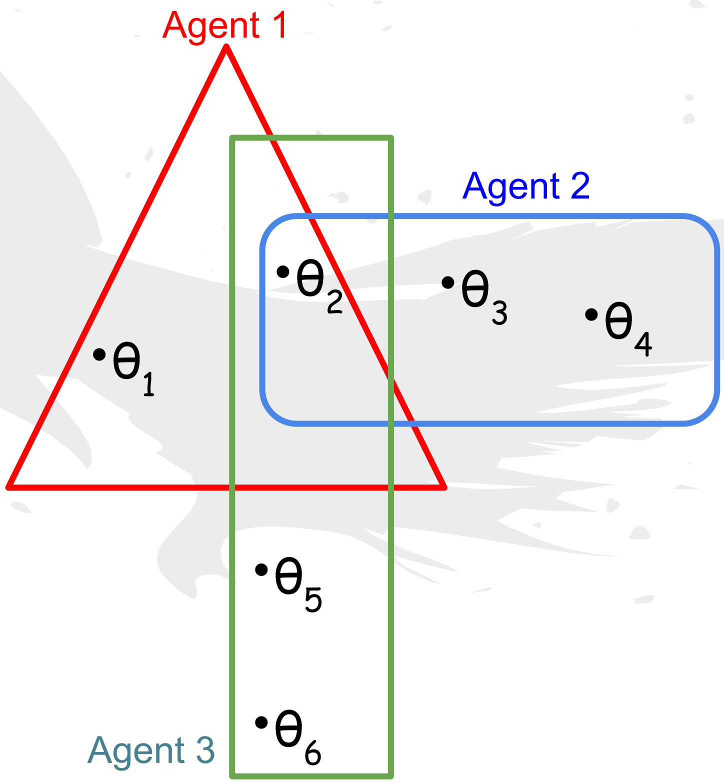
What have we accomplished?

We have proposed an algorithm that:

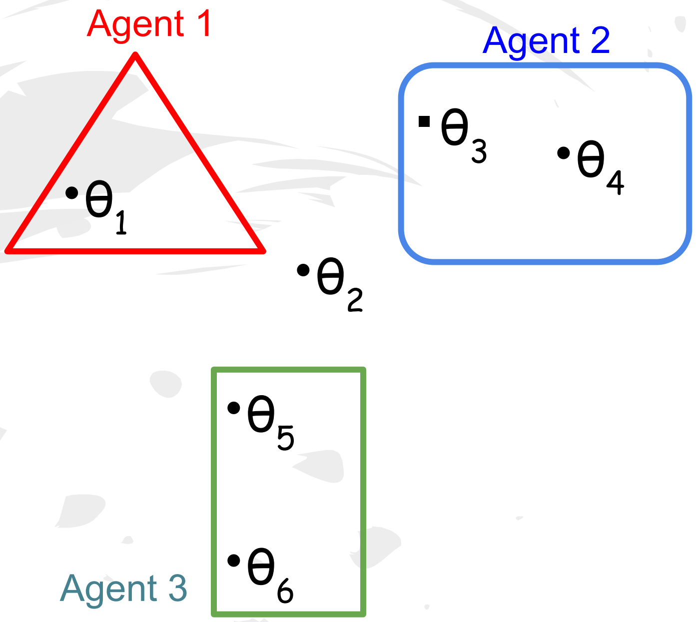
- Every agent stores three numbers.
- Always works in linear time on fixed graphs (this is optimal).
- Automatically robust to failing nodes.
- Simulations show it is robust to link failures.
- Simulations show it works in linear time on time-varying graphs.

Distributed (non)Bayesian Learning

- There is a finite set of hypotheses $\boldsymbol{\Theta}$.
- At time t, agent i receives i.i.d. measurements $s_i(t)$, lying in some finite set, having a distribution q_i.
- Under hypothesis θ, the measurements $s_i(t)$ have distribution $P_i(.|\theta)$.
- Nodes want to cooperate and identify the state of the world which best explains the observations.
- Call that state of the world θ^*.
- Formally: $\theta^* = \arg\min_\theta \sum_i D_{KL}(q_i, P_i(.|\theta))$



Here θ_2 is θ^* and is the true state of the world



Here θ_2 could be θ^* although it is not the best in terms of the observations of any individual agent

Distributed Bayesian Learning

- Agent i maintains a stochastic vector over Θ, which we will denote $b_i(t, \theta)$, initialized to be uniform. Stack these up into $b_i(t)$
- For a nonnegative vector x, define $N(x)$ to be $x/||x||_1$.
- Bayes rule may be written as

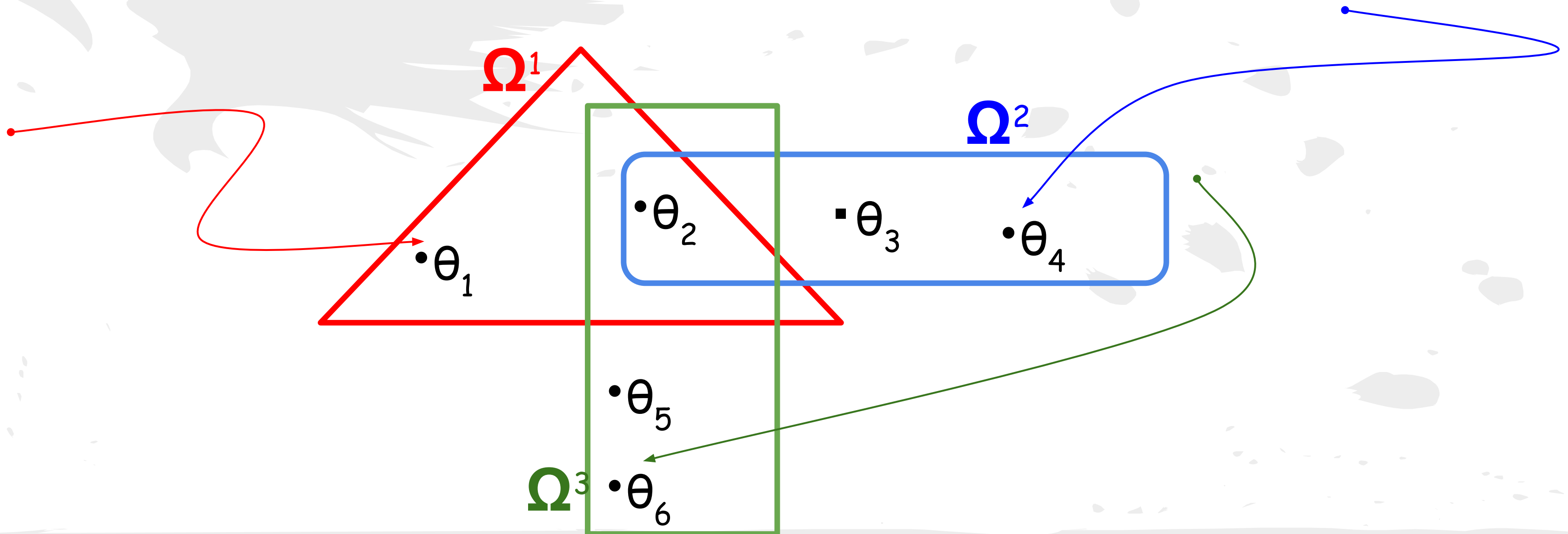
$$b_{i,\, temp}(t+1) = \quad b_i(t) \;.\!*\; P(s_i(t)|\theta))$$

$$b_i(t+1) \qquad = \quad N\big(b_{\,i,\, temp}(t+1)\big)$$

where.* is elementwise multiplication of vectors.

The Independent Bayes Update

Let $\mathbf{\Omega}^i$ be the set of hypotheses best for agent i. Well-known: if agents use above rule (i.e., ignore each other) then all $b_i(t, \theta)$ concentrate on $\mathbf{\Omega}^i$ as t -> $+\infty$.



Distributed (non)Bayesian Learning - II

- First attempt at an algorithm:

  $b_{i,\,temp}(t+1) = \quad b_i(t) \;.\!*\; P(s_i(t)|\theta)).\!*\; \prod_{j \in N(i,t)} b_j(t)^{a\_\{ij\}}$

  $b_i(t+1) \qquad = \quad N(b_{i,\,temp}(t+1))$

- Essentially proposed by [Alanyali, Saligrama, Savas, Aeron 2004]. Each node performs a weighted Bayes update treating the beliefs of neighbors as observations and ignoring dependencies.
- **Theorem** [Nedic, O., Uribe 2015], [Shahrampour, Rakhlin, Jadbabaie 2015], [Lalitha, Sarwate, Javidi 2015]: if $[a_{ij}]$ is any of the stochastic consensus matrices from before, and the graph is undirected and connected, then almost surely all $b_i(t, \theta)$ geometrically approach $\mathbf{1}(\theta^*)$ (i.e., indicator of θ^*)

Distributed (non)Bayesian Learning - III

- The update

$$b_{i,\,temp}(t+1) = \quad b_i(t) \;.\!*\; P(s_i(t)|\theta)).\!*\; \Pi_{j \in N(i,t)}\, b_j(t)^{a\_\{ij\}}$$

$$b_i(t+1) \qquad = \quad N\big(b_{i,\,temp}(t+1)\big)$$

is very similar to a consensus update after the nonlinear change of variables $y_i(t) = \log b_i(t)$.

- Similar idea to distributed optimization: each node ``pulls'' in favor of the explanations that favor its data and these pulls are reconciled through a consensus scheme.

Distributed (non)Bayesian Learning - IV

- Well if that is the case, then how about:

$$b_{i,\,temp}(t+1) = b_i(t) \;.\!*\; P_i(s_i(t)|\theta)).\!*\prod_{j \in N(i)} b_j(t)^{(1+\sigma)a\_\{ij\}}$$

$$v_{i,\,temp}(t+1) = \prod_{j \in N(i)} b_j(t-1) \;.\!*\; P_j(s_j(t)|\theta))$$

$$b_i(t+1) = N\left(b_{i,\,temp}(t+1) \;./\; v_{i,\,temp}(t+1)\right)$$

where a_{ij} are the lazy Metropolis weights and $\sigma = 1-(18n)^{-1}$.

- Intuition: each node pulls in favor its own beliefs, and these pulls are reconciled now using the AM method.

Distributed (non)Bayesian Learning - V

Theorem [Nedic, **O.**, Uribe 2015]: Suppose that under θ^* all events occur with probability at least $p_{min.}$

Then, for all $\theta \neq \theta^*$ and all t, we have with probability $1-\rho$ the bound

$$b_i(t, \theta) \leq e^{-(a/2)t+c}$$

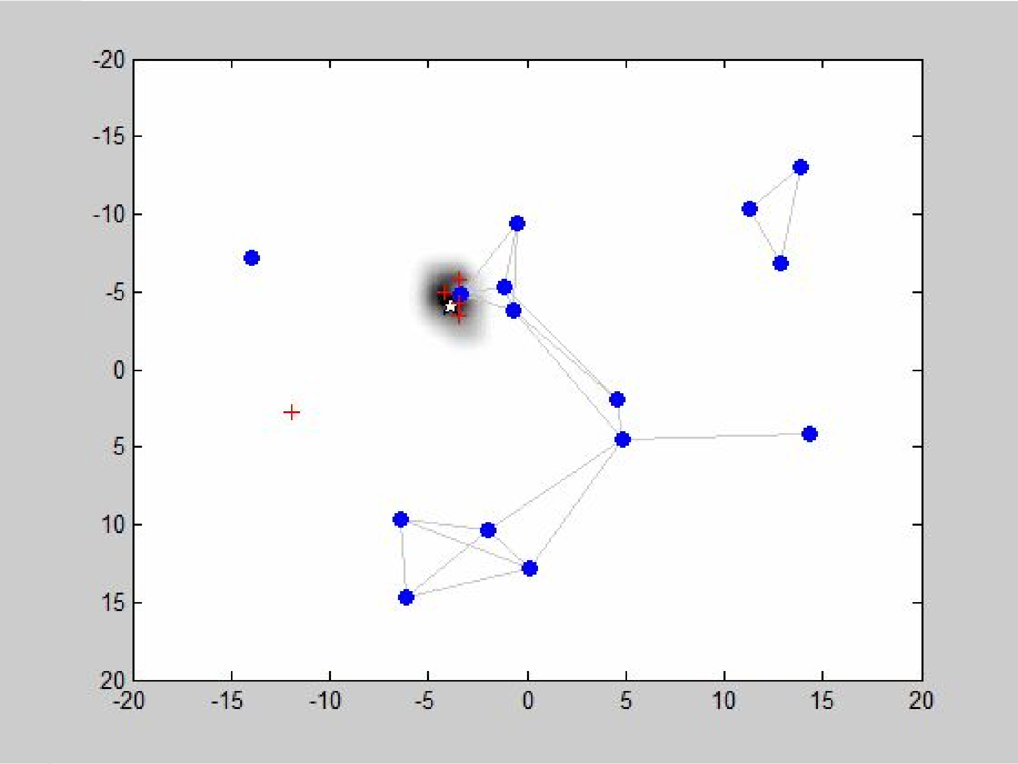
...holds for all $t \geq N(\rho)$ where

$$a = (1/n) \min_{\theta \neq \theta^*} [\textstyle\sum_j D_{KL}(q_j \,||\, P_j(s_j(t)|\theta)) - D_{KL}(q_j \,||\, P_j(s_j(t)|\theta^*))$$

$$c = O(n (\log n) (\log (1/p_{min}))$$

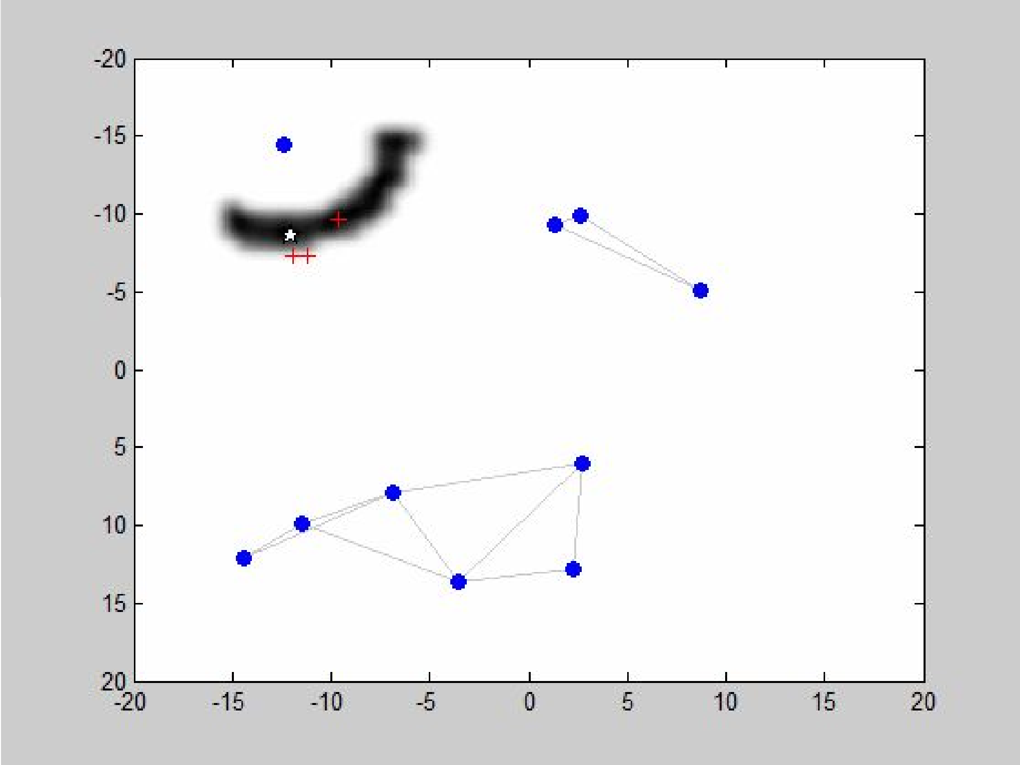
$$N(\rho) = O([\log (1/p_{min}) \log (1/\rho)] / a^2)$$

Learning for Target Localization



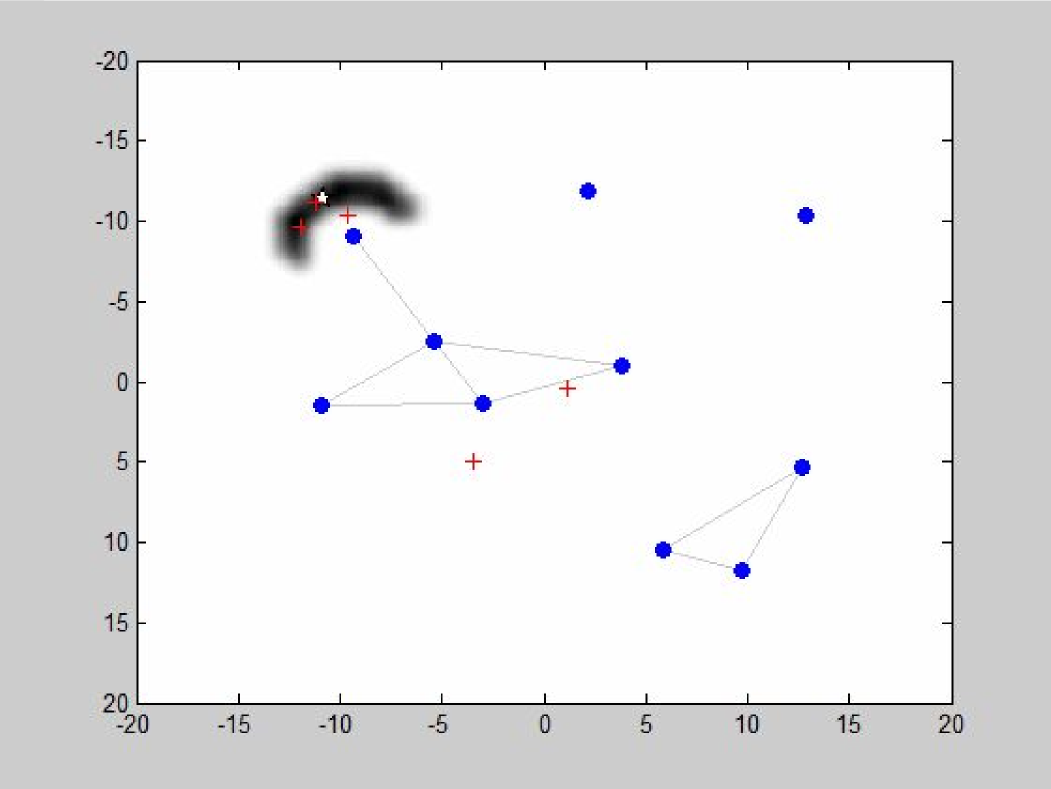
- Fixed target position.
- 15 sensors performing random motion.
- Gaussian noise
- Time-varying graph, often disconnected.
- Learning is very quick.

Learning for Target Tracking



- Target performs random motion.
- 10 sensors performing random motion.
- Gaussian noise
- Time-varying graph, often disconnected.

Following a target

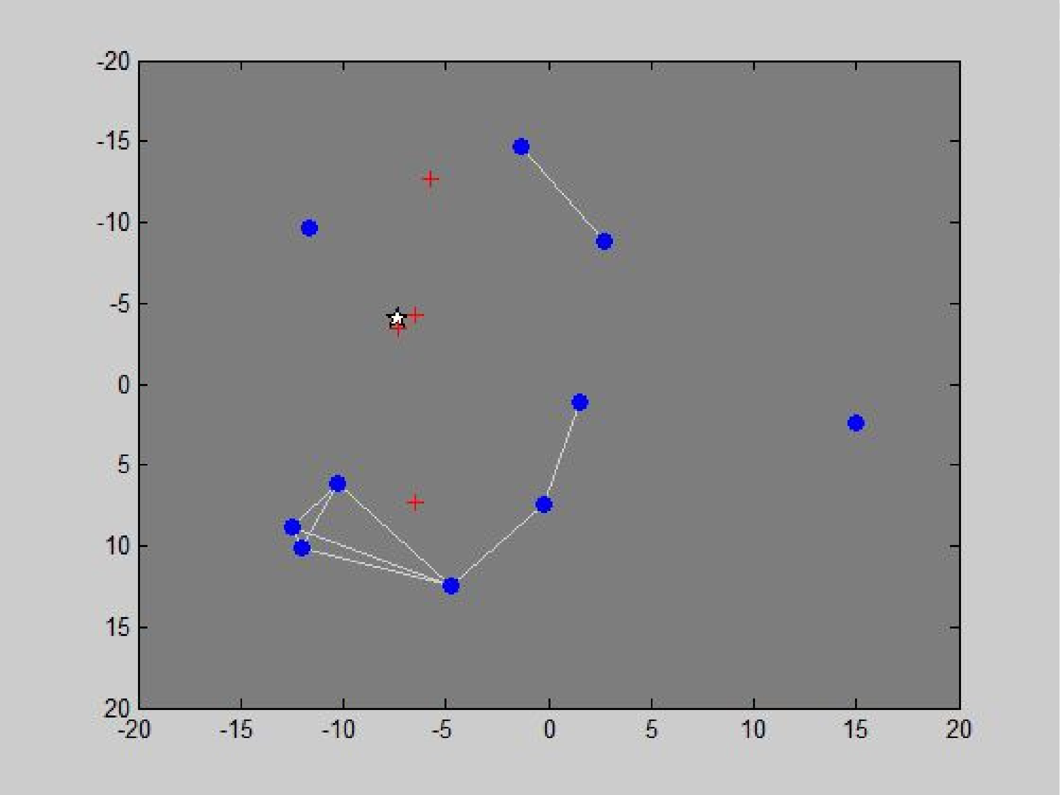


- Target performs random motion.
- 10 sensors:

  -- attracted to estimates of target position

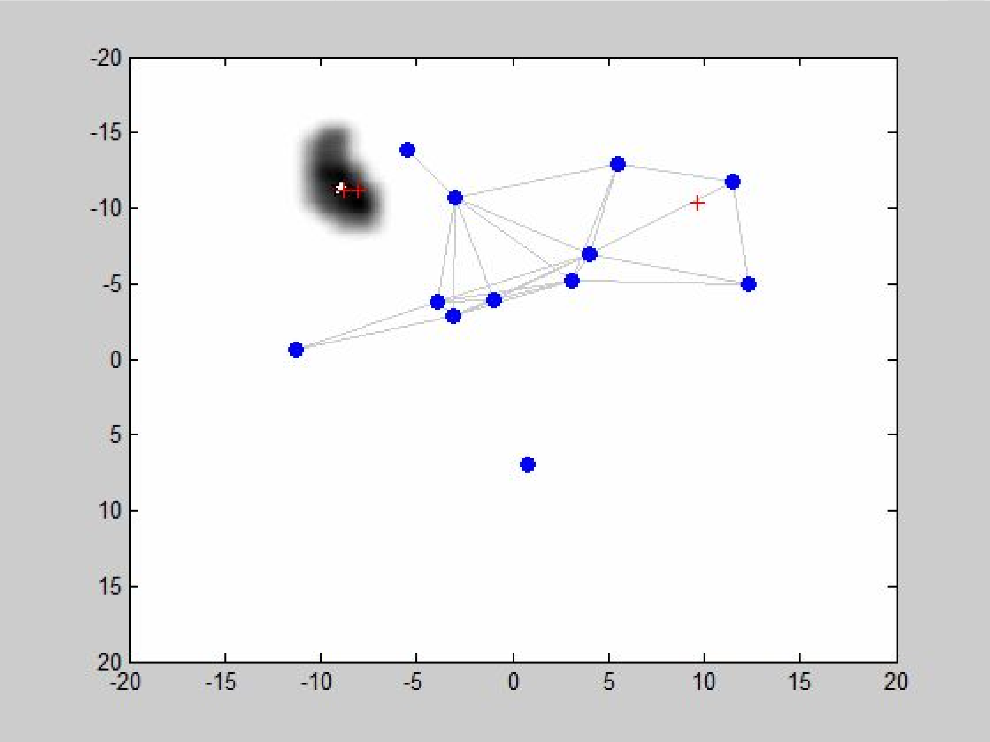
  --repulsed from each other
- Gaussian noise

Following a faster target: failure



- Target performs random motion.
- 10 sensors:
  -- attracted to estimates of target position
  --repulsed from each other
- Much faster target than before

Following a faster target: success



- Target performs random motion.
- 12 sensors:

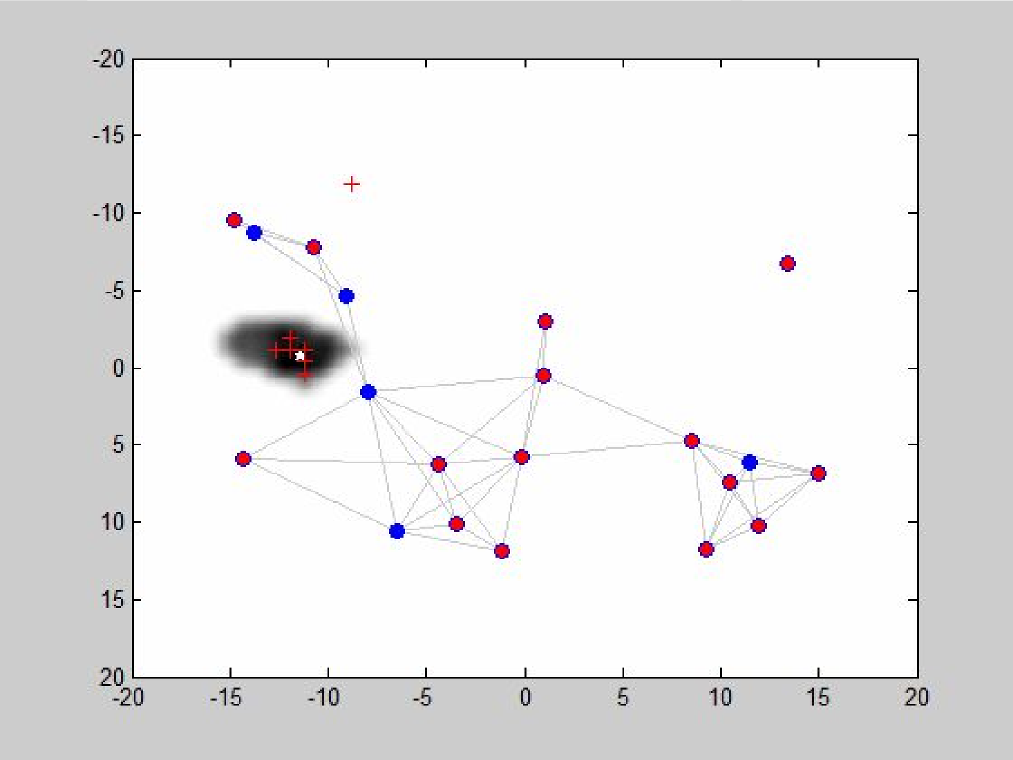
  8 are:

  -- attracted to estimates of target position

  --repulsed from each other

  -- 4 perform random motions

Tracking with incorrect measurements



- Both target and sensors perform random motion.
- Red sensors have random bias in addition to noise. Blue sensors are just noisy.
- Time-varying graph.
- Now takes longer for estimates to resolve.

Conclusion

- *One* (very simple) result: a consensus protocol with convergence time $O\left(n \log (n/\varepsilon)\right)$.
- *This talk*: linear-time algorithms for distributed optimization and distributed learning.
- **Main take-away**: every multi-agent problem that can be solved by coupling local objectives via consensus terms can be linearly scalable in network size with this method.