Algebraic Path Finding

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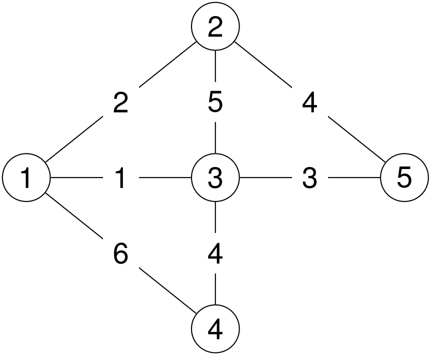
Outline

- Q: Can we separte the WHAT from the HOW in (current) network routing protocols?
- A : “Algebraic path problems” from operations research may help...
- ... but the notion of “global optimality” is too limited.
  - “Local optimality” for algebraic path problems is a new concept, and it may have widespread applicability beyond routing — operations research, combinatorics, ...
  - Thank you BGP.
- Using these abstractions to build tools.
- Routing vs. forwarding still needs work....

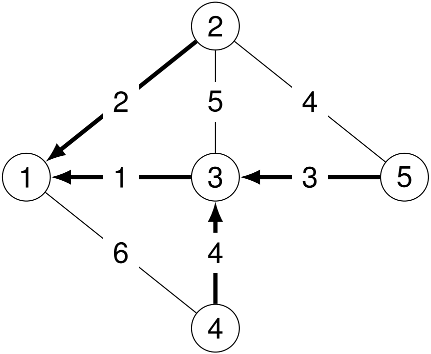
Shortest paths example, $\mathrm{sp} = (\mathbb{N}^{\infty},\ \min,\ +)$



The adjacency matrix

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{bmatrix} \end{array}$$

Shortest paths example, continued



Bold arrows indicate the shortest-path tree rooted at 1.

The routing matrix

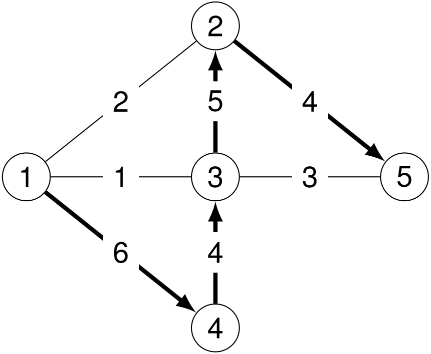
$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array}\right] \end{array}$$

Matrix $\mathbf{A}^*$ solves this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from i to j.

Widest paths example, $(\mathbb{N}^{\infty}, \max, \min)$



Bold arrows indicate the widest-path tree rooted at 1.

The routing matrix

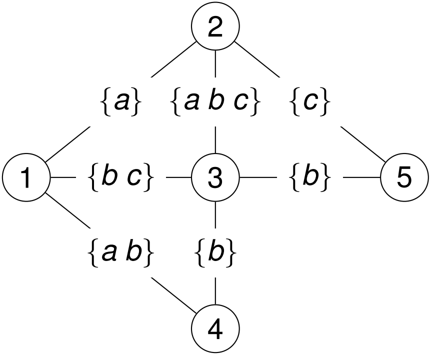
$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} \infty & 4 & 4 & 6 & 4 \\ 4 & \infty & 5 & 4 & 4 \\ 4 & 5 & \infty & 4 & 4 \\ 6 & 4 & 4 & \infty & 4 \\ 4 & 4 & 4 & 4 & \infty \end{array}\right] \end{array}$$

Matrix $\mathbf{A}^*$ solves this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \max_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the minimal edge weight in p.

Fun example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$



We want a Matrix $\mathbf{A}^*$ to solve this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \bigcup_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the intersection of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{A}^*(i,\ j)$ to mean that there is at least one path from i to j with x in every arc weight along the path.

Fun example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$

The matrix $\mathbf{A}^*$

	1	2	3	4	5
1	$\{a\ b\ c\}$	$\{a\ b\ c\}$	$\{a\ b\ c\}$	$\{a\ b\}$	$\{b\ c\}$
2	$\{a\ b\ c\}$	$\{a\ b\ c\}$	$\{a\ b\ c\}$	$\{a\ b\}$	$\{b\ c\}$
3	$\{a\ b\ c\}$	$\{a\ b\ c\}$	$\{a\ b\ c\}$	$\{a\ b\}$	$\{b\ c\}$
4	$\{a\ b\}$	$\{a\ b\}$	$\{a\ b\}$	$\{a\ b\ c\}$	$\{b\}$
5	$\{b\ c\}$	$\{b\ c\}$	$\{b\ c\}$	$\{b\}$	$\{a\ b\ c\}$

A few Semirings $(S, \oplus, \otimes, \overline{0}, \overline{1})$

A few examples

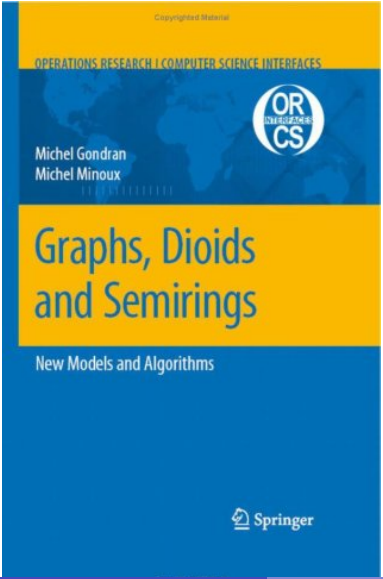
	S	$\oplus$	$\otimes$	$\overline{0}$	$\overline{1}$	possible applications
sp	$\mathbb{N} \cup \{\infty\}$	min	$+$	∞	0	minimum-weight routing
bw	$\mathbb{N} \cup \{\infty\}$	max	min	0	∞	greatest-capacity routing
rel	$[0, 1]$	max	$\times$	0	1	most-reliable routing
use	$\{0, 1\}$	max	min	0	1	usable-paths
links	2^W	$\cup$	$\cap$	$\{\}$	W	shared link attributes
paths	2^W	$\cap$	$\cup$	W	$\{\}$	shared path attributes

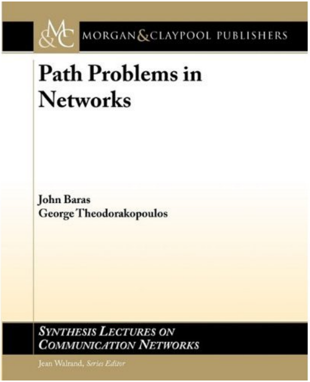
Historically, a focus on global optimality

$$\mathbf{A}^*(i, j) = \bigoplus_{p \,:\, i \rightsquigarrow j} w(p)$$

where $w(p)$ is an $\otimes$-product of arc weights.

Recommended Reading





Assumptions

Semiring Axioms

$$
\begin{array}{rcrcl}
\text{ADD.ASSOCIATIVE} & : & a \oplus (b \oplus c) & = & (a \oplus b) \oplus c \\
\text{ADD.COMMUTATIVE} & : & a \oplus b & = & b \oplus a \\
\text{ADD.LEFT.ID} & : & \overline{0} \oplus a & = & a \\
\text{MULT.ASSOCIATIVE} & : & a \otimes (b \otimes c) & = & (a \otimes b) \otimes c \\
\text{MULT.LEFT.ID} & : & \overline{1} \otimes a & = & a \\
\text{MULT.RIGHT.ID} & : & a \otimes \overline{1} & = & a \\
\text{MULT.LEFT.ANN} & : & \overline{0} \otimes a & = & \overline{0} \\
\text{MULT.RIGHT.ANN} & : & a \otimes \overline{0} & = & \overline{0} \\
\text{L.DISTRIBUTIVE} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\
\text{R.DISTRIBUTIVE} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c)
\end{array}
$$

Distributivity in $\mathrm{sp} = (\mathbb{N}^{\infty},\ \text{min},\ +)$:

$$
\begin{array}{rcrcl}
\text{L.DISTRIBUTIVE} & : & a + (b \text{ min } c) & = & (a + b) \text{ min } (a + c), \\
\text{R.DISTRIBUTIVE} & : & (a \text{ min } b) + c & = & (a + c) \text{ min } (b + c).
\end{array}
$$

Additional assumptions

Some subset of these axioms are typically assumed.

$$
\begin{array}{rcrcl}
\text{ADD.IDEMPOTENT} & : & a \oplus a & = & a \\
\text{ADD.SELECTIVE} & : & a \oplus b & \in & \{a,\ b\} \\
\text{ADD.LEFT.ANN} & : & \overline{1} \oplus a & = & \overline{1} \\
\text{ADD.RIGHT.ANN} & : & a \oplus \overline{1} & = & \overline{1} \\
\text{RIGHT.ABSORBTION} & : & a \oplus (a \otimes b) & = & a \\
\text{LEFT.ABSORBTION} & : & a \oplus (b \otimes a) & = & a
\end{array}
$$

With idempotency, $\oplus$ induces natural (partial) orders

$$
\begin{array}{rcl}
a \leq_l b & \equiv & a = a \oplus b \\
a \leq_r b & \equiv & b = a \oplus b
\end{array}
$$

If $\overline{1}$ is a $\oplus$-annihilator (Semiring is *bounded*)

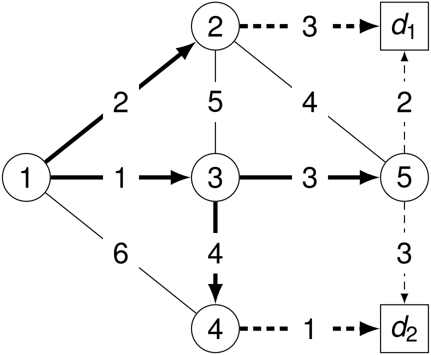
$$
\begin{array}{ccc}
\overline{1} & \leq_l & \overline{0} \\
\overline{0} & \leq_r & \overline{1}
\end{array}
$$

Many variations on basic structures....

weight computation	weight summarization: **algebraic**	**ordered**
algebraic	$(S, \oplus, \otimes)$	$(S, \lesssim, \otimes)$
functional	$(S, \oplus, F \subseteq S \rightarrow S)$	$(S, \lesssim, F \subseteq S \rightarrow S)$

... and many variations on the basic algorithms (Dijkstra's, Bellman-Ford, ...).

Let's model LISP!



routing = path finding + mapping
$\mathbf{R} = \mathbf{A}^*\mathbf{M}$

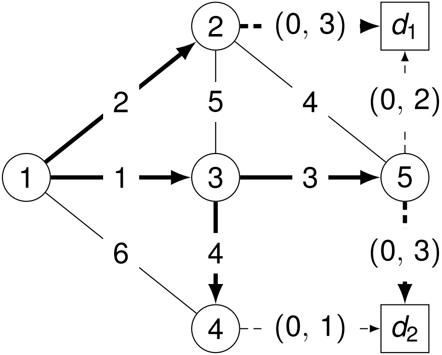
$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} \infty & \infty \\ 3 & \infty \\ \infty & \infty \\ \infty & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

Mapping matrix

$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} 5 & 6 \\ 3 & 7 \\ 5 & 5 \\ 9 & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

Routing/Forwarding matrix

More Interesting Example : Hot-Potato Idiom



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} \infty & \infty \\ (0,\ 3) & \infty \\ \infty & \infty \\ \infty & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{bmatrix} \end{array}$$

Mapping matrix

$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (2,\ 3) & (4,\ 3) \\ (0,\ 3) & (4,\ 3) \\ (3,\ 2) & (3,\ 3) \\ (7,\ 2) & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{bmatrix} \end{array}$$

Routing/Forwarding matrix

$$\mathbf{R} = \mathbf{A}^* \triangleright \mathbf{M}$$

$$\mathbf{R}(\mathrm{i},\ \mathrm{d}) = \bigoplus_{q} \mathbf{A}^*(i,\ q) \triangleright \mathbf{M}(q,\ d)$$

Working out the algebraic details

- A model of Internet routing using semi-modules. John N. Billings and Timothy G. Griffin. RelMiCS11/AKA6, November 2009.
- Application to route redistribution and administrative distance.
  - On the interaction of multiple routing algorithms. M. Abdul Alim, Timothy G. Griffin. ACM CoNEXT 2011, December 2011.

Semiring limitations — some realistic metrics are not distributive!

Two ways of forming "lexicographic" combination of shortest paths sp and bandwidth bw.

Widest shortest paths

- metric values of form (d, b)
- d in sp
- b in bw
- consider d first, break ties with b
- is distributive (some details ignored ...)

Shortest Widest paths

- metric values of form (b, d)
- d in sp
- b in bw
- consider b first, break ties with d

Left-Local Optimality

Say that $\mathbf{L}$ is a left locally-optimal solution when

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{L}(i,\ j) = \bigoplus_{q \in V} \mathbf{A}(i,\ q) \otimes \mathbf{L}(q,\ j)$$

- $\mathbf{L}(i,\ j)$ is the best possible value given the values $\mathbf{L}(q,\ j)$, for all out-neighbors q of source i.
- Rows $\mathbf{L}(i,\ \_)$ represents **out-trees <u>from</u>** i (think Bellman-Ford).
- Columns $\mathbf{L}(\_,\ i)$ represents **in-trees <u>to</u>** i.
- Works well with hop-by-hop forwarding from i.

Right-Local Optimality

Say that $\mathbf{R}$ is a right locally-optimal solution when

$$\mathbf{R} = (\mathbf{R} \otimes \mathbf{A}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{R}(i,\ j) = \bigoplus_{q \in V} \mathbf{R}(i,\ q) \otimes \mathbf{A}(q,\ j)$$

- $\mathbf{R}(i,\ j)$ is the best possible value given the values $\mathbf{R}(q,\ j)$, for all in-neighbors q of destination j.
- Rows $\mathbf{L}(i,\ \_)$ represents **out-trees from** i (think Dijkstra).
- Columns $\mathbf{L}(\_,\ i)$ represents **in-trees to** i.

With and Without Distributivity

With distributivity

For (bounded) semirings, the three optimality problems are essentially the same — locally optimal solutions are globally optimal solutions.

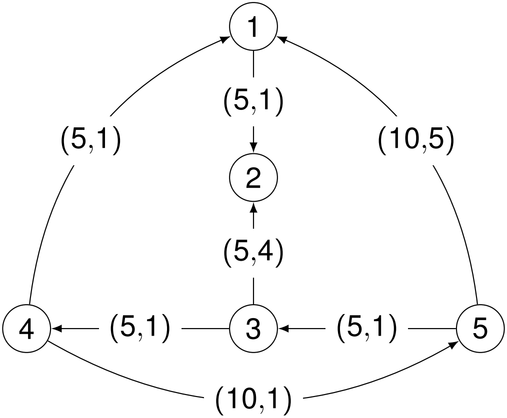
$$\mathbf{A}^* = \mathbf{L} = \mathbf{R}$$

Without distributivity

It may be that $\mathbf{A}^*$, $\mathbf{L}$, and $\mathbf{R}$ exists but are all distinct.

Health warning : matrix multiplication over structures lacking distributivity is not associative!

Example



(bandwidth, distance) with lexicographic order (bandwidth first).

Global optima

$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} (\infty, 0) & (5, 1) & (0, \infty) & (0, \infty) & (0, \infty) \\ (0, \infty) & (\infty, 0) & (0, \infty) & (0, \infty) & (0, \infty) \\ (5, 2) & (5, 3) & (\infty,\ 0) & (5, 1) & (5, 2) \\ (10, 6) & (5, 2) & (5, 2) & (\infty, 0) & (10, 1) \\ (10, 5) & (5, 4) & (5, 1) & (5, 2) & (\infty, 0) \end{array}\right] \end{array},$$

Left local optima

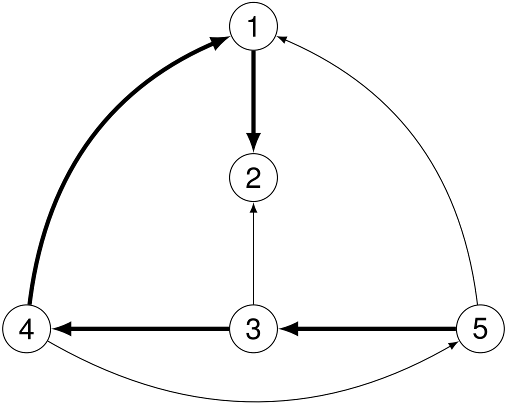
$$\mathbf{L} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} (\infty, 0) & (5, 1) & (0, \infty) & (0, \infty) & (0, \infty) \\ (0, \infty) & (\infty, 0) & (0, \infty) & (0, \infty) & (0, \infty) \\ (\mathbf{5}, \mathbf{7}) & (5, 3) & (\infty, 0) & (5, 1) & (5, 2) \\ (10, 6) & (5, 2) & (5, 2) & (\infty, 0) & (10, 1) \\ (10, 5) & (5, 4) & (5, 1) & (5, 2) & (\infty, 0) \end{array}\right] \end{array},$$

Entries marked in **bold** indicate those values which are not globally optimal.

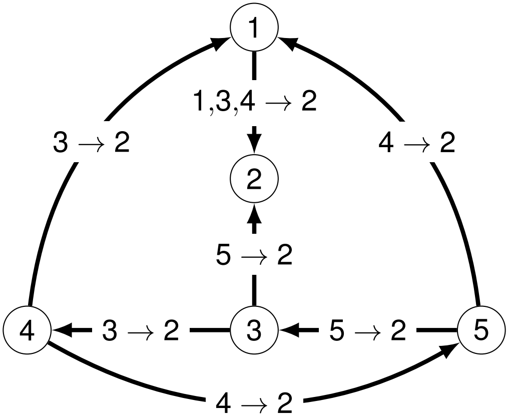
Right local optima

$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} (\infty,0) & (5,1) & (0,\infty) & (0,\infty) & (0,\infty) \\ (0,\infty) & (\infty,0) & (0,\infty) & (0,\infty) & (0,\infty) \\ (5,2) & (5,3) & (\infty,\ 0) & (5,\ 1) & (5,2) \\ (10,6) & (\mathbf{5},\mathbf{6}) & (5,2) & (\infty,0) & (10,1) \\ (10,5) & (\mathbf{5},\mathbf{5}) & (5,1) & (5,2) & (\infty,0) \end{bmatrix} \end{array},$$

Left-locally optimal paths to node 2



Right-locally optimal paths to node 2



Bellman-Ford can compute left-local solutions

(Unmodified) Bellman-Ford iterations

$$\mathbf{A}^{[0]} = \mathbf{I}$$
$$\mathbf{A}^{[k+1]} = (\mathbf{A} \otimes \mathbf{A}^{k}) \oplus \mathbf{I},$$

Bellman-ford iterations must be modified to ensure only cycle-free paths are inspected.

- $(S, \oplus, \overline{0})$ is a commutative, idempotent, and selective monoid,
- $(S, \otimes, \overline{1})$ is a monoid,
- $\overline{0}$ is the annihilator for $\otimes$,
- $\overline{1}$ is the annihilator for $\oplus$,
- Left strictly inflationarity, L.S.INF : $\forall a, b : a \neq \overline{0} \implies a < a \otimes b$
- Here $a \leq b \equiv a = a \oplus b$.

Convergence to a unique left-local solution is guaranteed. Currently no polynomial bound is known on the number of iterations required.

MTNS observation : Dijkstra's algorithm computes local optima!

Input : adjacency matrix $\mathbf{A}$ and source vertex $i \in V$,
Output : the i-th row of $\mathbf{R}$, $\mathbf{R}(i, \_)$.

begin
$S \leftarrow \{i\}$
$\mathbf{R}(i, i) \leftarrow \overline{1}$
for each $q \in V - \{i\} : \mathbf{R}(i, q) \leftarrow \mathbf{A}(i, q)$
while $S \neq V$
begin
find $q \in V - S$ such that $\mathbf{R}(i, q)$ is $\leq_{\oplus}^{L}$ -minimal
$S \leftarrow S \cup \{q\}$
for each $j \in V - S$
$\mathbf{R}(i, j) \leftarrow \mathbf{R}(i, j) \oplus (\mathbf{R}(i, q) \otimes \mathbf{A}(q, j))$
end
end

The goal

Given adjacency matrix **A** and source vertex $i \in V$, Dijkstra's algorithm will compute $\mathbf{R}(i, \_)$ such that

$$\forall j \in V : \mathbf{R}(i, j) = \mathbf{I}(i, j) \oplus \bigoplus_{q \in V} \mathbf{R}(i, q) \otimes \mathbf{A}(q, j).$$

Main invariant

$$\forall k : 1 \leq k \leq \mid V \mid \implies \forall j \in S_k : \mathbf{R}_k(i, j) = \mathbf{I}(i, j) \oplus \bigoplus_{q \in S_k} \mathbf{R}_k(i, q) \otimes \mathbf{A}(q, j)$$

Routing in Equilibrium. JoÃ£o LuÃs Sobrinho and Timothy G. Griffin. The 19th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2010).

Minimal subset of semiring axioms needed right-local Dijkstra

~~Semiring~~ Axioms

ADD.ASSOCIATIVE	:	$a \oplus (b \oplus c)$	$=$	$(a \oplus b) \oplus c$
ADD.COMMUTATIVE	:	$a \oplus b$	$=$	$b \oplus a$
ADD.LEFT.ID	:	$\overline{0} \oplus a$	$=$	a
~~MULT.ASSOCIATIVE~~	:	~~$a \otimes (b \otimes c)$~~	~~$=$~~	~~$(a \otimes b) \otimes c$~~
MULT.LEFT.ID	:	$\overline{1} \otimes a$	$=$	a
~~MULT.RIGHT.ID~~	:	~~$a \otimes \overline{1}$~~	~~$=$~~	~~a~~
~~MULT.LEFT.ANN~~	:	~~$\overline{0} \otimes a$~~	~~$=$~~	~~$\overline{0}$~~
~~MULT.RIGHT.ANN~~	:	~~$a \otimes \overline{0}$~~	~~$=$~~	~~$\overline{0}$~~
~~L.DISTRIBUTIVE~~	:	~~$a \otimes (b \oplus c)$~~	~~$=$~~	~~$(a \otimes b) \oplus (a \otimes c)$~~
~~R.DISTRIBUTIVE~~	:	~~$(a \oplus b) \otimes c$~~	~~$=$~~	~~$(a \otimes c) \oplus (b \otimes c)$~~

Additional axioms needed right-local Dijkstra

$$
\begin{array}{rcrcl}
\text{ADD.SELECTIVE} & : & a \oplus b & \in & \{a,\ b\} \\
\text{ADD.LEFT.ANN} & : & \overline{1} \oplus a & = & \overline{1} \\
\text{ADD.RIGHT.ANN} & : & a \oplus \overline{1} & = & \overline{1} \\
\text{RIGHT.ABSORBTION} & : & a \oplus (a \otimes b) & = & a
\end{array}
$$

Need left-local optima?

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I} \quad \Longleftrightarrow \quad \mathbf{L}^T = (\mathbf{L}^T \hat{\otimes}^T \mathbf{A}^T) \oplus \mathbf{I}$$

where $\otimes^T$ is matrix multiplication defined with as

$$a \otimes^T b = b \otimes a$$

and we assume left-inflationarity holds, L.INF : $\forall a, b : a \leq b \otimes a$.

Tools? Metarouting Project

- Language of combinators for algebraic structures + library of verified algorithms.
- Vilius Naudžiūnas implemented prototype in Coq
- Allows users to instantiate generic algorithms with custom built algebras
- No theorem proving required of users — correctness check of instantiation is done by “type checking”

Our approach to defining a language of combinators

Starting with an initial set of properties $\mathcal{P}_0$ ...

- Define a language $\mathcal{L}$ of combinators,
- a well-formedness condition $\text{WF}(E)$, for $E \in \mathcal{L}$,
- and a set of properties $\mathcal{P}$, with $\mathcal{P}_0 \subseteq \mathcal{P}$

so that properties are decidable for well-formed expressions:

$$\forall \text{Q} \in P : \forall E \in \mathcal{L} : \text{WF}(E) \implies (\text{Q}([\![E]\!]) \vee \neg \text{Q}([\![E]\!]))$$

(The logic is constructive!)

Difficulty: increase expressive power while preserving decidability ...

Example: A bottleneck semiring[1]

The idea ...

- arc weights from a partial order $\leq$
- path weight $w(p) \;=\;$ set of edges in p with maximal weight.
- $w(p) \leq w(q) \iff \forall x \in w(p), \exists y \in w(q), x \leq y$

... in Coq (so far abstract syntax only)

```
Definition s1 := sProduct sNatMin sNatMin
Definition s2 :=
    sFMinSetsUnion (pRightNaturalOrder s1)
Definition bottleneck :=
    bFMinSets (oRightNaturalOrder s2)
```

[1] Originally defined in **Bottleneck shortest paths on a partially ordered scale.**, Monnot, J. and Spanjaard, O., 4OR: A Quarterly Journal of Operations Research, 2003

The language design methodology

For every combinator C and every property P

find $\mathrm{wf}_{P,C}$ and $\beta_{P,C}$ such that

$$\mathrm{wf}_{P,C}(\vec{a}) \Rightarrow (P(C(\vec{a})) \Leftrightarrow \beta_{P,C}(\vec{a}))$$

... which is then turned into two "bottom-up rules" ...

$$\begin{array}{rcl} \mathrm{wf}_{P,C}(\vec{a}) \wedge \beta_{P,C}(\vec{a}) & \Rightarrow & P(C(\vec{a})) \\ \mathrm{wf}_{P,C}(\vec{a}) \wedge \neg\beta_{P,C}(\vec{a}) & \Rightarrow & \neg P(C(\vec{a})), \end{array}$$

Current development snapshot

name	signature	(positive) properties	constructors
Sets	(S)	3	9
Semigroups	$(S, \oplus)$	14	17
Preorders	$(S, \leq)$	4	5
Bisemigroups	$(S, \oplus, \otimes)$	22	20
Order semigroups	$(S, \leq, \oplus)$	17	6
Transforms	$(S, L, \rhd)$	2	8
Order transforms	$(S, L, \leq, \rhd)$	3	2
Semigroup transforms	$(S, L, \oplus, \rhd)$	4	10

where $\rhd \in L \to S \to S$.

This represents over 1700 bottom-up rules ...

One open problem

Relationship of Routing and Forwarding

- simple: routing $=$ path finding $+$ mapping
- reality: routing $=$ path finding $+$ mapping $+$ forwarding
- The data plane uses paths in many different ways
  - exact match
  - best match
  - tunnels
  - ...
- We don't understand relationship
  - eBGP should be done with tunnels
  - are 2547 VPNs broken ?
  - subnetting
  - overlay/underlay