Towards an Algebraic Network Information Theory

Bobak Nazer (BU)

Joint work with Sung Hoon Lim (EPFL), Chen Feng (UBC), and Michael Gastpar (EPFL).

DIMACS Workshop on Network Coding: The Next 15 Years

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- State-of-the-art elegantly captured in the recent textbook of **El Gamal and Kim.**
- Codes with algebraic structure are sought after to mimic the performance of random i.i.d. codes.

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This Talk: We build on previous work and propose a joint typicality approach to algebraic network information theory.

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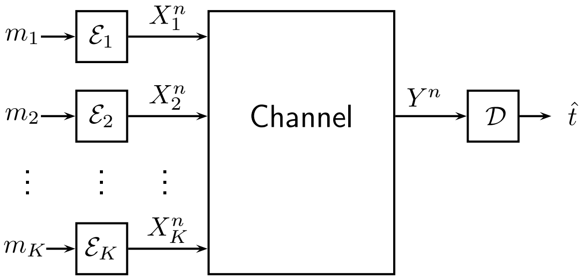

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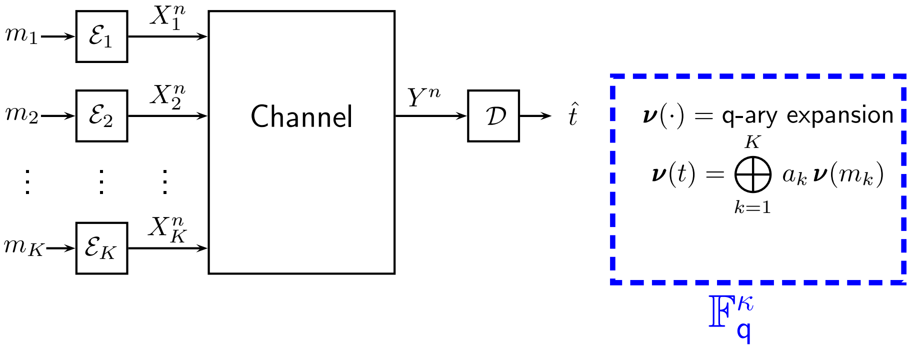

Compute-and-Forward

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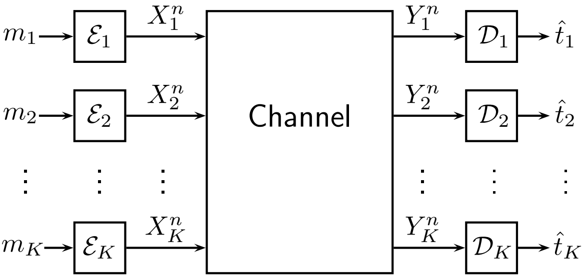

$\boldsymbol{\nu}(\cdot) =$ q-ary expansion

$$\boldsymbol{\nu}(t_\ell) = \bigoplus_{k=1}^{K} a_{\ell,k}\, \boldsymbol{\nu}(m_k)$$

$\mathbb{F}_{\mathsf{q}}^{\kappa}$

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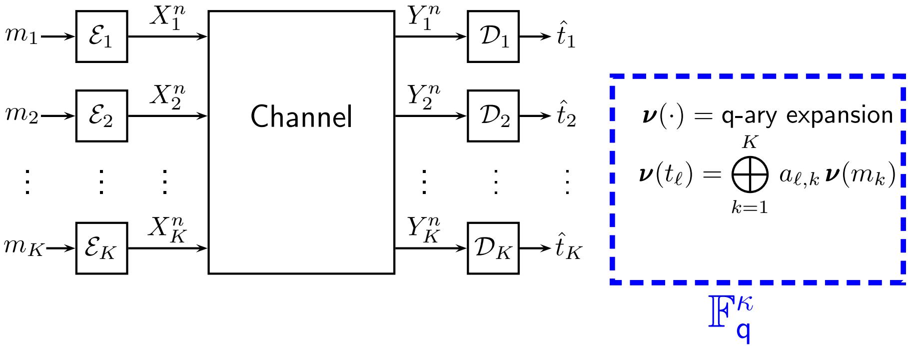

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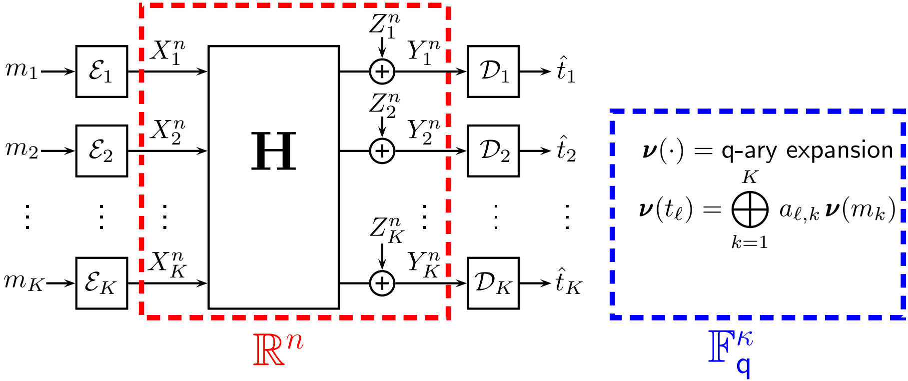

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- Equal power constraints:
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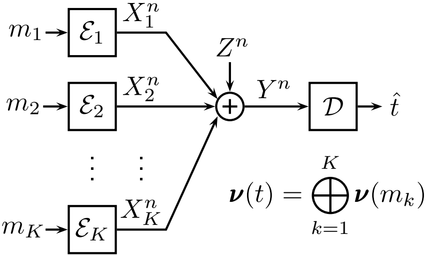

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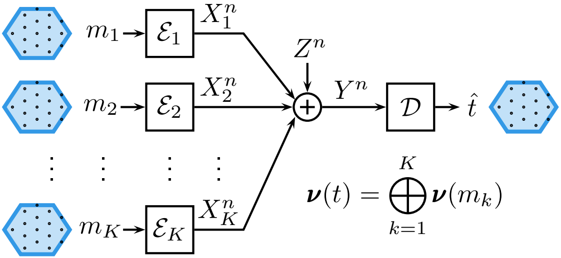

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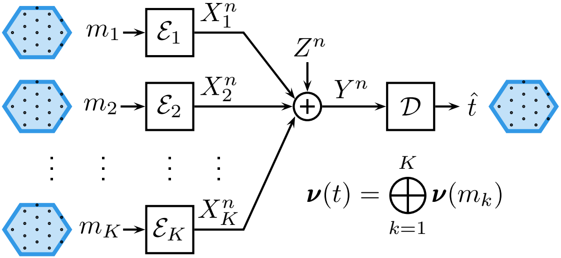

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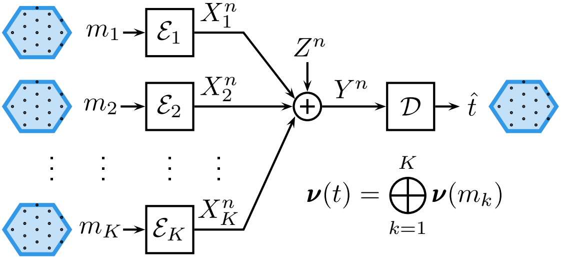

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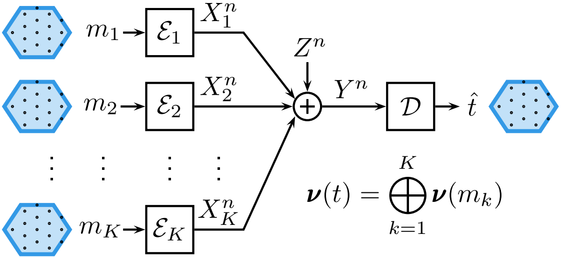

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- What about the "1+"? Still open! (Ice wine problem.)

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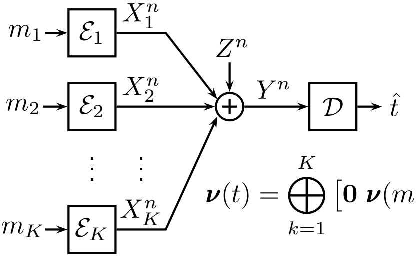

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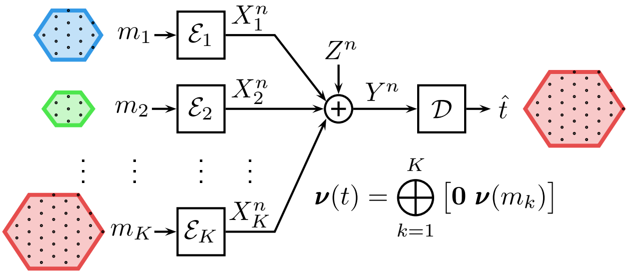

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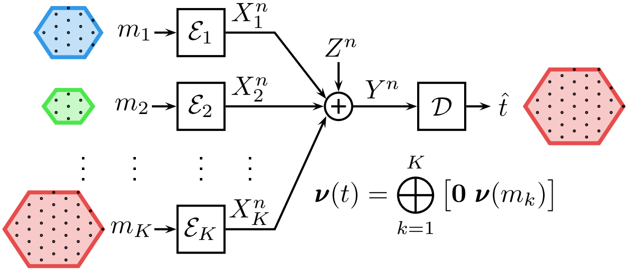

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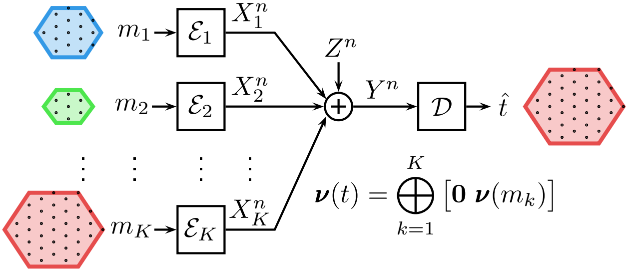

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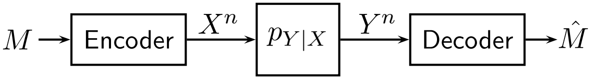
- **Nazer-Cadambe-Ntranos-Caire '15:** Expanded compute-and-forward framework to link unequal power setting to finite fields.

Point-to-Point Channels



- Messages: $m \in [2^{nR}] \triangleq \{0, \ldots, 2^{nR} - 1\}$
- Encoder: a mapping $x^n(m) \in \mathcal{X}^n$ for each $m \in [2^{nR}]$
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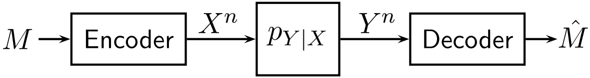

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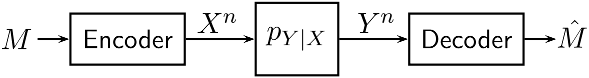

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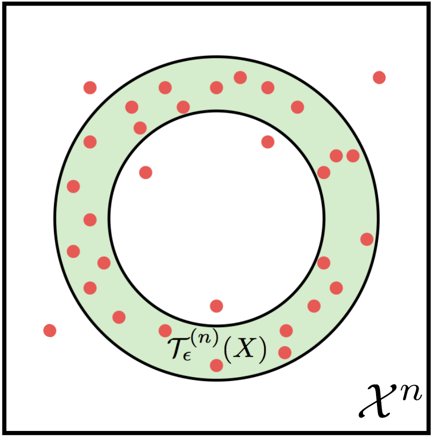
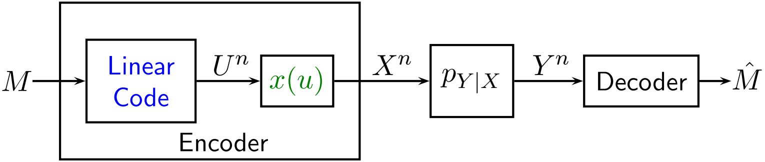
- Proof relies on random i.i.d. codebooks combined with joint typicality decoding.



Random i.i.d. Codes

- Codewords are independent of one another.
- Can directly target an input distribution $p_X(x)$.

Point-to-Point Channels: Linear Codes



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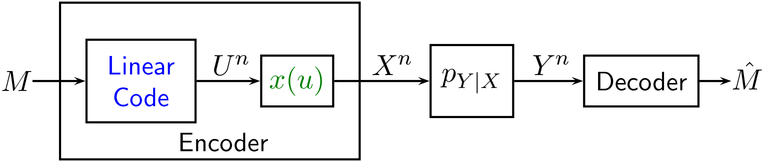

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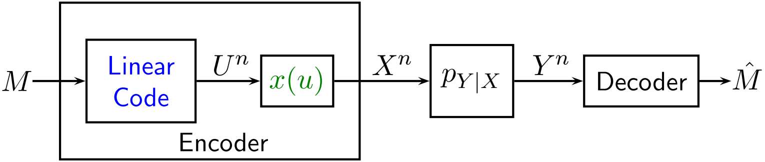

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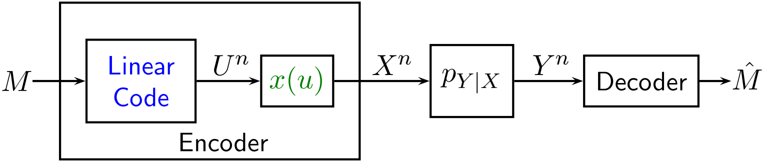

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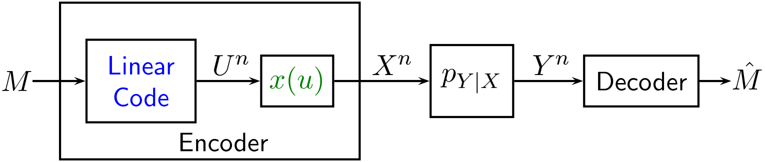

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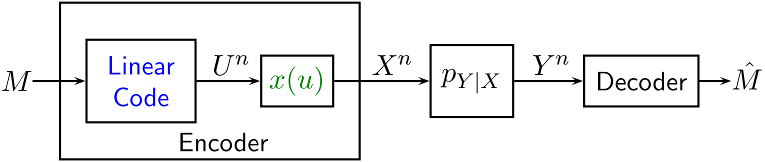

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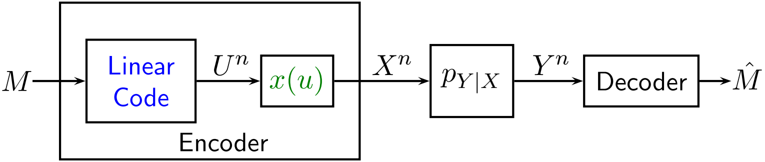

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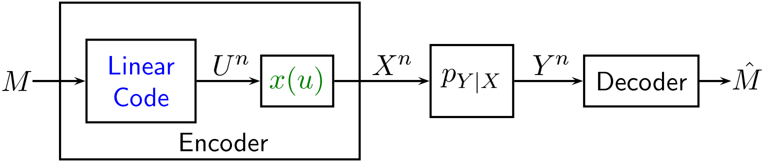

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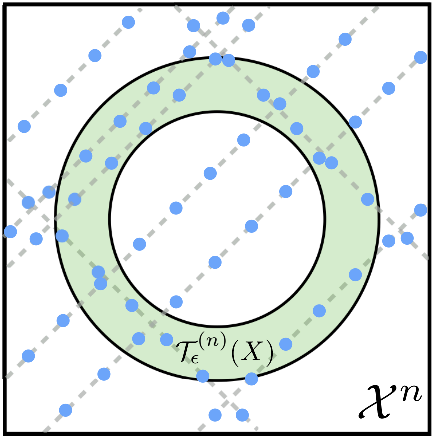
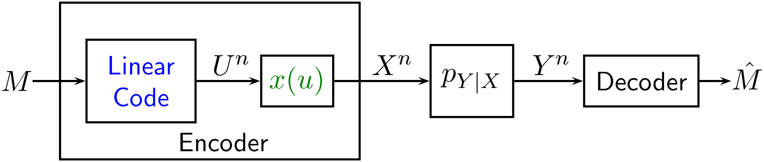
- Channel input at time i is $x_i(m) = x(u_i(m))$.



Random Linear Codes

- Codewords are pairwise independent of one another.
- Codewords are uniformly distributed over $\mathbb{F}_q^n$.

Point-to-Point Channels: Linear Codes



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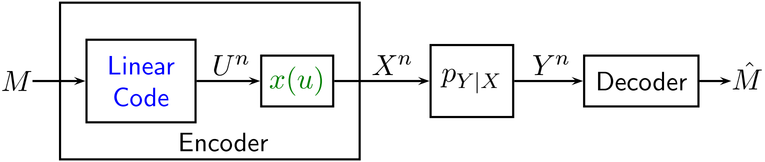

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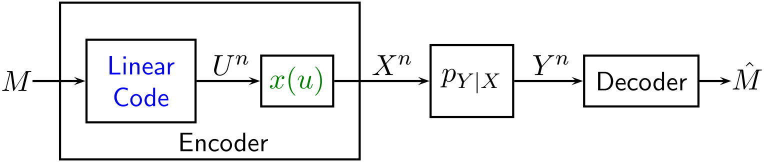

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- This will not work for us. Roughly speaking, if each encoder has a different input distribution, the symbol mappings may be quite different, which will disrupt the linear structure of the codebook.

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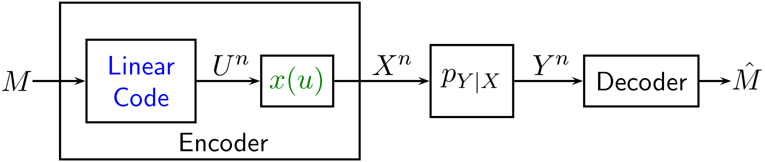

- Well known that a direct application of linear coding is not sufficient to reach the point-to-point capacity, **Ahlswede '71.**
- **Gallager '68:** Pick $\mathbb{F}_\mathsf{q}$ with q $\gg \mathcal{X}$ and choose symbol mapping $x(u)$ to reach c.a.i.d. from $\mathrm{Unif}(\mathbb{F}_\mathsf{q})$. This can attain the capacity.
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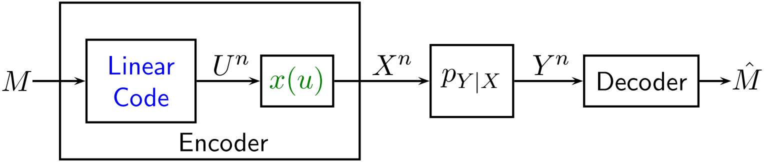

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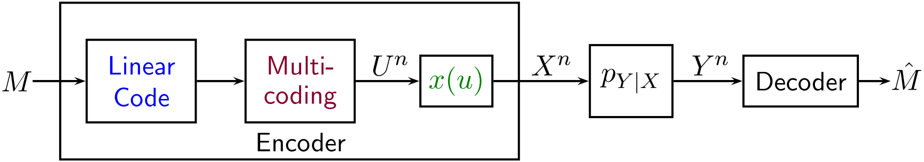
- **Padakandla-Pradhan '13:** It is possible to shape the input distribution using nested linear codes.
- Basic idea: Generate many codewords to represent one message. Search in this "bin" to find a codeword with the desired type, i.e., multicoding.



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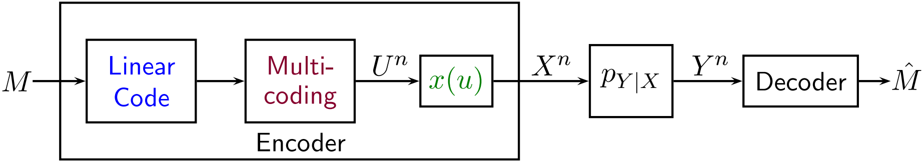

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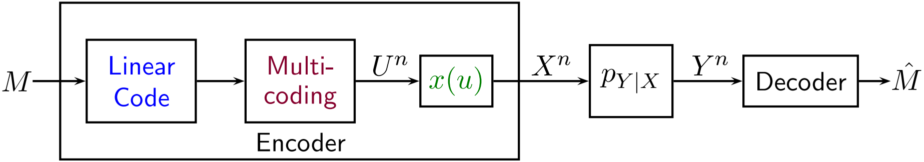
- Messages $m \in [2^{nR}]$ and auxiliary indices $l \in [2^{n\hat{R}}]$.



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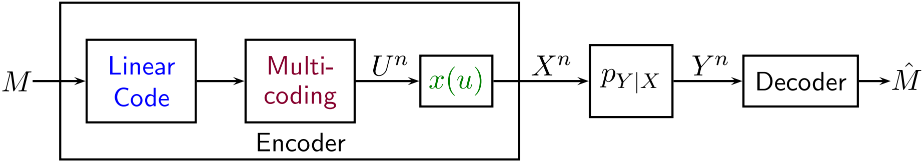

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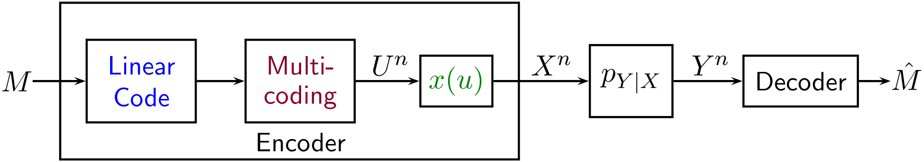

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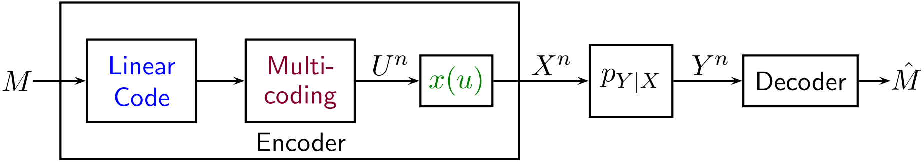

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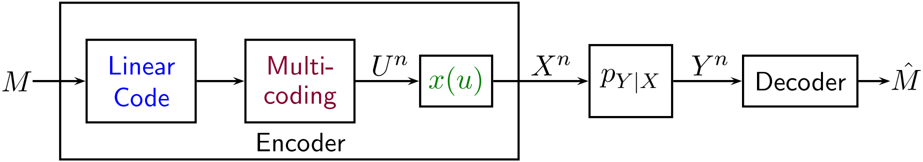
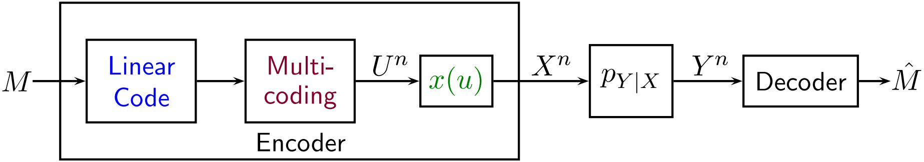

M
Linear Code
Multi-coding
U^n
$x(u)$
Encoder
X^n
$p_{Y|X}$
Y^n
Decoder
$\hat{M}$

Encoding:

Point-to-Point Channels: Linear Codes + Multicoding



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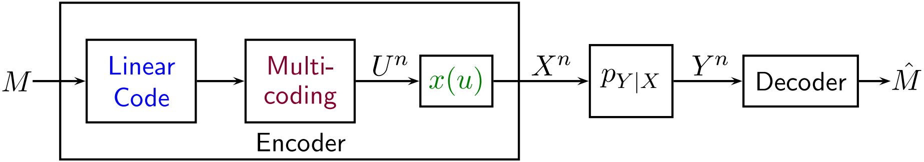

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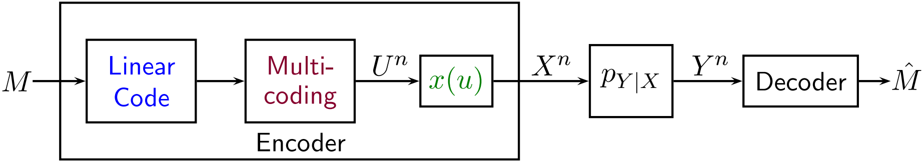

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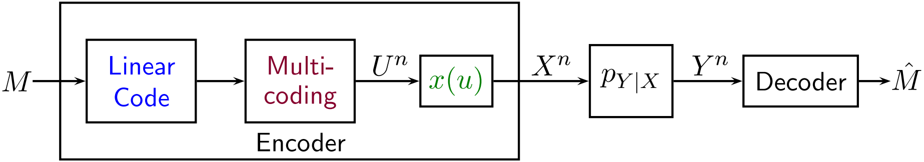

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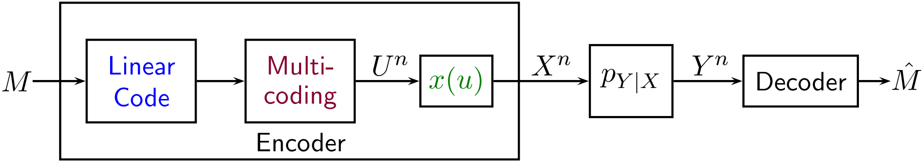

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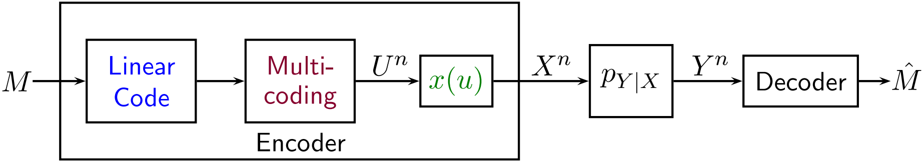
Decoding:

Point-to-Point Channels: Linear Codes + Multicoding



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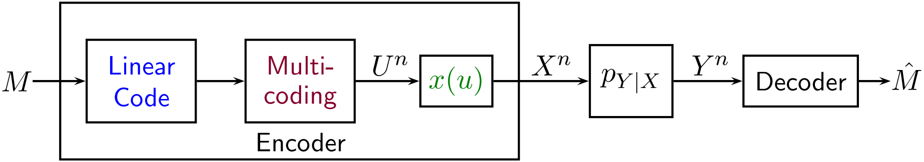
- Joint Typicality Decoding: Find the unique index $\hat{m}$ such that $\big(u^n(\hat{m},\hat{l}),\, y^n\big) \in \mathcal{T}_{\epsilon}^{(n)}(U,Y)$ for some index $\hat{l}$.

Point-to-Point Channels: Linear Codes + Multicoding



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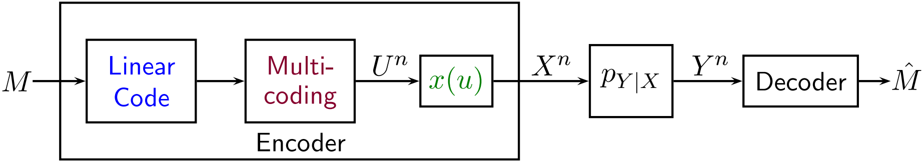


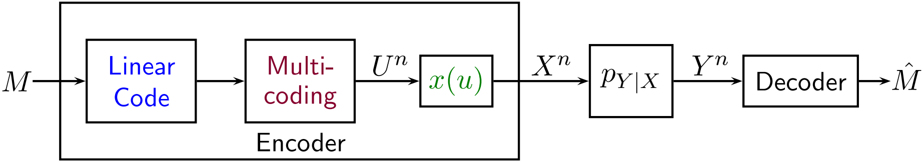
Theorem (Padakandla-Pradhan '13)

Any rate R satisfying

$$R < \max_{p(u),\, x(u)} I(U;Y)$$

is achievable. This is equal to the capacity if $\mathsf{q} \geq |\mathcal{X}|$.

Point-to-Point Channels: Linear Codes + Multicoding



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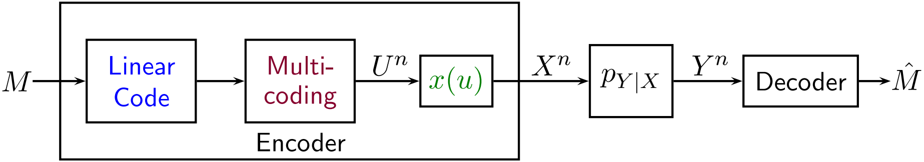

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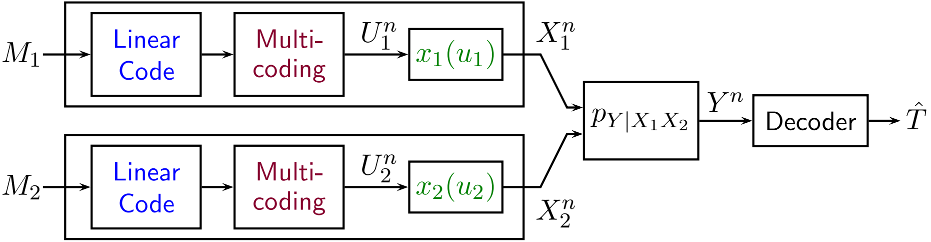
- Next, let's examine a two-transmitter, one-receiver "compute-and-forward" network.

Nested Linear Coding Architecture



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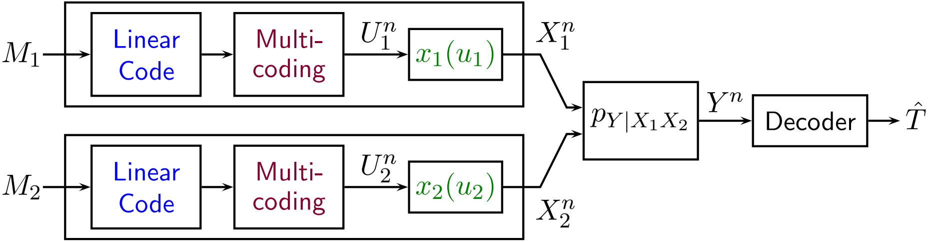

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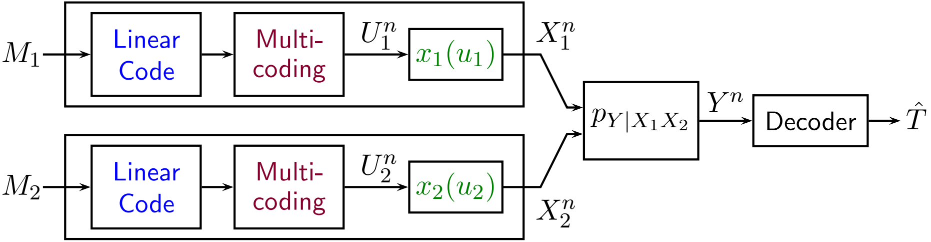

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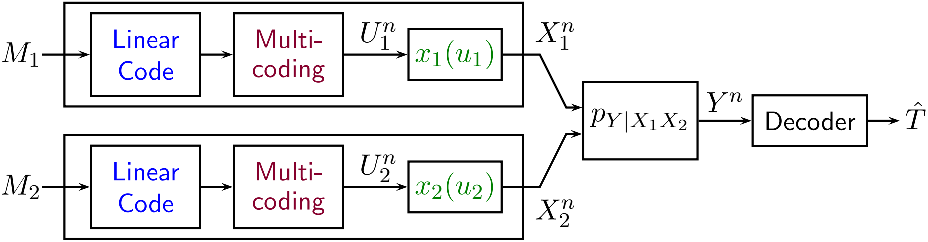

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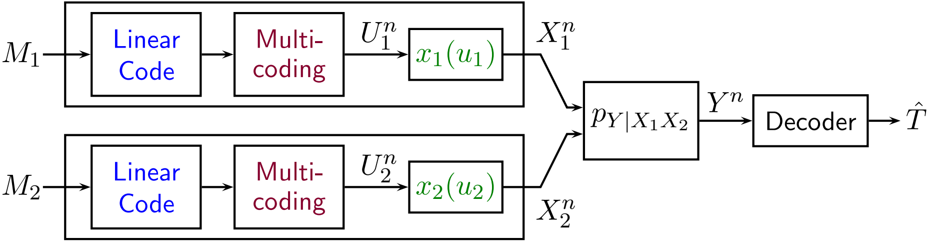
- Take q-ary expansions $\begin{bmatrix}\boldsymbol{\nu}(m_1) & \boldsymbol{\nu}(l_1)\end{bmatrix} \in \mathbb{F}_{\mathsf{q}}^{\kappa}$

  $\begin{bmatrix}\boldsymbol{\nu}(m_2) & \boldsymbol{\nu}(l_2) & \mathbf{0}\end{bmatrix} \in \mathbb{F}_{\mathsf{q}}^{\kappa}$ Zero-padding

Nested Linear Coding Architecture



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- Take q-ary expansions $\left[\boldsymbol{\eta}(m_1, l_1)\right] \in \mathbb{F}_{\mathsf{q}}^{\kappa}$

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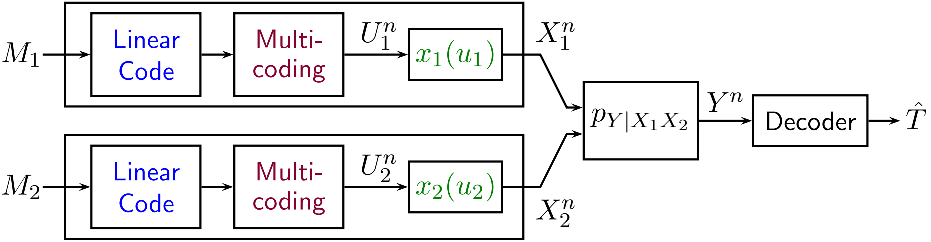
Nested Linear Coding Architecture



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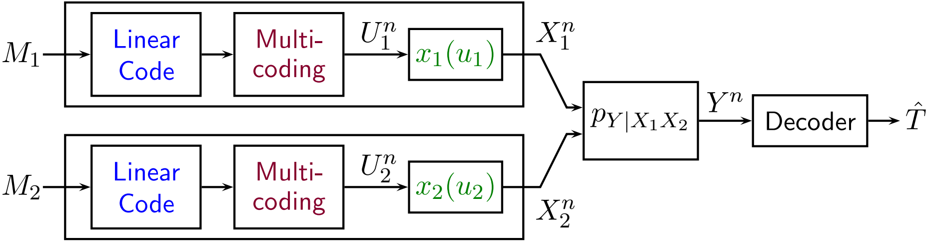

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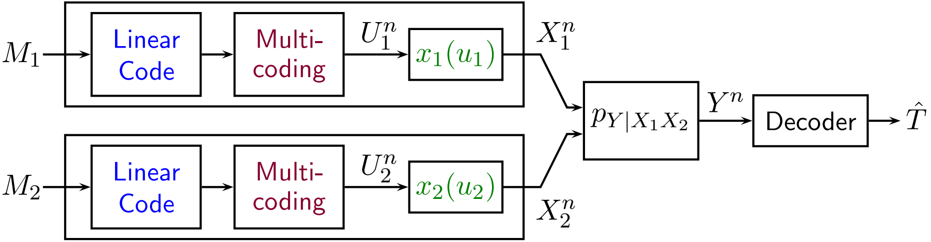

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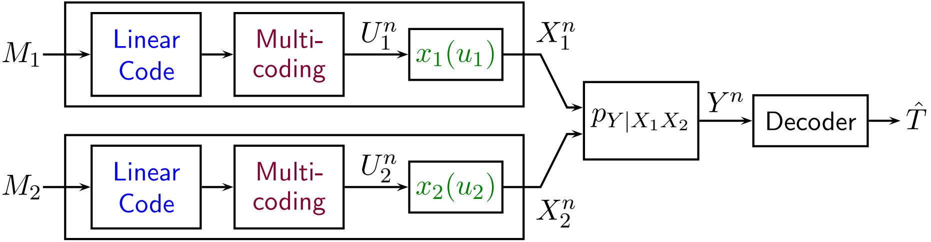

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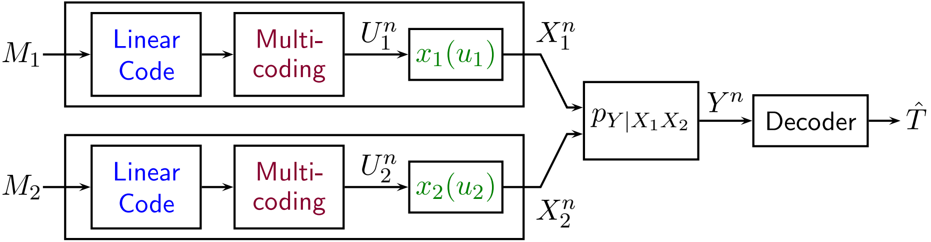

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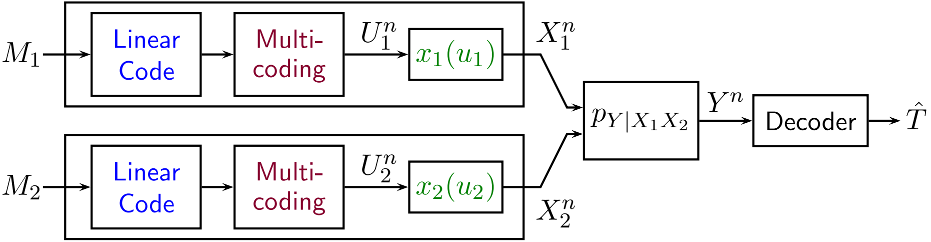

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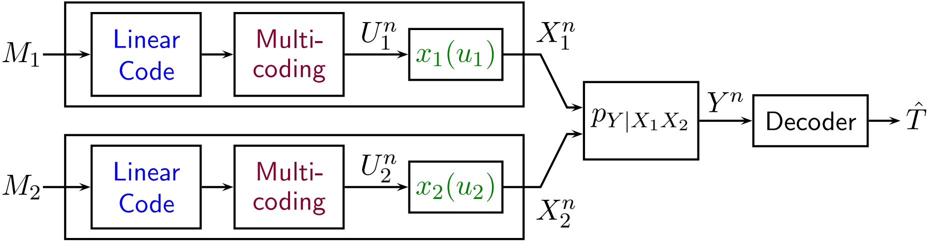

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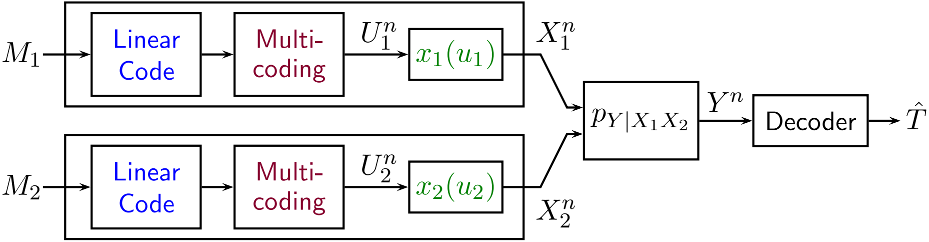
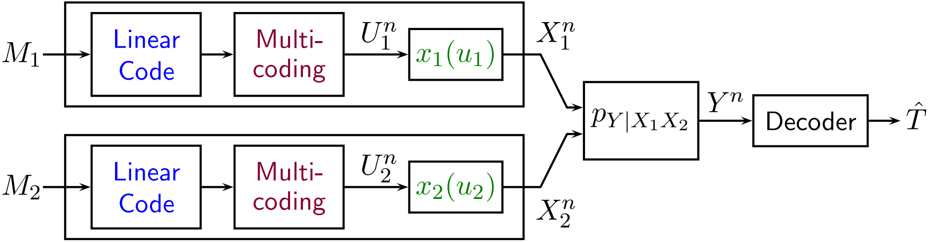


Computation Problem:



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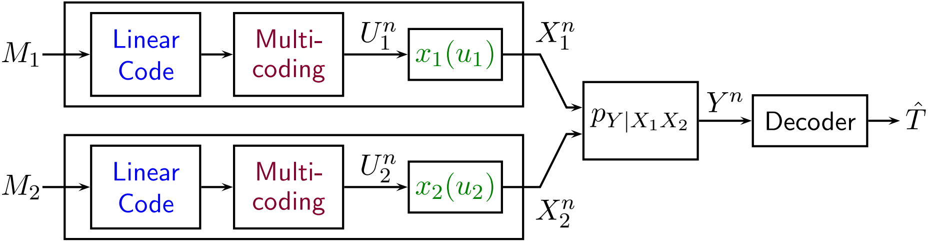
- Consider the coefficients $\mathbf{a} \in \mathbb{F}_q^2$, $\mathbf{a} = [a_1,\ a_2]$

Nested Linear Coding Architecture



Computation Problem:

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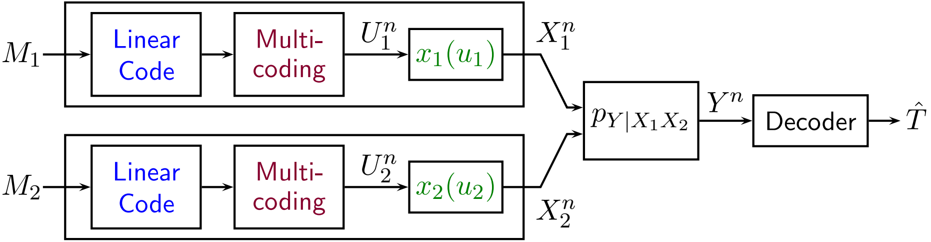
- For $m_k \in [2^{nR_k}]$, $l_k \in [2^{n\hat{R}_k}]$, the linear combination of codewords with coefficient vector $\mathbf{a}$ is

$$\begin{aligned} & a_1 u_1^n(m_1, l_1) \oplus a_2 u_2^n(m_2, l_2) \\ & = \big[a_1 \boldsymbol{\eta}(m_1, l_1) \oplus a_2 \boldsymbol{\eta}(m_2, l_2)\big] \mathsf{G} \oplus a_1 d_1^n \oplus a_2 d_2^n \\ & = \boldsymbol{\nu}(t) \mathsf{G} \oplus d_w^n \\ & = w^n(t), \quad t \in [2^{n \max\{R_1 + \hat{R}_1, R_2 + \hat{R}_2\}}] \end{aligned}$$



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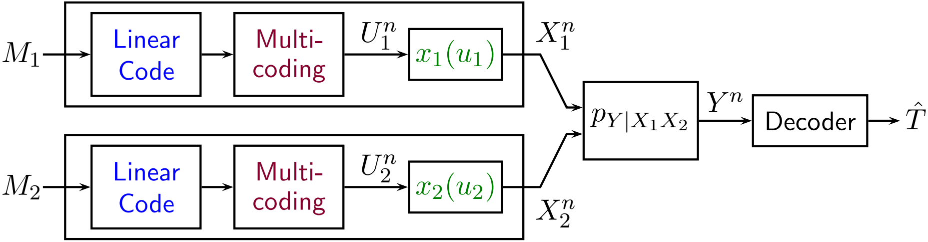

Computation Problem:

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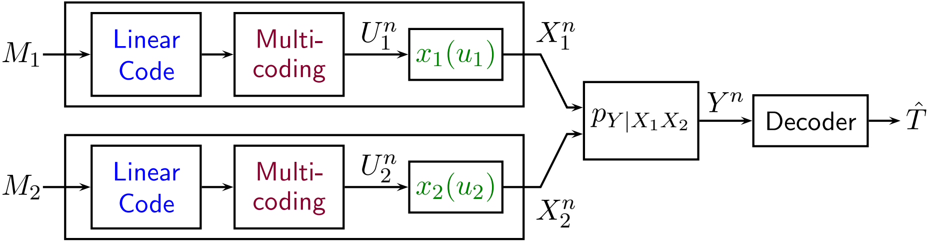
- Decoder wants a linear combination of the codewords:

$$W^n(T) = a_1 U_1^n(M_1, L_1) \oplus a_2 U_2^n(M_2, L_2)$$

Nested Linear Coding Architecture



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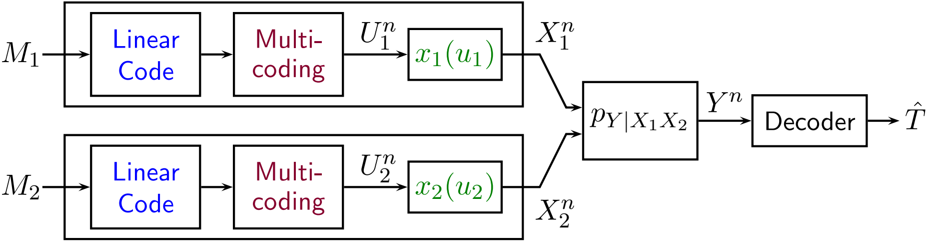
Nested Linear Coding Architecture



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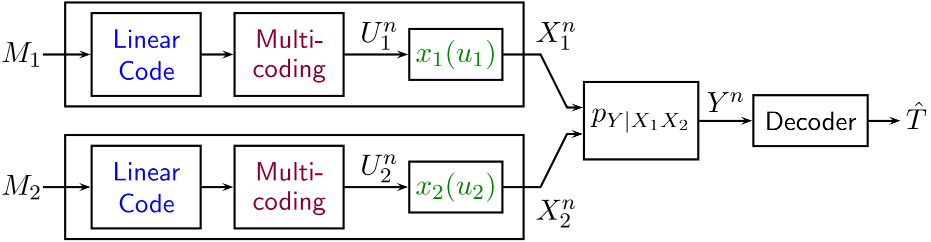

Decoding:

- Joint Typicality Decoding: Find an index $t \in [2^{n\max(R_1+\hat{R}_1, R_2+\hat{R}_2)}]$ such that $(w^n(t), y^n) \in \mathcal{T}_\epsilon^{(n)}$.

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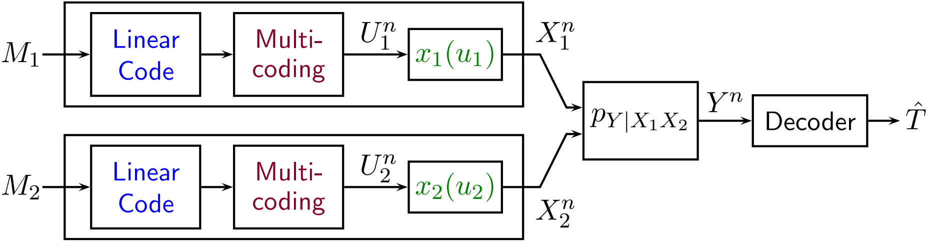

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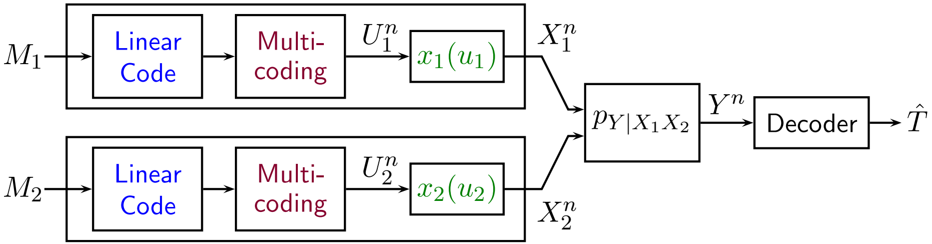

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- **Padakandla-Pradhan '13:** Special case where $R_1 = R_2$.

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Theorem (Nazer-Gastpar '11)

For any channel vector $\mathbf{h}$ *and integer coefficient vector* $\mathbf{a}$*, any rate tuple satisfying* $R_k < R_{\textit{comp}}(\mathbf{h}, \mathbf{a})$ *for* k *s.t.* $a_k \neq 0$ *is achievable where*

$$R_{\textit{comp}}(\mathbf{h}, \mathbf{a}) = \frac{1}{2} \log^+ \left(\frac{P}{\mathbf{a}^\mathsf{T} \left(P^{-1} \mathbf{I} + \ \mathbf{h}\mathbf{h}^\mathsf{T} \right)^{-1} \mathbf{a}} \right)$$

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- **Ordentlich-Erez '13** derived bounds for lattice-based codes.
- **This talk:** We can analyze this via joint typicality decoding to get an achievable rate region.

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  with vanishing probability of error.
- **Key Technical Issue:** Random linear codewords are pairwise independent, but not 4-wise independent!

Jointly Decoding Two Linear Combinations of K Codewords

Theorem (Lim-Chen-Nazer-Gastpar Allerton '15)

A rate tuple $(R_1, \ldots, R_K)$ is achievable for computing two linear combinations if

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\begin{aligned}
R_k &< \min\{H(U_k) - H(V|Y), H(U_k) - H(W_1, W_2|Y, V)\}, \quad k \in \mathcal{K}_1 \\
R_j &< I(W_2; Y, W_1) - H(W_2) + H(U_j), \quad j \in \mathcal{K}_2, \\
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for some $\prod_{k=1}^{K} p(u_k)$ and $x_k(u_k)$ and non-zero vector $\boldsymbol{b} \in \mathbb{F}_{\mathsf{q}}^2$,
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- The auxiliary linear combination V plays a key role in classifying dependent competing pairs in the error analysis.

Theorem (Lim-Chen-Nazer-Gastpar Allerton '15)

A rate pair (R_1, R_2) *is achievable for the discrete memoryless multiple-access channel if*

$$R_1 < \max_{\mathbf{a} \neq \mathbf{0}} \min\{H(U_1) - H(W|Y),\ H(U_1) - H(U_1, U_2|Y, W)\},$$

$$R_2 < I(X_2; Y|X_1),$$

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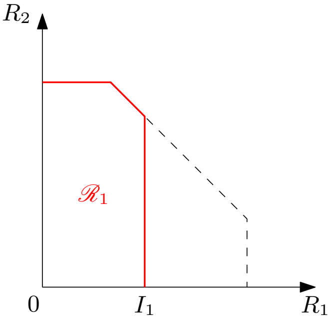
for some $p(u_1)p(u_2)$ *and* $x_1(u_1)$, $x_2(u_2)$, *where* $W = a_1 U_1 \oplus a_2 U_2$.

Multiple-Access Rate Region



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\begin{aligned}
R_1 &< I_1, \\
R_2 &< I(X_2; Y | X_1), \\
R_1 + R_2 &< I(X_1, X_2; Y),
\end{aligned}
$$

where $I_1 = \max_{\mathbf{a} \neq \mathbf{0}} \min\{H(U_1) - H(W|Y),\ H(U_1) - H(U_1, U_2|Y, W)\}$

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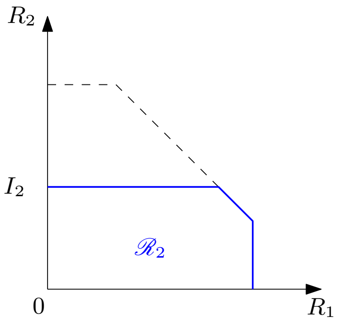
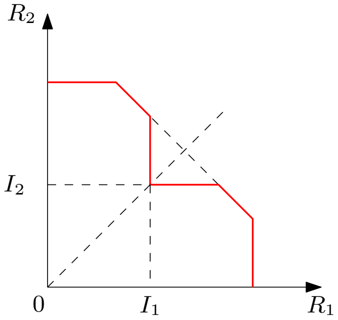


$$R_1 < I(X_1; Y | X_2),$$
$$R_2 < I_2,$$
$$R_1 + R_2 < I(X_1, X_2; Y),$$

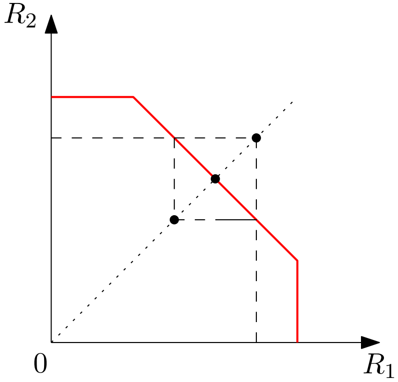
where $I_2 = \max_{\mathbf{a} \neq \mathbf{0}} \min\{H(U_2) - H(W|Y),\ H(U_2) - H(U_1, U_2|Y, W)\}$

Multiple-Access Rate Region



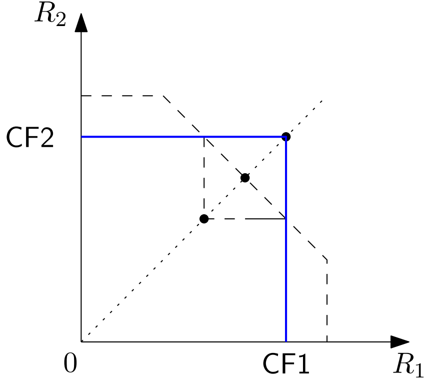
- Multiple-access rate region via nested linear codes:

$$\mathscr{R}_1 \cup \mathscr{R}_2$$



MAC Capacity Region

- Even if the receiver is only interested in recovering one linear combination it can sometimes help to decode two!



Decode One Linear Combination

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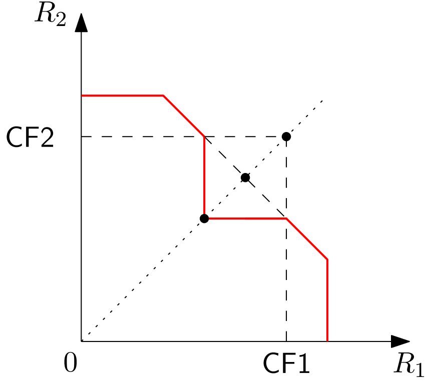

Multiple-Access via Nested Linear Codes

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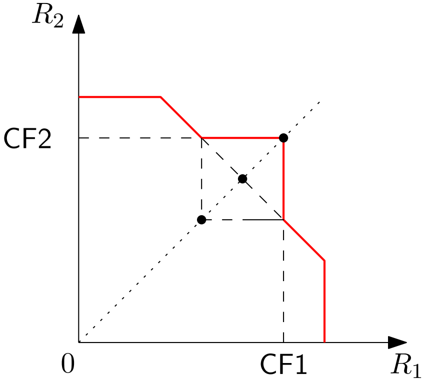
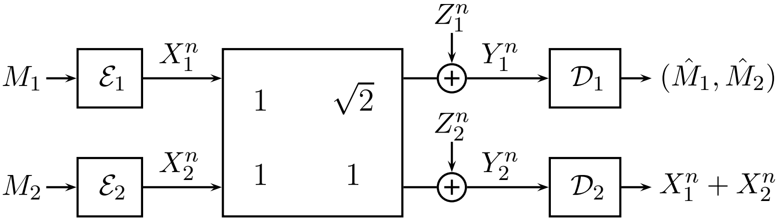


Union

- Even if the receiver is only interested in recovering one linear combination it can sometimes help to decode two!

Case Study: Two-Sender, Two-Receiver Network



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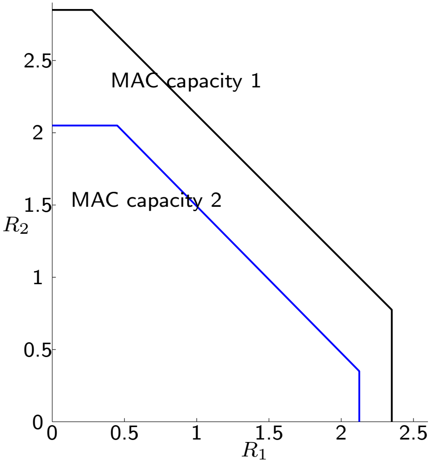

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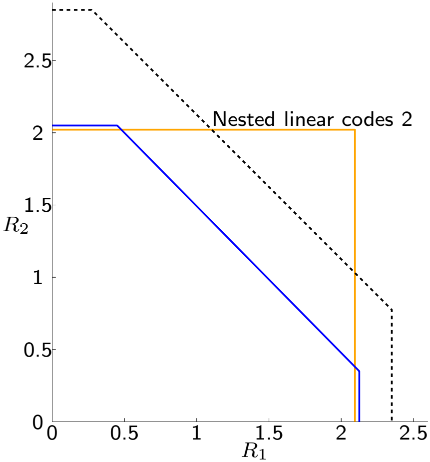

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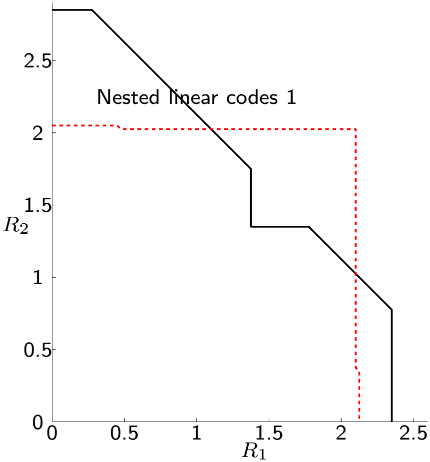

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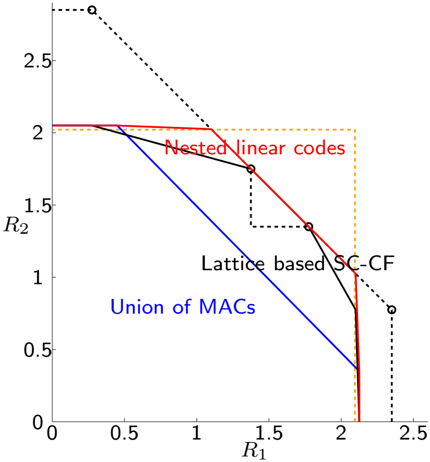

Concluding Remarks

- First steps towards bringing algebraic network information theory back into the realm of joint typicality.
- Joint decoding rate region for compute-and-forward that outperforms parallel and successive decoding.