Some Combinatorial Aspects of WDM Cross Connect Design

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August 2007

Non-blocking Properties

Strictly non-blocking:

- Can route any new demand no matter how existing demands are routed.

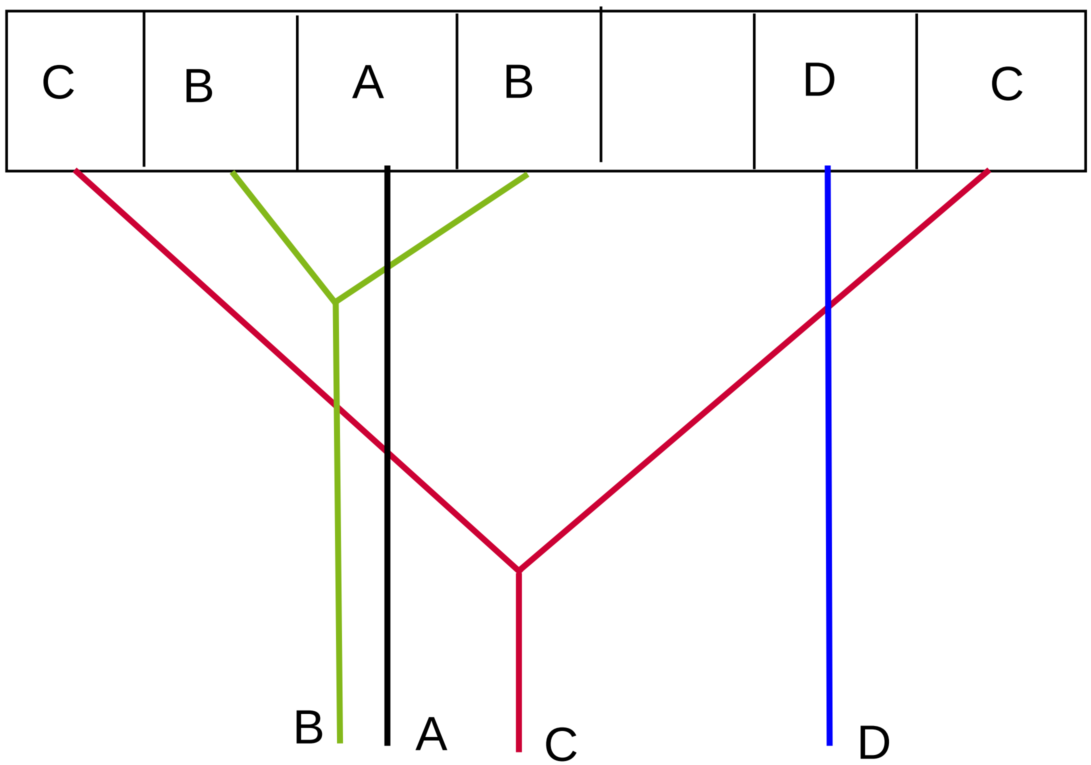
Wide-sense non-blocking:

- Algorithm *A* can find a route that does not disrupt any demands previously routed by *A*.

Rearrangeably non-blocking:

- May require other demands to be rerouted.

4-Arm ROADM



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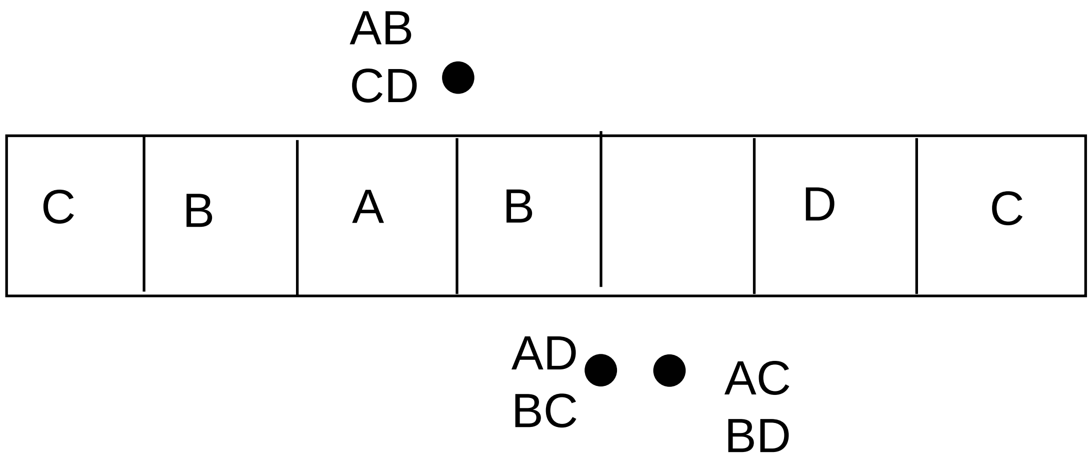
4-Arm ROADM (2 splitters, 7 ports), strictly non-blocking



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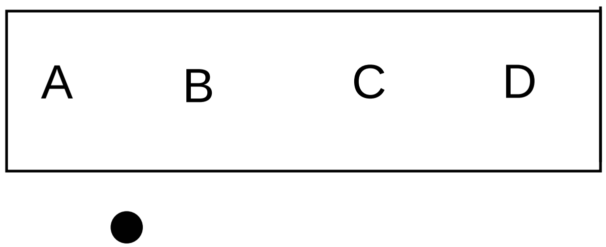
4-Arm ROADM (0 splitters?)



Connecting AB prohibits connecting CD.

4-Arm ROADM (1 splitter, WLOG assume 2 C’s)

A B D

- To connect AB, implies C to left of A to get CD

C A B D

- To connect BD, implies C to right of D to get AC

C_1 A X B D C_2

4-Arm ROADM (1 splitter, WLOG assume 2 C's)

Consider ● connecting AD.

C_1 A X B D C_2

● Between A and B, implies BX, X between A and B

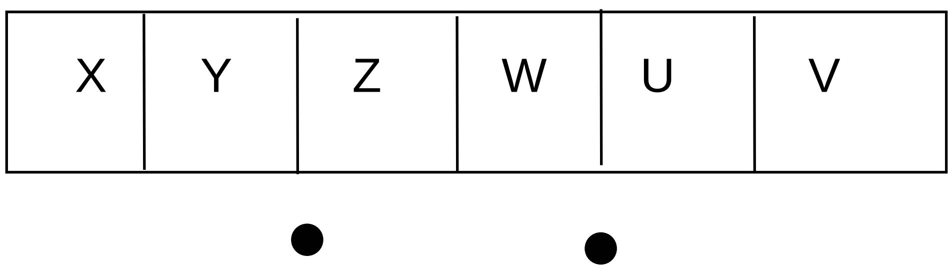
C_1 A B X D C_2

● Between B and D, implies BX, X between B and D

Therefore, must be at least 2 splitters.

4-Arm ROADM (2 splitters, 6 ports?)

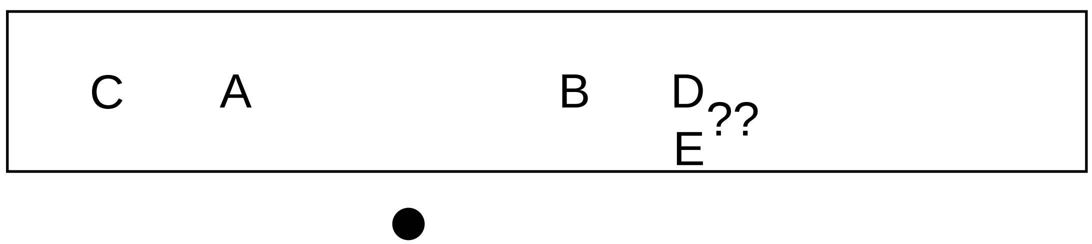
Need 3 matchings (AB,CD), (AC,BD), (AD,BC).
Each ● needs exactly 2 (nonempty) ports one side.



k-arm Roadms

Is there a systematic way to design “small” k-arm roadms (k>4) where small means
(1) optimal number of splitters and/or
(2) optimal number of ports?

No. If goal is strictly non-blocking designs.



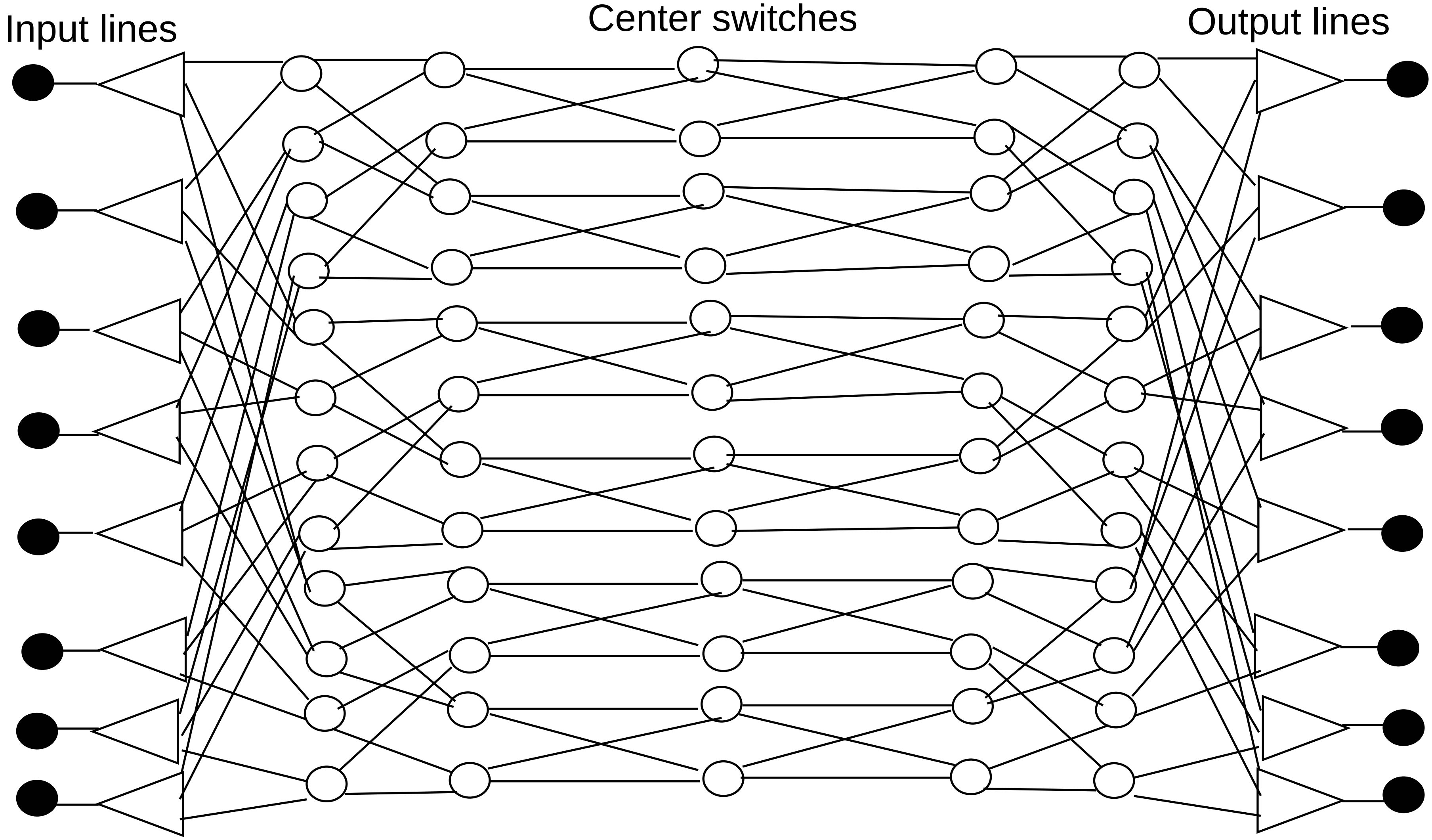
Maybe. If goal is rearrangeably non-blocking.

k-arm Roadms

Is there a systematic way to design a network of small roadms to form a large (k>4) strictly non-blocking crossconnect?

Yes .....

8x8 (Directed) Cantor Network



Cantor Network

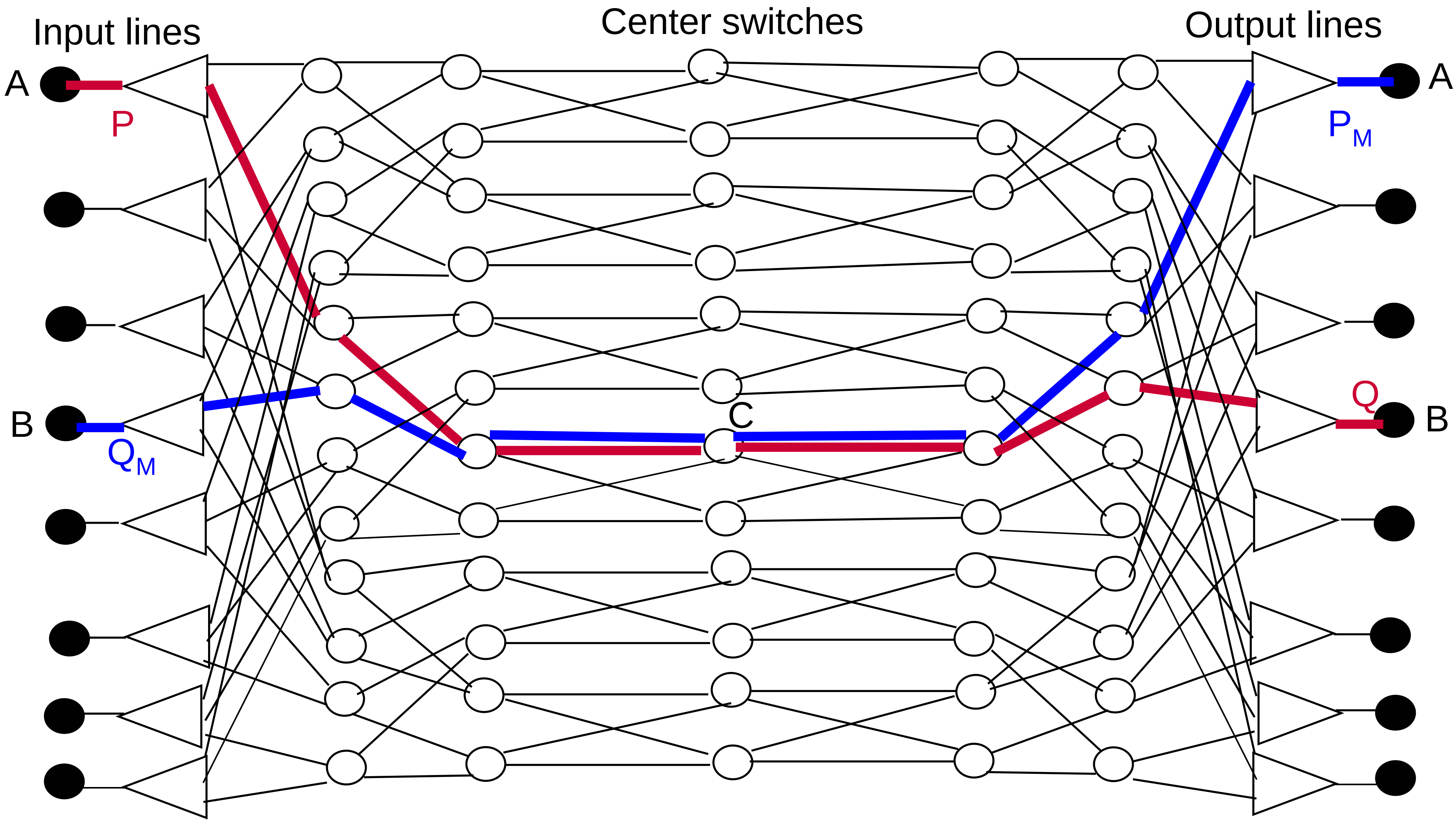
Demand: connect A and B in undirected network
Request directed demands A_{In} to B_{Out} and
B_{In} to A_{Out} in Cantor network

There exists a center switch C where:

(1) Path P from A_{In} to C is edge disjoint from other connections
(2) Path Q from C to B_{Out} is edge disjoint from other connections
(3) Path P_M from C to A_{Out} is edge disjoint from other connections
(3) Path Q_M from B_{In} to C is edge disjoint from other connections

Note that P and Q_M (or Q and P_M) might not be edge disjoint.

8x8 Cantor Network



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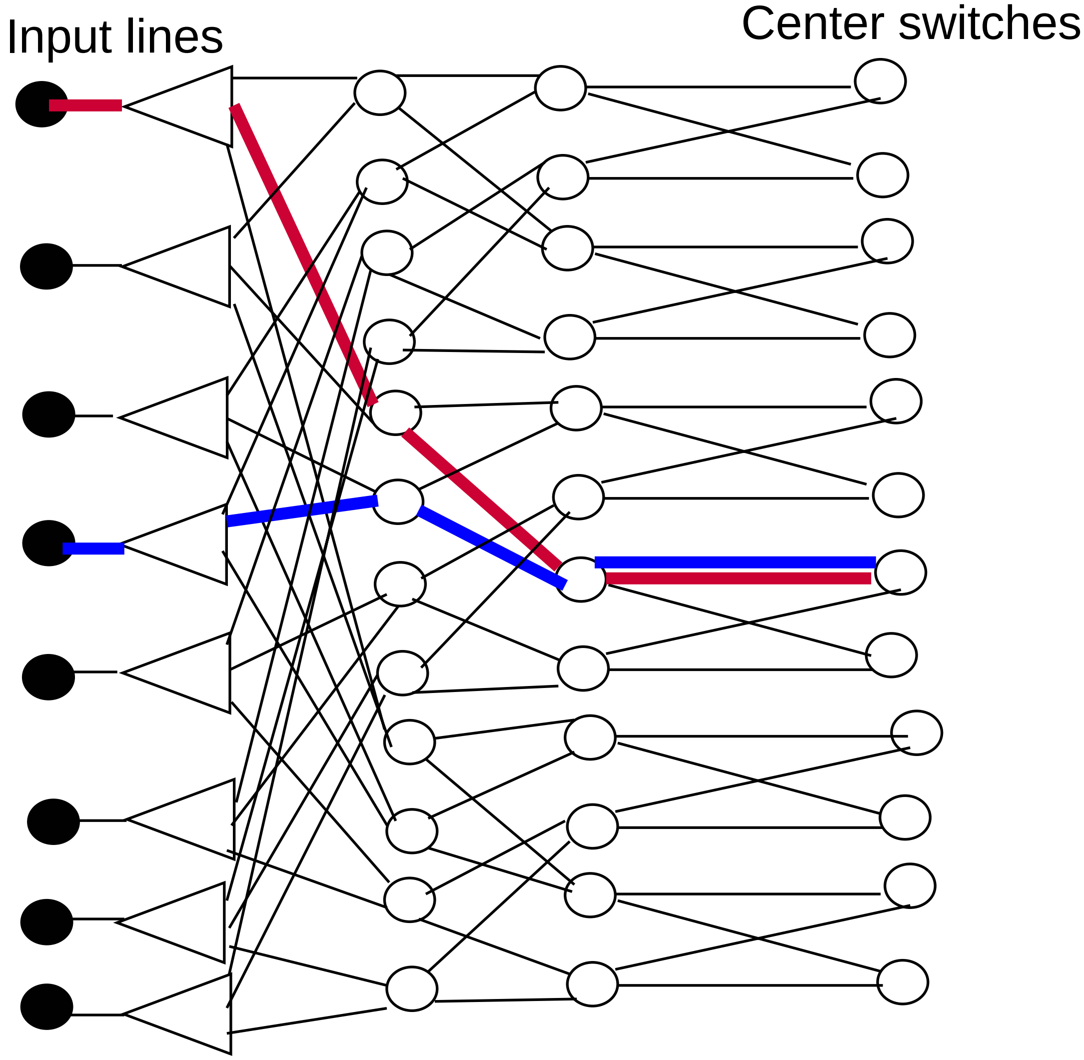
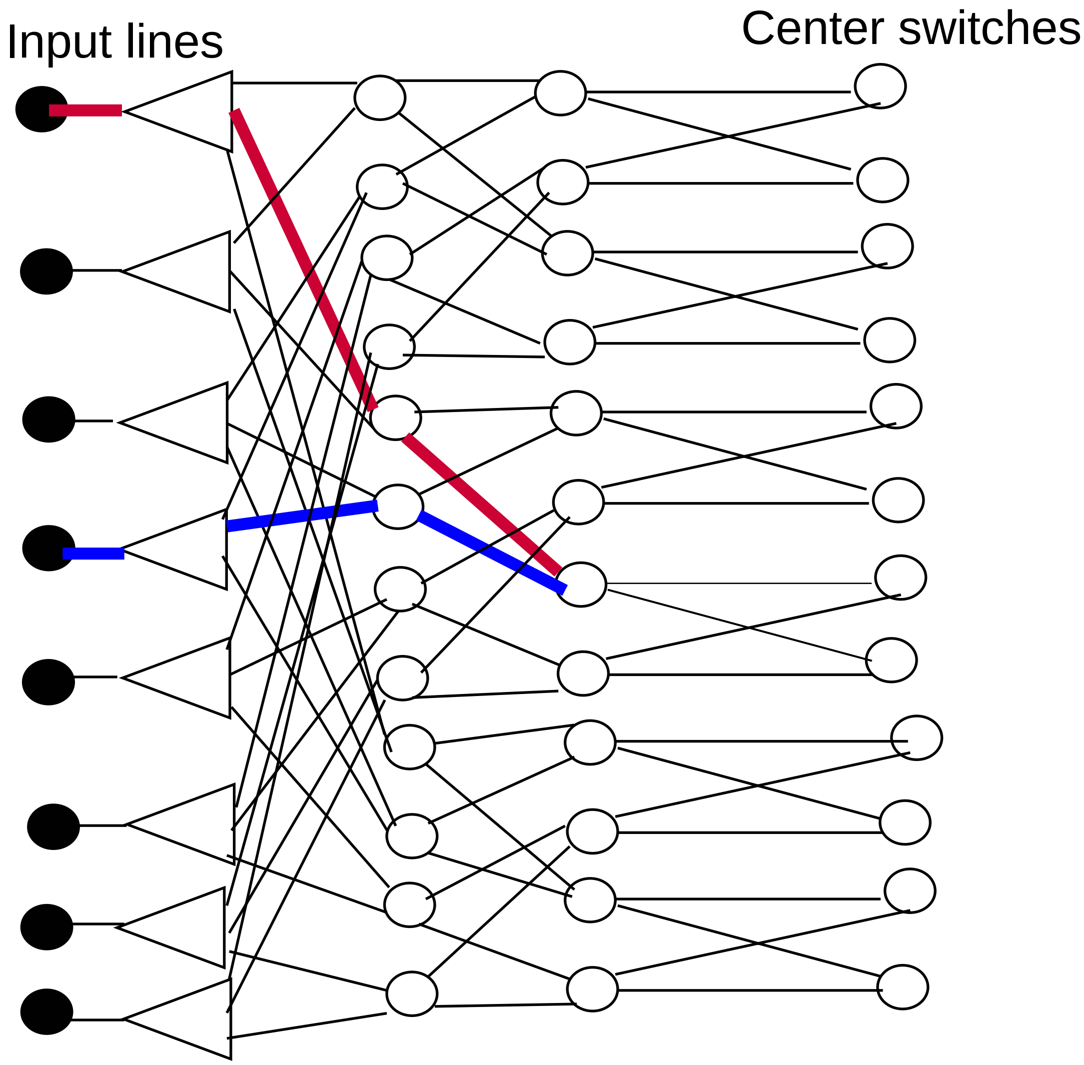
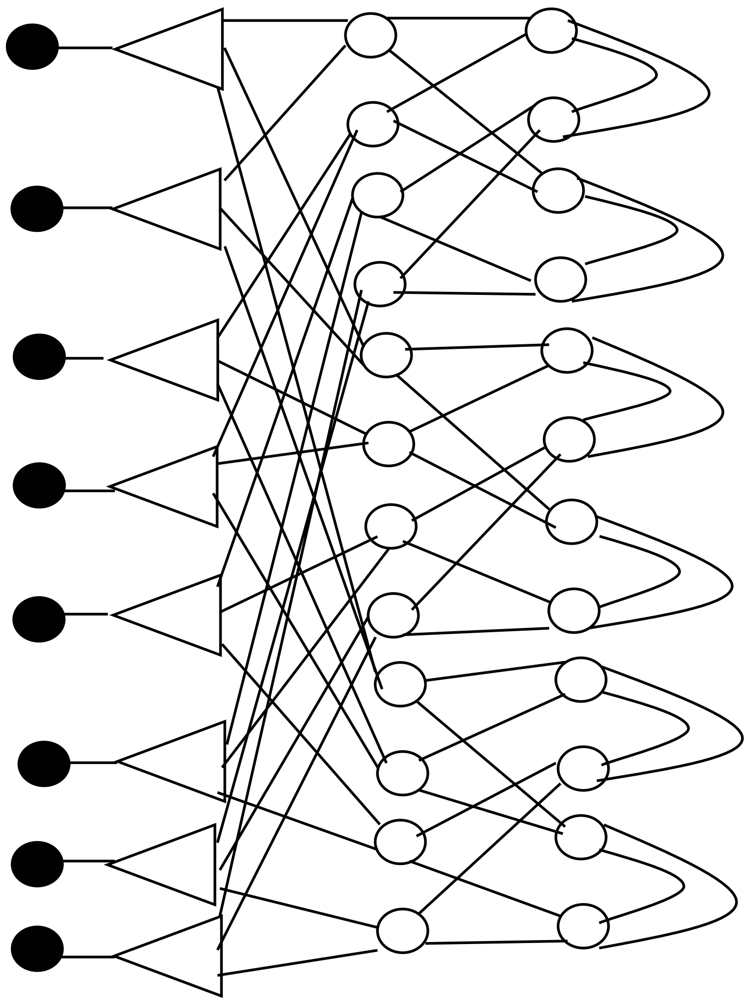
8x8 Cantor Network



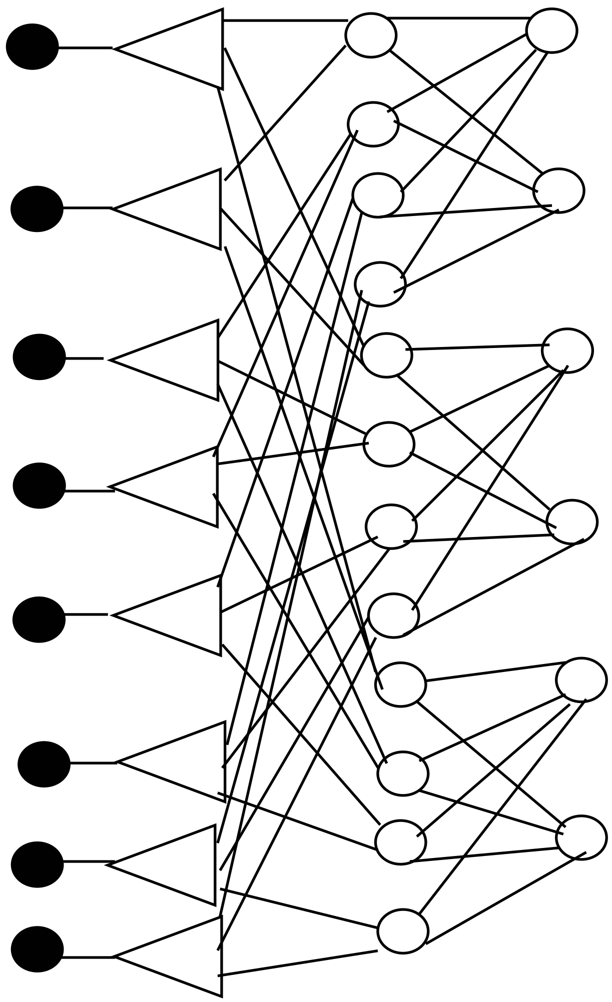
8x8 Cantor Network



Simplified 8x8 Undirected Cantor Network



Simplified Simplified 8x8 Undirected Cantor Network



Optimal Undirected (Symmetric) Networks

Cantor design uses roughly $k \log^2 k$ switches.

Is this optimal?

Probably not.

Widesense Non-blocking WDM Cross-connects
Or
Dynamic Edge Coloring of Bipartite Multigraphs

Joint work with:

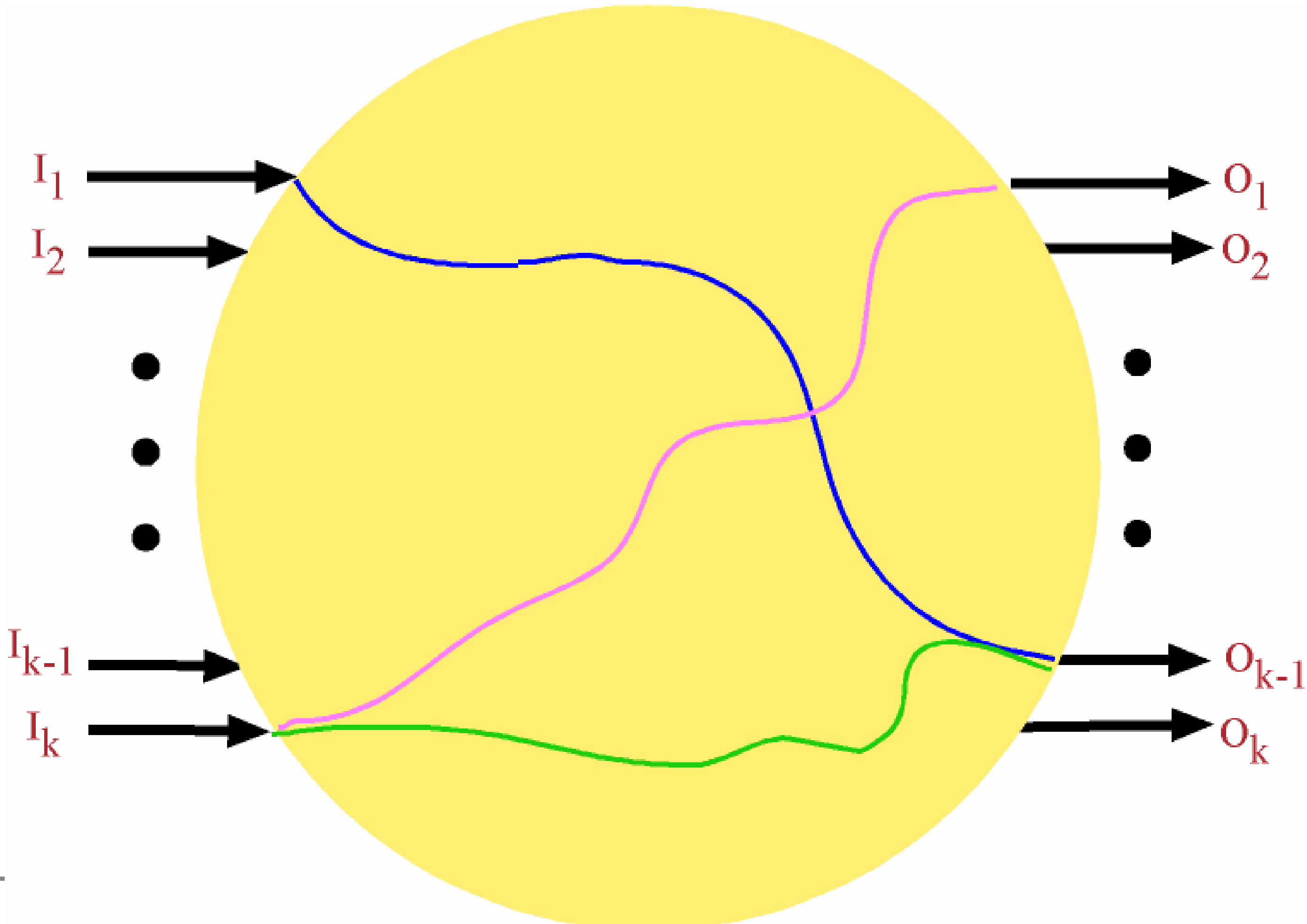
Penny Haxell (U. Waterloo), April Rasala (Google), Peter Winkler (Dartmouth)

WDM: (Wavelength Division Multiplexed)

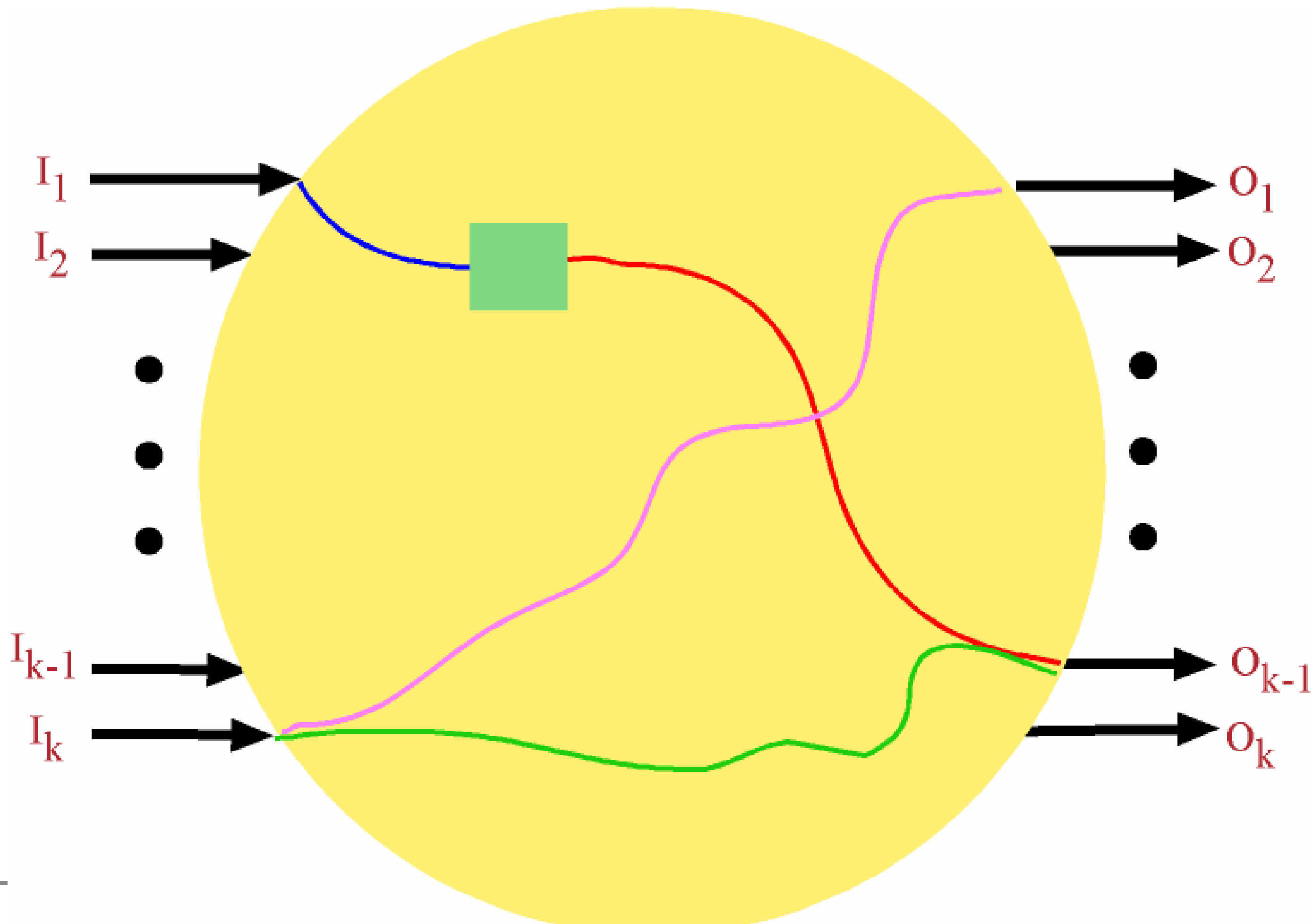
A WDM optical fiber carries multiple signals simultaneously with each signal on a distinct wavelength.

WDM fiber with 4 wavelengths.

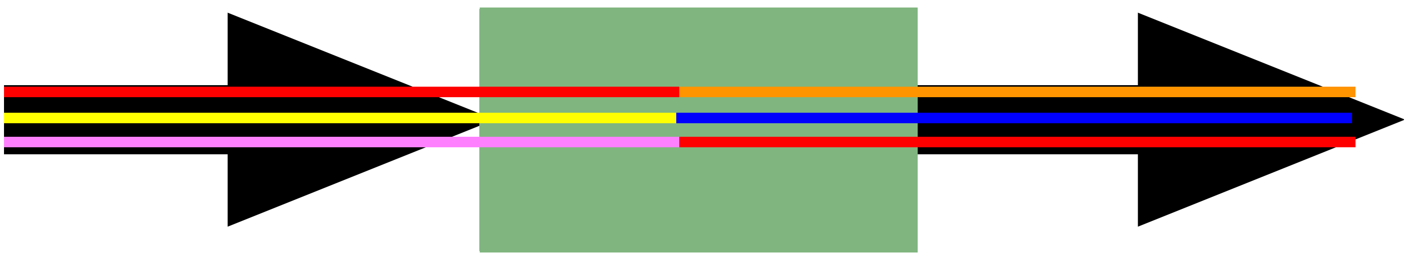
WDM Cross-connects



Wavelength Interchanging WDM Cross-connects



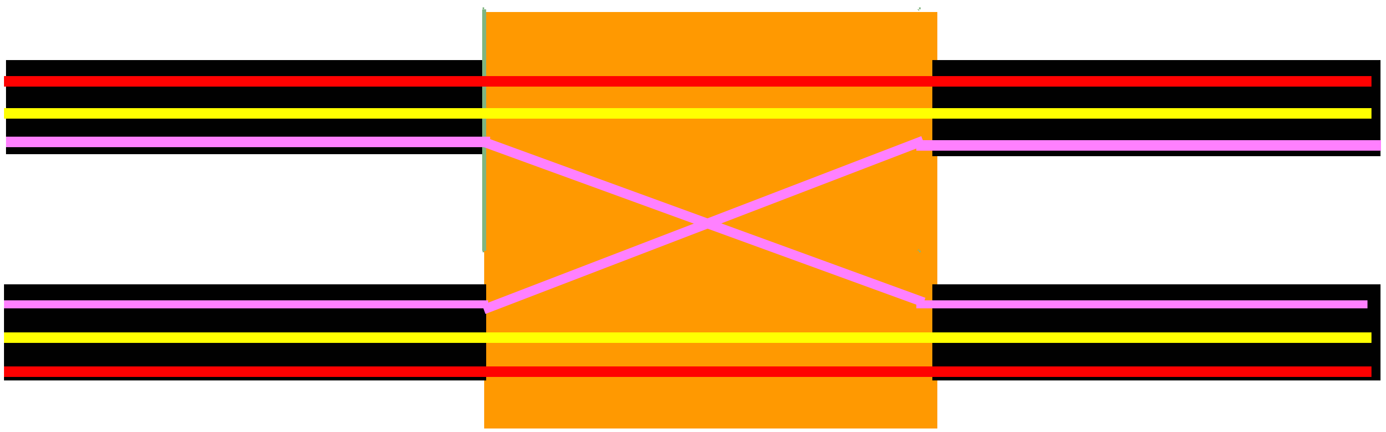
Wavelength Interchangers (WIs)



One input fiber and one output fiber.

Can move any set of signals on any subset of the *n* incoming wavelengths onto any set of distinct outgoing wavelengths.

Wavelength Selective Switch



two input fibers and two output fibers

can switch input signals to either output fiber provided no two signals on the same wavelength end up on same output fiber

Traditional Cross-connects

Benès 1935, Shannon 1950, Clos 1953, Pippenger 1982, etc.

- traditional (i.e. non-WDM) cross-connects
- goal was to minimize number of switches

WDM Optimization

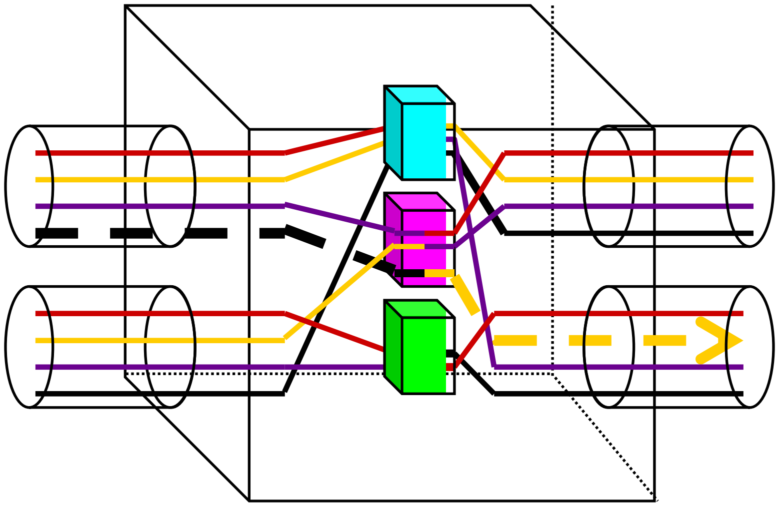
Wavelength interchangers are much more expensive than switches

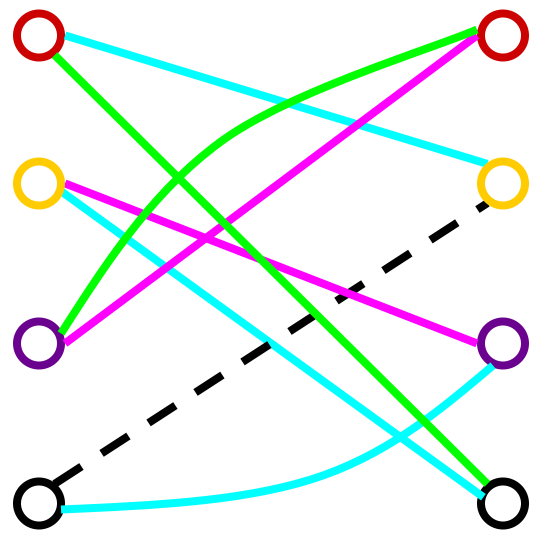
- goal is to minimize number of WIs
- can assume that switches are “free”

Known Results (for Δ input and output fibers)

	# wavelength interchangers	
	Necessary	Sufficient
Strictly nonblocking	2Δ-1	2Δ-1 [RW]
Wide-sense nonblocking	$2\Delta-1$	2Δ-1 [HRWW]
Rearrangeably nonblocking	Δ	Δ **[DMWZ]**

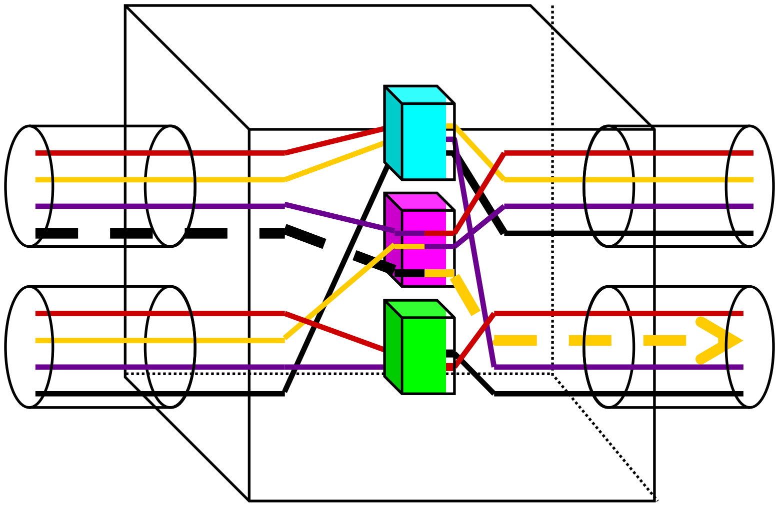
Dynamic Edge Coloring

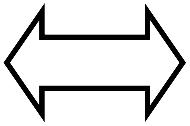


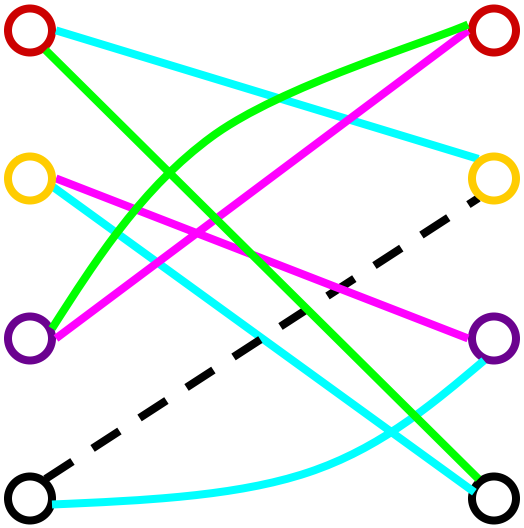


n=Input/output wavelengths	n=Left/Right nodes
# of fibers: k=Δ	Maximum degree: Δ
Demands are added/removed	Edges are added/removed
Wavelength Interchanger	Edge Color

Dynamic Edge Coloring







Minimize: # of wavelength interchangers.

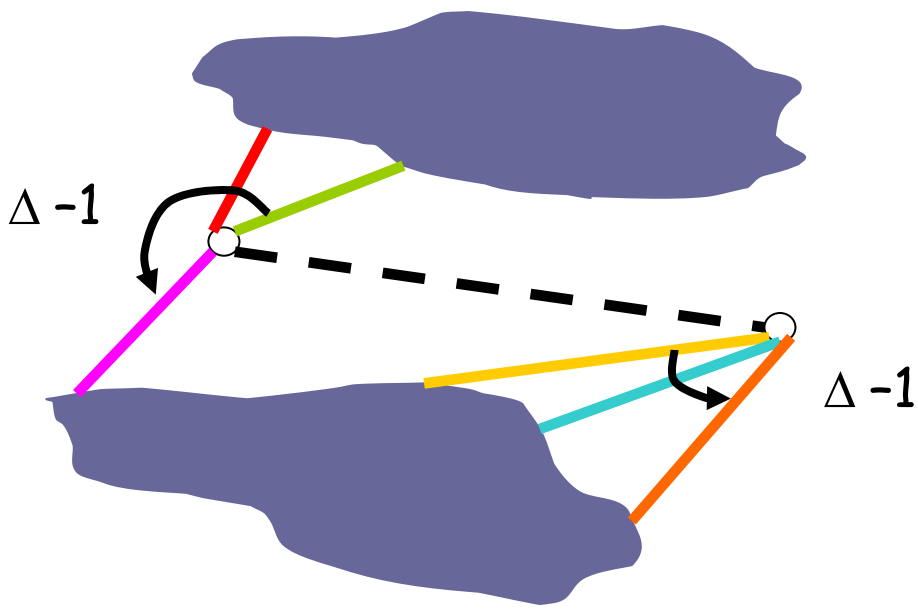
Subject to the existence of an algorithm that can always route future demands.

Minimize: # of colors.

Subject to existence of an algorithm that can always legally color future edges.

2Δ-1 colors are sufficient

- Worst case: new edge needs new color.
- Maximum of 2Δ – 2 previously used colors.
- Need at most 2Δ – 1 colors.



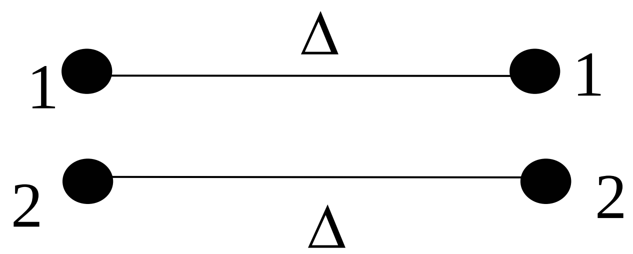
Are 2Δ-1 colors necessary?

- Sometimes not!
  - Very small graphs: There exist dynamic edge-coloring algorithms that use fewer than 2Δ-1 colors.
- Sometimes!
  - For every on-line edge-coloring algorithm, there exist graphs with $\Theta(2^{\Delta})$ nodes that require 2Δ-1 colors. [Bar-Noy, Motwani, Naor]
  - For every dynamic edge-coloring algorithm, there exist graphs with $\Theta(\Delta^2)$ nodes that require 2Δ-1 colors.

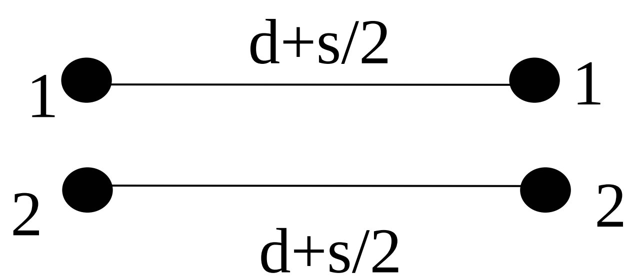
Small Number of Wavelengths (Nodes)

n	**Upper bound**	**Lower bound**
2	$3\Delta/2$	$3\Delta/2$
3	$15\Delta/8$	$7\Delta/4$

Lower Bound, n=2

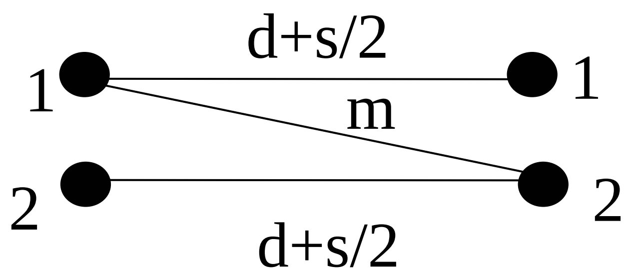


d = no. colors appearing once (say on 11 edges)
s = no. colors appearing twice (once on both edge types)
Then d+s= Δ.
Delete half the edges from each set with same color so that now all colors used only once.



Lower Bound, n=2

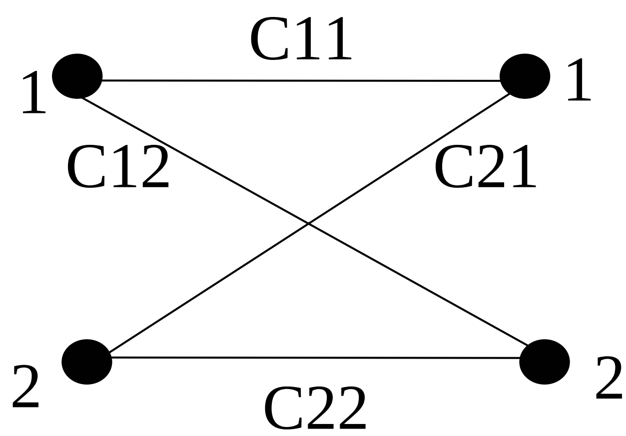
Add $m = \Delta - (d + s/2)$ 12 edges



No. colors used is

$$
\begin{aligned}
m+2d+s &= \Delta + d + s/2 \\
&= \Delta + (\Delta - s) + s/2 \text{ (since } d+s = \Delta) \\
&= 2\Delta - s/2 \\
&\geq 3\Delta/2 \text{ (since } s \leq \Delta)
\end{aligned}
$$

Upper bound, n=2



Cij=set of colors used to color edges ij

Invariants:

$$(1)\ |C11 \cup C22| \le \Delta$$

$$(2)\ |C12 \cup C21| \le \Delta$$

$$(3)\ |C11 \cup C22 \cup C12 \cup C21| \le 3\Delta/2$$

Invariants:

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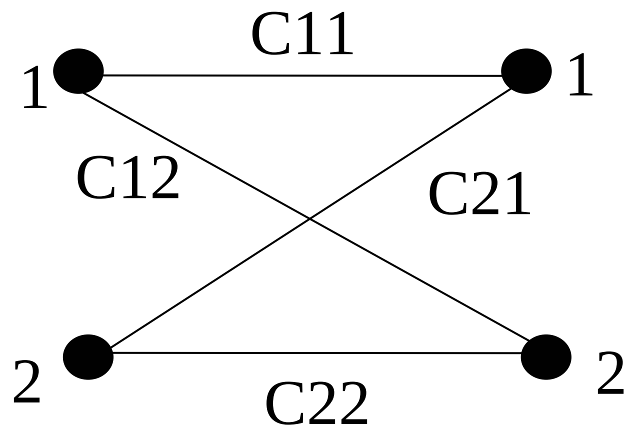


Without loss of generality, new edge is 11:

(1) $\exists$ c$\in$C22, c$\notin$C11

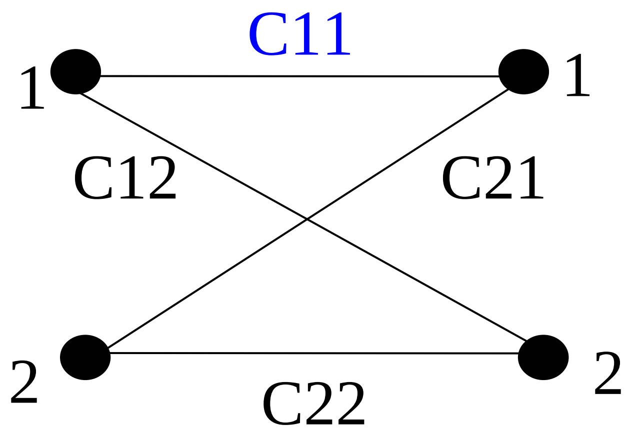
Then color new edge with color c

(2) $\forall$c $\in$ C22, c $\in$C11

$\Rightarrow$ C22 $\subseteq$ C11

Color with new color c.

$C11 = C11 \cup \{c\}$



(1) $|C11 \cup C21| \leq \Delta$ (max degree)

(2) $|C11| + |C12| \leq \Delta$ (max degree)

(3) $|C12 \cup C21| \leq \Delta$ (invariant)

(4) $|C11 \cup C22| = |C11| \leq \Delta$ (max degree)

$2\,|C11 \cup C22 \cup C12 \cup C21|$

$= 2\,|C11 \cup C12 \cup C21|$

$\leq |C11| + |C12 \cup C21| + |C12| + |C11 \cup C21| \leq 3\Delta$

$|C11 \cup C22 \cup C21 \cup C12| \leq 3\Delta/2$

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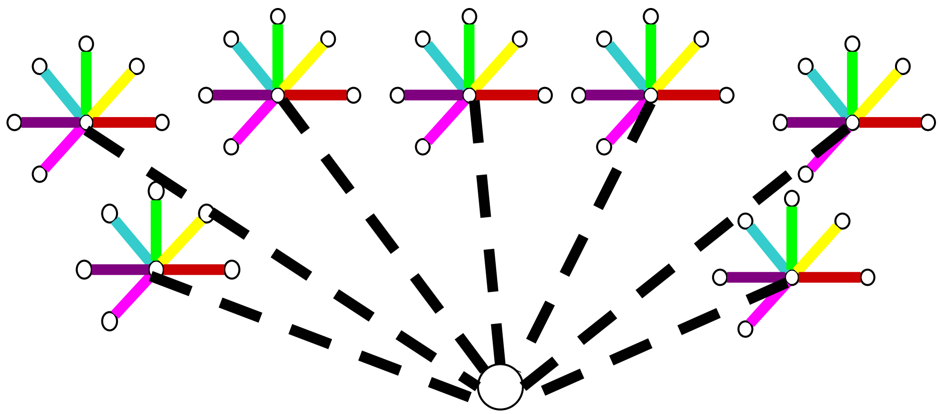
Lower Bounds:

Create $\Delta \binom{m}{\Delta-1}$ "star"-nodes with Δ-1 edges each,

$\Delta \leq m < 2\Delta - 1$.

Must exist Δ "stars" with the same Δ-1 colors.



- Add edge from each such star to a new node.
- Stars used the same Δ -1 colors. New edges must use Δ new colors.
- 2 Δ -1 colors necessary.

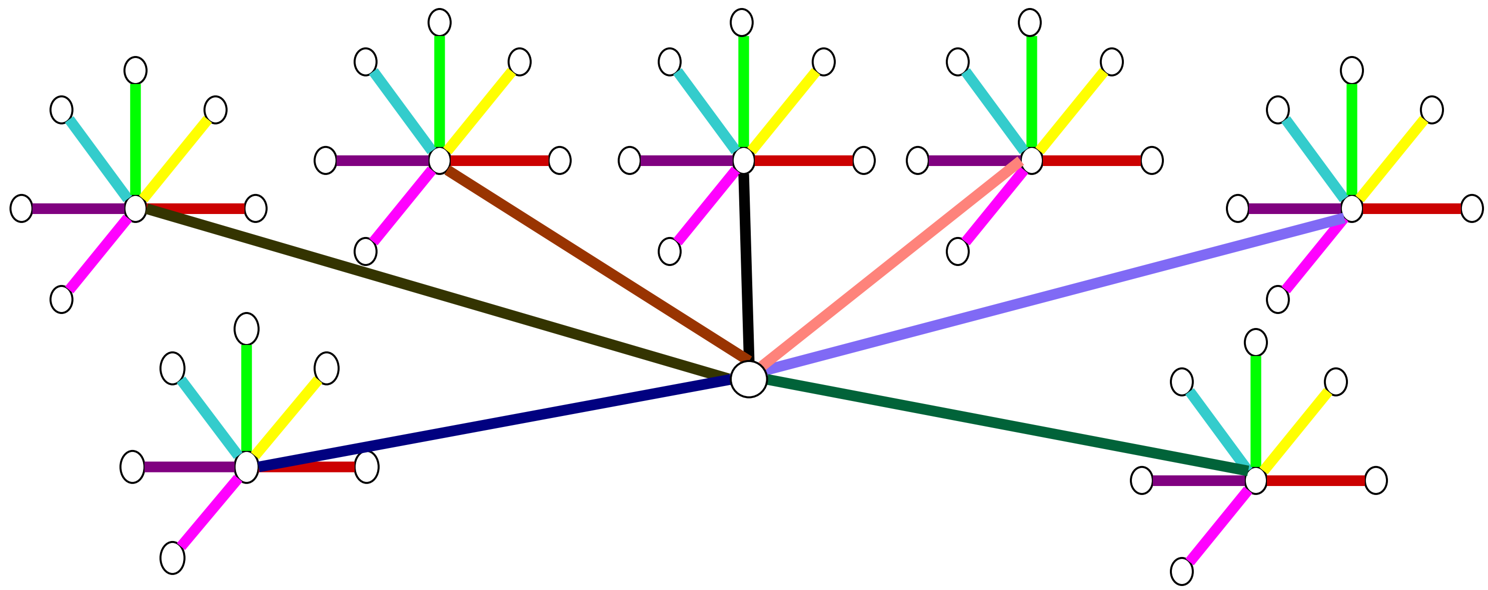
Question?

Lower bound required graph with exponentially many nodes.

- Doesn't use power of dynamic graph.
- Implies only that cross-connects with exponentially many wavelengths require 2Δ-1 wavelength interchangers.

Are 2Δ-1 colors necessary to dynamically edge-color graphs with $|V| = o(2^{\Delta})$?

Lower Bound: $|V| = \Omega(\Delta^2)$



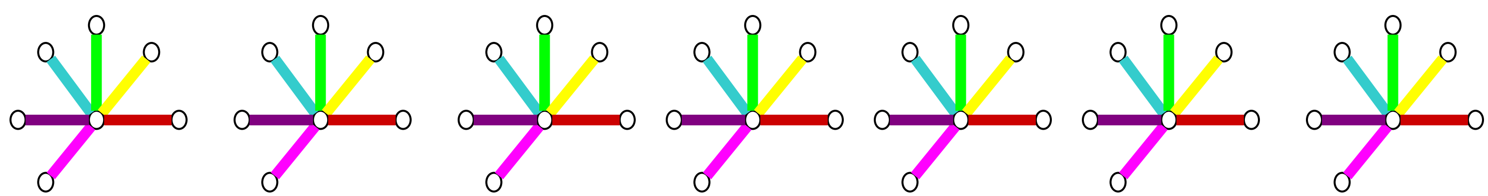
Goal: Reach this step with only $O(\Delta^2)$ nodes.

Lower Bound: $|V| = \Omega(\Delta^2)$

- Assume (for contradiction): 2Δ-2 colors.
- Partition into two disjoint sets of Δ-1 colors each.

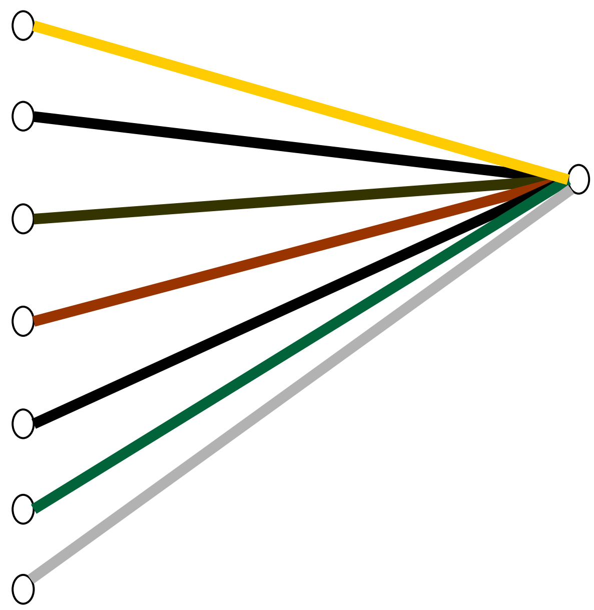
Light Colors Dark Colors

- Goal: Create Δ light stars.



Creating light stars (idea)

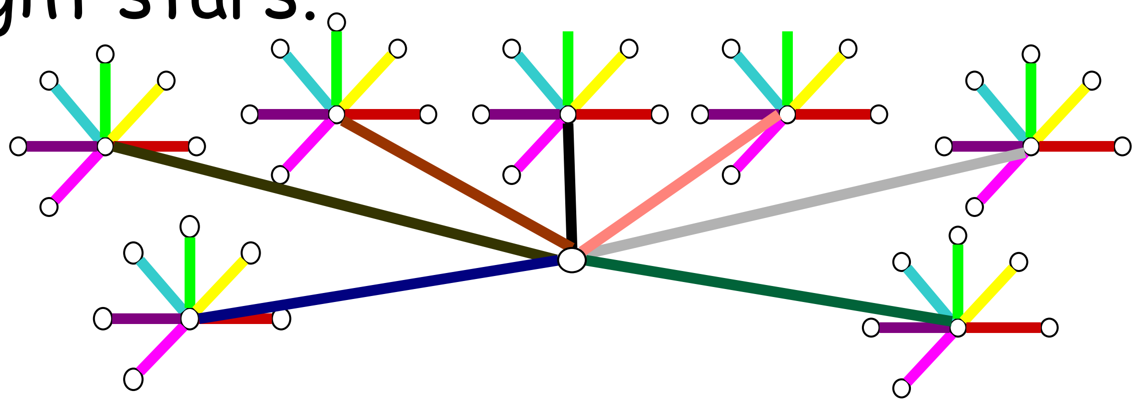
- Bipartite graph with 2Δ nodes on left.
- Repeat until there are Δ light stars on left.
    - Add a new node on right with Δ edges.
    - Keep all light edges.



- Only Δ-1 dark colors
    - Fact: node with Δ edges must have at least one light edge.

Lower Bound: $|V| = \Omega(\Delta^2)$

- Each new node on right adds at least one light edge to one of the 2Δ nodes on the left.
- Using only $\Delta(\Delta-1)$ nodes on the right and 2Δ nodes on the left, we can produce one light star.
- In fact, using only $O(\Delta^2)$ nodes we can produce Δ light stars.



- Thus $\Delta-1 + \Delta = 2\Delta-1$ colors are necessary.

General Lower Bounds

Theorem: Any widesense non-blocking $\Delta x \Delta$ WDM cross-connect with $n=(1/4+o(1))\Delta^2$ wavelengths requires $2\Delta-1$ wavelength interchangers.

Theorem: For any wavelength interchanger assignment algorithm and for any $\varepsilon>0$ and $\Delta>1/2\varepsilon$, there is a widesense non-blocking $\Delta x \Delta$ WDM cross-connect with fewer than $1/\varepsilon^2$ wavelengths that requires more than $2(1-\varepsilon)\Delta$ wavelength interchangers.

Summary

2Δ-1 colors are necessary and sufficient to dynamically edge color every bipartite multi-graph with $\Omega(\Delta^2)$ nodes.

Wide-sense non-blocking WDM cross-connects with $\Omega(\Delta^2)$ wavelengths and Δ input/output fibers must have 2Δ-1 wavelength interchangers.

Strictly non-blocking cross-connects are optimal for this situation.

Open Question

Recall that small graphs (i.e. $|V| = 4,6$) can be dynamically edge-colored with fewer than 2Δ-1 colors.

How many colors are necessary and sufficient to dynamically edge-color graphs with $|V| = o(\Delta^2)$?

Multicasting/multiplexing cross-connects give rise to related hyperedge colorings of bipartite multi-hyper graphs