On Weighted Graphs Yielding Facets of the Linear Ordering Polytope

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Definition

For any finite set Z,

- for $R \subseteq Z \times Z$, the vector x^R is the characteristic vector of R, that is,

$$x^R_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in R \\ 0 & \text{otherwise} \end{cases}$$

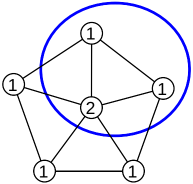
- the linear ordering polytope $P^Z_{LO} \subset \mathbb{R}^{Z \times Z}$ is

$$P^Z_{LO} = \text{conv}\{x^L : L \text{ linear order on } Z\}$$

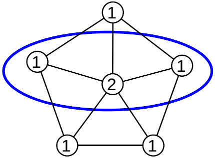
Definition

For a vertex-weighted graph (G, μ) and $S \subseteq V(G)$,

- $\mu(S) := \sum_{v \in S} \mu(v)$ (weight of S)
- $\mathrm{w}(S) := \mu(S) - |E(G[S])|$ (worth of S)
- $\alpha(G, \mu) := \max_{S \subseteq V(G)} \mathrm{w}(S)$
- S is tight if $\mathrm{w}(S) = \alpha(G, \mu)$



- weight $= 4$
- worth $= 1$



- weight $= 4$
- worth $= 2$
- tight

Suppose

- (G, μ) is any weighted graph
- Y is a set s.t. $|Y| = |V(G)|$ and $Y \cap V(G) = \emptyset$
- $f : V(G) \to Y$ is a bijection
- Z is a finite set s.t. $V(G) \cup Y \subseteq Z$

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Definition

- The graphical inequality of (G, μ), which is valid for P_{LO}^{Z}, is

$$\sum_{v \in V(G)} \mu(v) \cdot x_{v,f(v)} - \sum_{\{v,w\} \in E(G)} (x_{v,f(w)} + x_{f(v),w}) \leq \alpha(G, \mu)$$

- (G, μ) is facet-defining if its graphical inequality defines a facet of P_{LO}^{Z}

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- (G, μ) is facet-defining if its graphical inequality defines a facet of P_{LO}^Z

N.B. (G, μ) being facet-defining is a property of the graph solely, i.e. it is independent of the particular choice of Y, f and Z

A characterization of facet-defining graphs

Definition

- For any *tight set* T of (G, μ), a corresponding affine equation is defined:
$$\sum_{v \in T} y_v + \sum_{e \in E(T)} y_e = \alpha(G, \mu)$$
- The system of (G, μ) is obtained by putting all these equations together

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Theorem (Christophe, Doignon and Fiorini, 2004)

(G, μ) is facet-defining $\Leftrightarrow$ the system of (G, μ) has a unique solution

- Basically rephrases the fact that the dimension of the face of P^Z_{LO} defined by the graphical inequality must be high enough
- We lack a 'good characterization' of these graphs...

A few results

(assuming from now on that all graphs have at least 3 vertices)

Definition

G is stability critical if G has no isolated vertex and $\alpha(G \setminus e) > \alpha(G)$ for all $e \in E(G)$

Theorem (Koppen, 1995)

$(G, \mathbb{1})$ is facet-defining $\Leftrightarrow$ G is connected and stability critical

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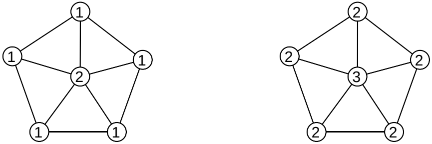
$(G, \mathbb{1})$ *is facet-defining* $\Leftrightarrow$ G *is connected and stability critical*

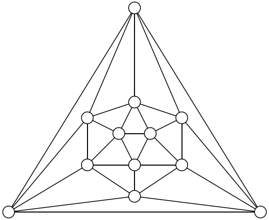
Theorem (Christophe, Doignon and Fiorini, 2004)

(G, μ) *is facet-defining* $\Leftrightarrow$ *its 'mirror image'* $(G, \deg - \mu)$ *is facet-defining*



Definition

- The defect of G is $|V(G)| - 2\alpha(G)$



a stability critical graph

$|V(G)| = 12$
$\alpha(G) = 3$
$\longrightarrow$ defect $= 6$

Definition

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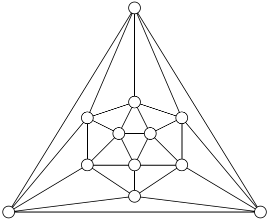
- The defect of (G, μ) is $\mu(V(G)) - 2\alpha(G, \mu)$



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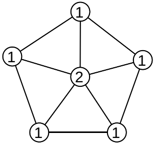

a facet-defining graph

$\mu(V(G)) = 7$
$\alpha(G, \mu) = 2$
$\longrightarrow$ defect $= 3$

Theorem

- *The defect δ of a connected stability critical graph G is always positive* (Erdős and Gallai, 1961)
- *Moreover,* $\delta \geq \deg(v) - 1$ *for all* $v \in V(G)$ (Hajnal, 1965)

Theorem

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Theorem (Doignon, Fiorini, J.)

- *The defect δ of any facet-defining graph (G, μ) is positive*
- (G, μ) *and* $(G, \deg - \mu)$ *have the same defect*
- *For all $v \in V(G)$, we have*

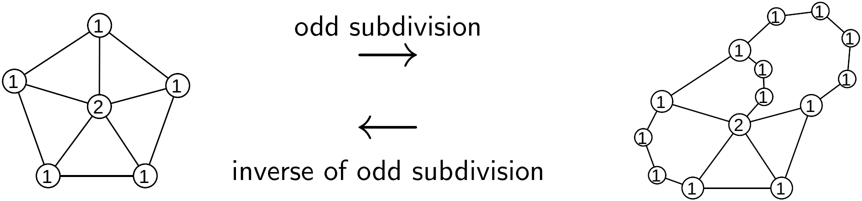
$$\delta \geq \deg(v) - \mu(v) \geq 1$$

and, because of the mirror image, also

$$\delta \geq \mu(v) \geq 1$$

Odd subdivision

Here is an extension of a classical operation on stability-critical graphs:

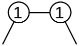


Theorem (Christophe, Doignon and Fiorini, 2004)

The odd subdivision operation and its inverse keep both a graph facet-defining. Moreover, the defect does not change

Lemma

An inclusionwise minimal cutset of a facet-defining graph cannot span "○" or "○—○"

Thus when we have 

we can always contract both edges by using the inverse of odd subdivision operation

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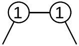
Thus when we have 

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Definition

A facet-defining graph is minimal if no two adjacent vertices have degree 2

Classification of stability critical graphs

Theorem (Lovász, 1978)

For every positive integer δ, the set $\mathcal{S}_\delta$ of minimal connected stability critical graphs with defect δ is finite

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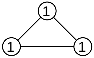
Research problem

Is there a finite number of minimal facet-defining graphs with defect δ, for every $\delta \geq 1$?

- It turns out to be true for $\delta \leq 3$
  $\longrightarrow$ an overview of the proofs is given in the next few slides
- The problem is wide open for $\delta \geq 4$

Notice first that the only minimal facet-defining graph with defect $\delta = 1$ is 

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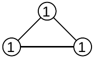
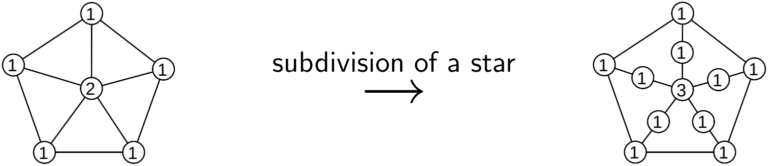
Notice first that the only minimal facet-defining graph with defect $\delta = 1$ is , because $\delta \geq \mu(v) \geq 1$

Let's look at another operation:



Theorem

The subdivision of a star operation keeps a graph facet-defining. Moreover, the defect does not change

Definition

(G_1, μ_1) and (G_2, μ_2) are equivalent if one can be obtained from the other by using the

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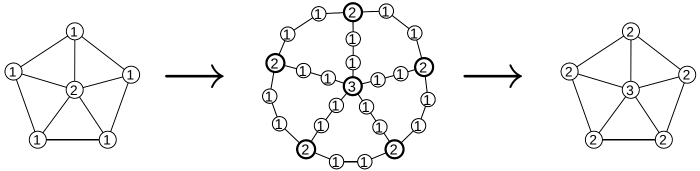
Notice

- two equivalent graphs have the same defect
- (G, μ) and $(G, \deg - \mu)$ are equivalent:



Facet-defining graphs with defect 2

Recall

$$\begin{cases} \delta \geq \mu(v) \geq 1 \\ \delta \geq \deg(v) - \mu(v) \geq 1 \end{cases}$$

for any vertex v of a facet-defining graph with defect δ

$\Rightarrow \deg(v) \leq 2\delta$

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$\deg(v) \leq 2\delta - 1$ *for any vertex* v *of a facet-defining graph with defect* $\delta \geq 2$

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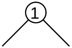
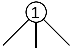
$\deg(v) \leq 2\delta - 1$ *for any vertex* v *of a facet-defining graph with defect* $\delta \geq 2$

Thus, every vertex of a facet-defining graph with defect 2 is either



or


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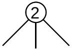
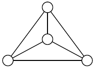


$\Rightarrow$ Any facet-defining graph with defect 2 is equivalent to some stability critical graph

Theorem (Andrásfai, 1967)

The only minimal connected stability critical graph with defect 2 is



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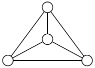
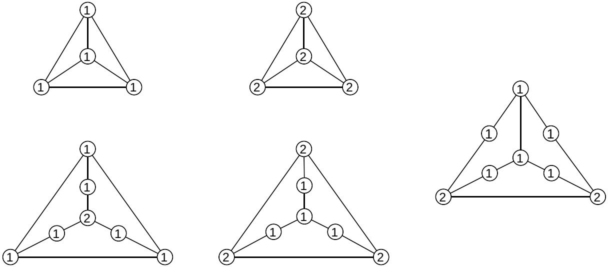


→ we derive:

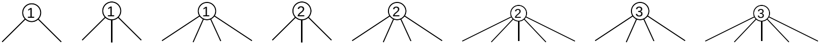
Theorem

There are exactly five minimal facet-defining graphs with defect 2:



Facet-defining graphs with defect 3

By previous bounds, any vertex falls in one of these cases when $\delta = 3$:



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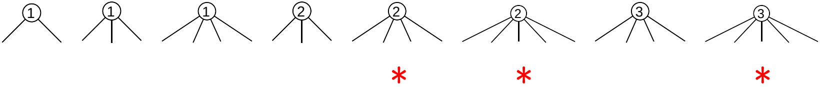

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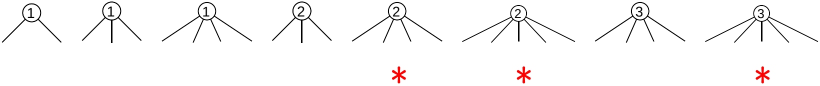

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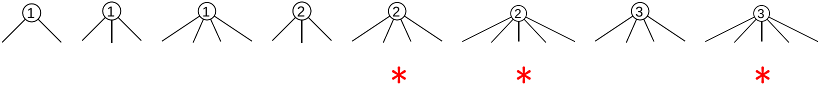

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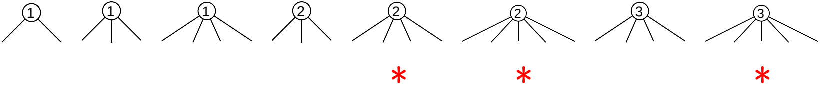

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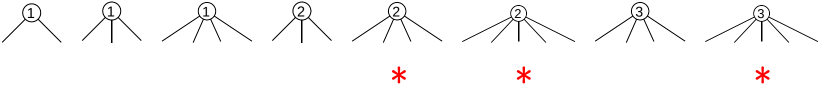

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- By the subdivision of a star operation, w.l.o.g. $\nexists$ $(2, 3)$-, $(3, 4)$-, or $(3, 5)$-vertices in (G, μ)

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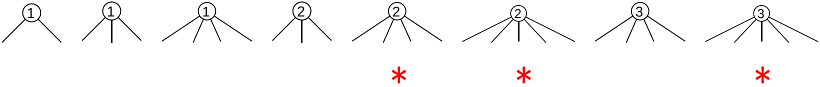

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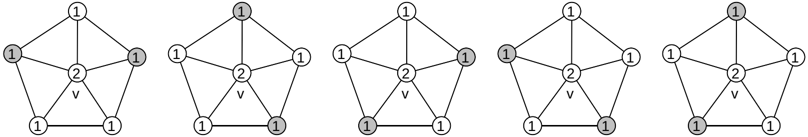
- By the subdivision of a star operation, w.l.o.g. $\nexists$ $(2, 3)$-, $(3, 4)$-, or $(3, 5)$-vertices in (G, μ)
- Main issue: how to get rid of the $(2, 4)$-vertices and $(2, 5)$-vertices?

Suppose v is a $(2,4)$- or $(2,5)$-vertex and look at those tight sets including exactly two neighbors of v but avoiding v:



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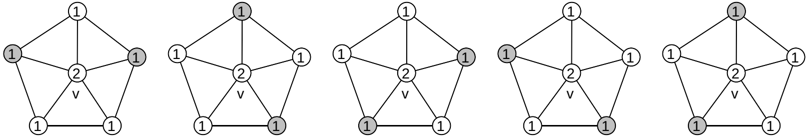
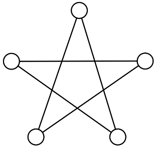
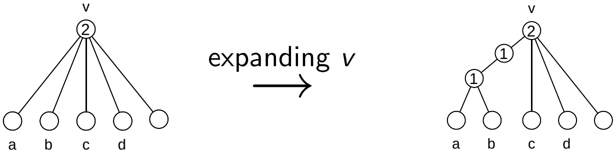


$\longrightarrow$ defines a graph on the neighborhood $N(v)$ of v, denoted H_v:



Expanding a vertex

Assume $\exists a, b, c, d \in V(H_v)$ s. t. $\{a, b\} \in E(H_v)$ and $\{c, d\} \notin E(H_v)$



Lemma

- *Expanding v keeps (G, μ) facet-defining and does not change the defect*
- *Any $(2, 5)$-vertex of (G, μ) is expandable*

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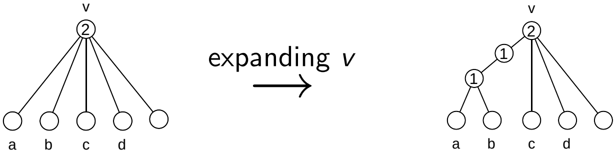

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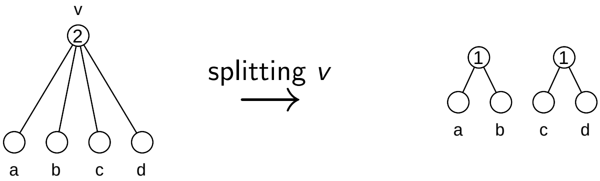
- *Expanding v keeps (G, μ) facet-defining and does not change the defect*
- *Any $(2, 5)$-vertex of (G, μ) is expandable*

$\rightarrow$ w.l.o.g. (G, μ) has no expandable vertices, as expanding a vertex increases the number of vertices with degree at least 3

Splitting a vertex

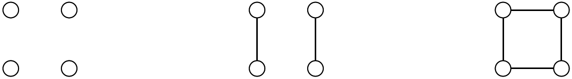
Suppose that v is a $(2,4)$-vertex and that $\{a,b\},\{c,d\} \notin E(H_v)$



Lemma

- *Splitting v keeps (G,μ) facet-defining and does not change the defect*
- *Every nonexpandable $(2,4)$-vertex is splittable*

Assume now that v is a nonexpandable $(2,4)$-vertex. As v is splittable, H_v is isomorphic to one of these 3 graphs:



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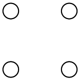
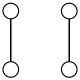
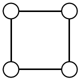




v is "thin"



v is "thick"

Lemma

- *v must be thin or thick, i.e. H_v cannot be isomorphic to the leftmost graph*
- *(G, μ) has at most 5 thick vertices*

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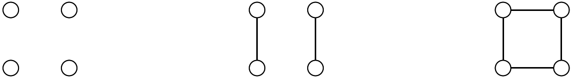

v is "thin" v is "thick"

Lemma

- *v must be thin or thick, i.e. H_v cannot be isomorphic to the leftmost graph*
- *(G,μ) has at most 5 thick vertices*

$\rightarrow$ it remains to show that (G,μ) has not too many thin vertices...

Key lemma

(G, μ) has at most $\frac{3}{2}N$ thin vertices, where N is the number of vertices with weight 1 and degree at least 3

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Thus we obtain:

Theorem

There is a finite number of minimal facet-defining graphs with defect 3

As a (brief) conclusion

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Thank you!