Mathematical Representations of Preference and Utility (& their role in Social Choice)

DIMACS Tutorial

Social Choice & Computer Science

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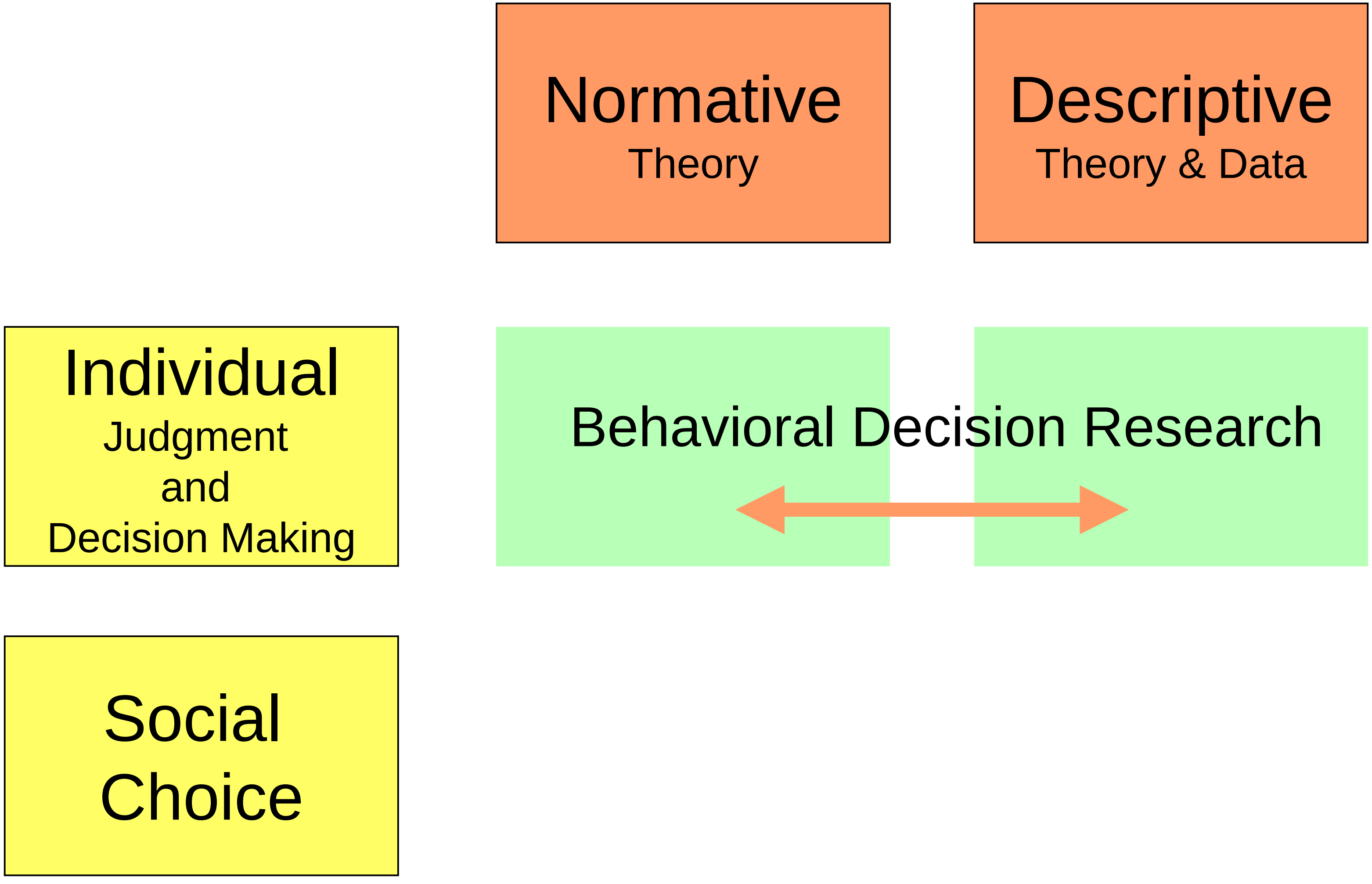
Multi-Year Interdisciplinary Effort

- Collaborators:
  Doignon, Falmagne,Grofman,
  Marley, Rykhlevskaia, Tsetlin
- Past NSF SBR 9730076, Duke B-School
- Past UIUC Research Board
- Book forthcoming with
  Cambridge University Press

2 Conceptual Distinctions in the Decision Sciences



2 Conceptual Distinctions in the Decision Sciences

Normative
Theory

Descriptive
Theory & Data

Individual
Judgment
and
Decision Making

Social Choice

???

???

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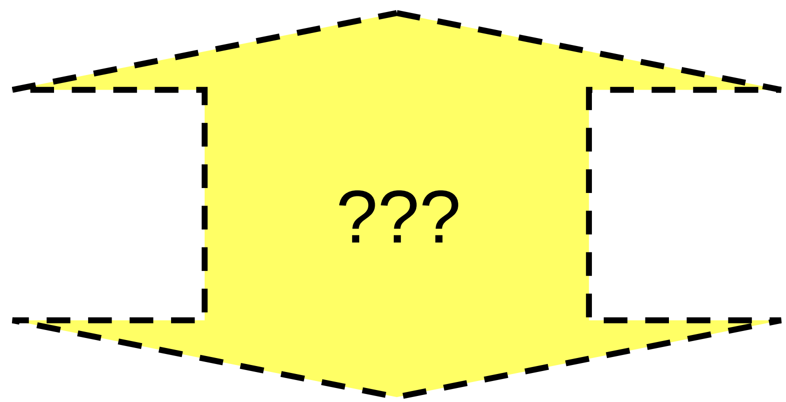
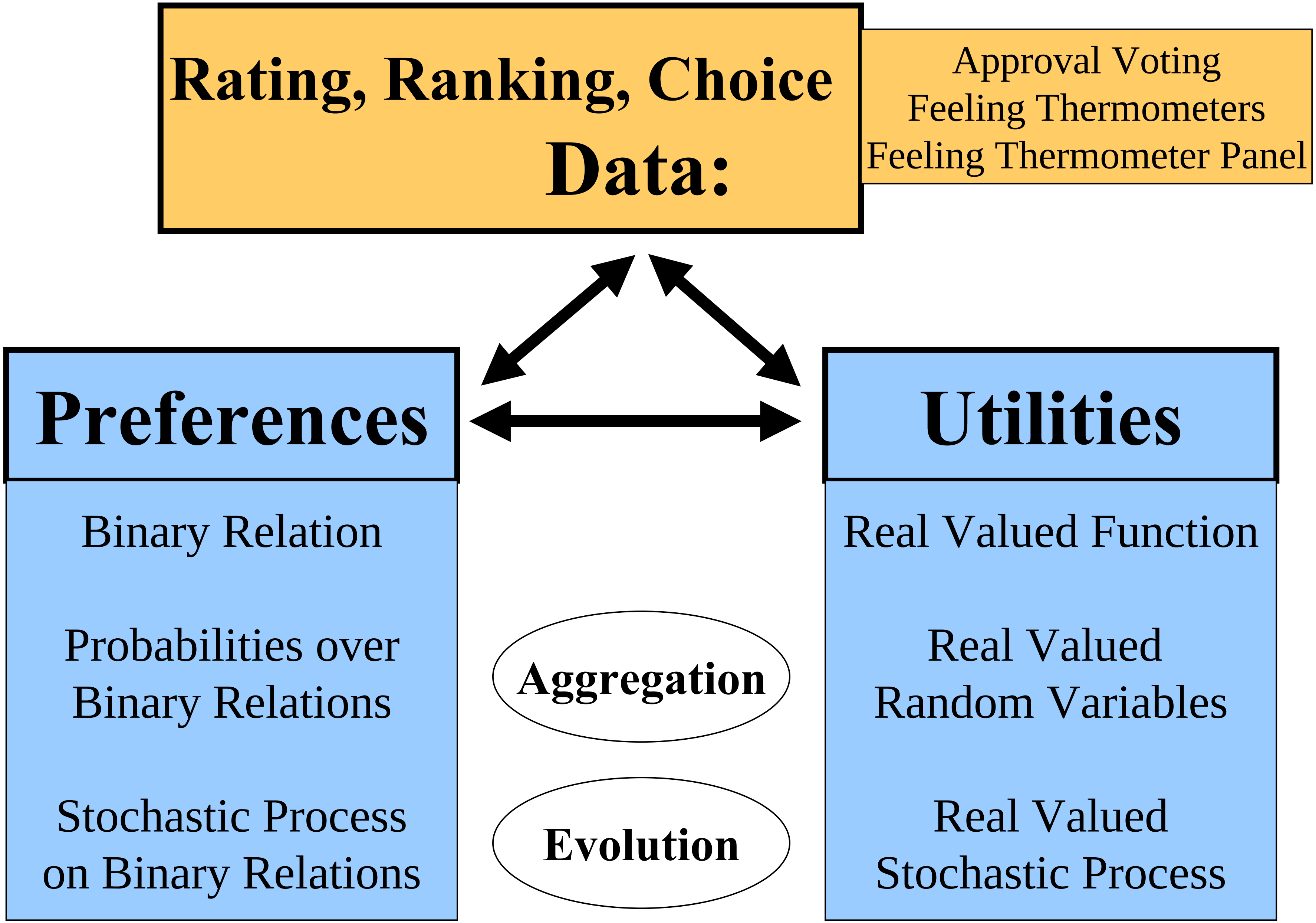


Criteria for a Unified Theory of Decision Making

(Inspired by Luce and Suppes, Handbook of Math Psych,1965)

- ✓ Treat individual & group decision making in a unified way
- ✓ Reconcile normative & descriptive work
- ✓ Integrate & compare competing normative benchmarks
- ✓ Reconcile theory & data
- ✓ Encompass & integrate multiple choice, rating and ranking paradigms
- ✓ Integrate & compare multiple representations of preference, utilities & choices
- ✓ Develop dynamic models as extensions of static models
- ▪ Systematically incorporate statistics as a scientific decision making apparatus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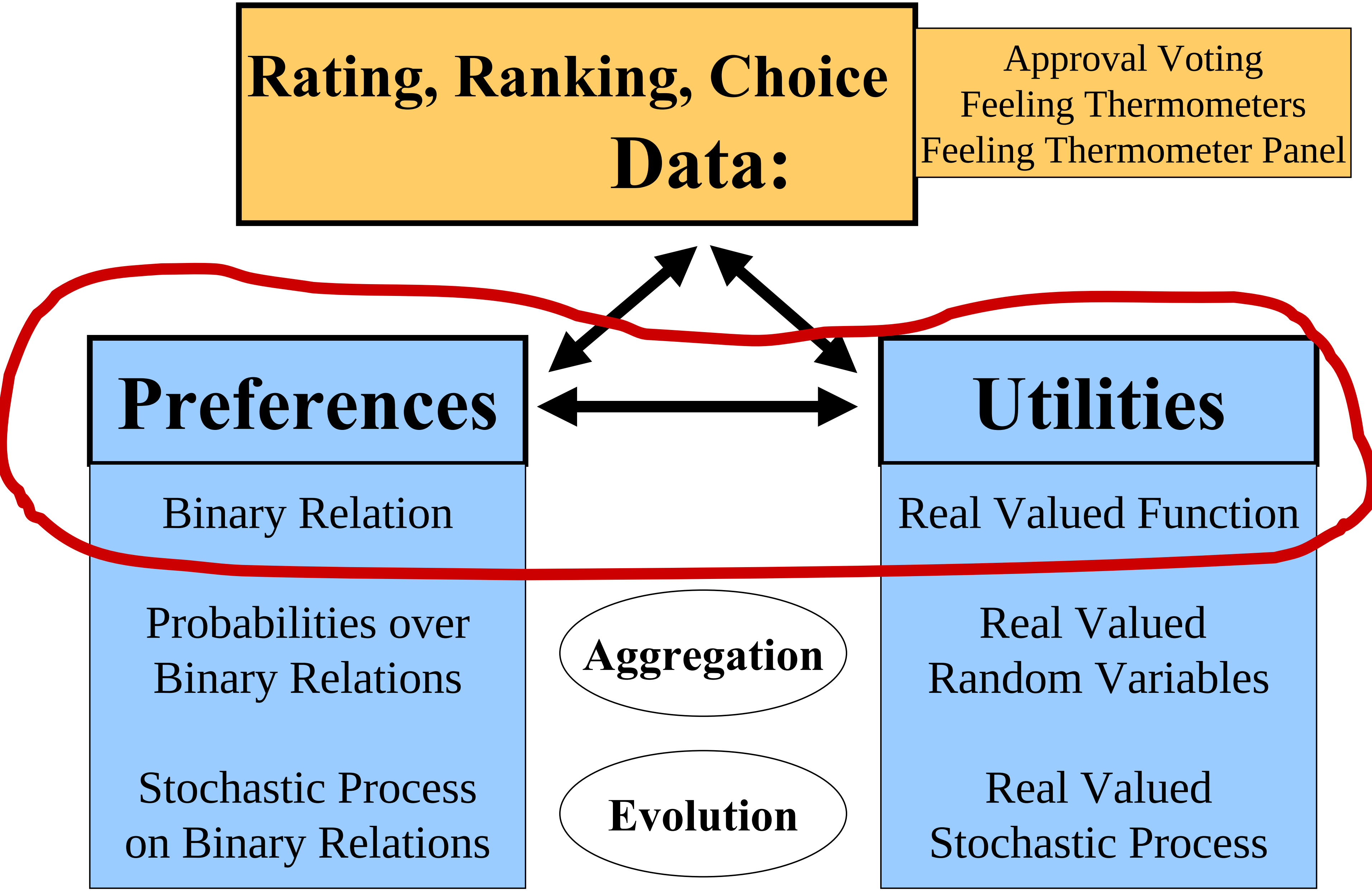
Stochastic Process on Binary Relations
Utilities
Real Valued Function
Real Valued Random Variables
Real Valued Stochastic Process
Aggregation
Evolution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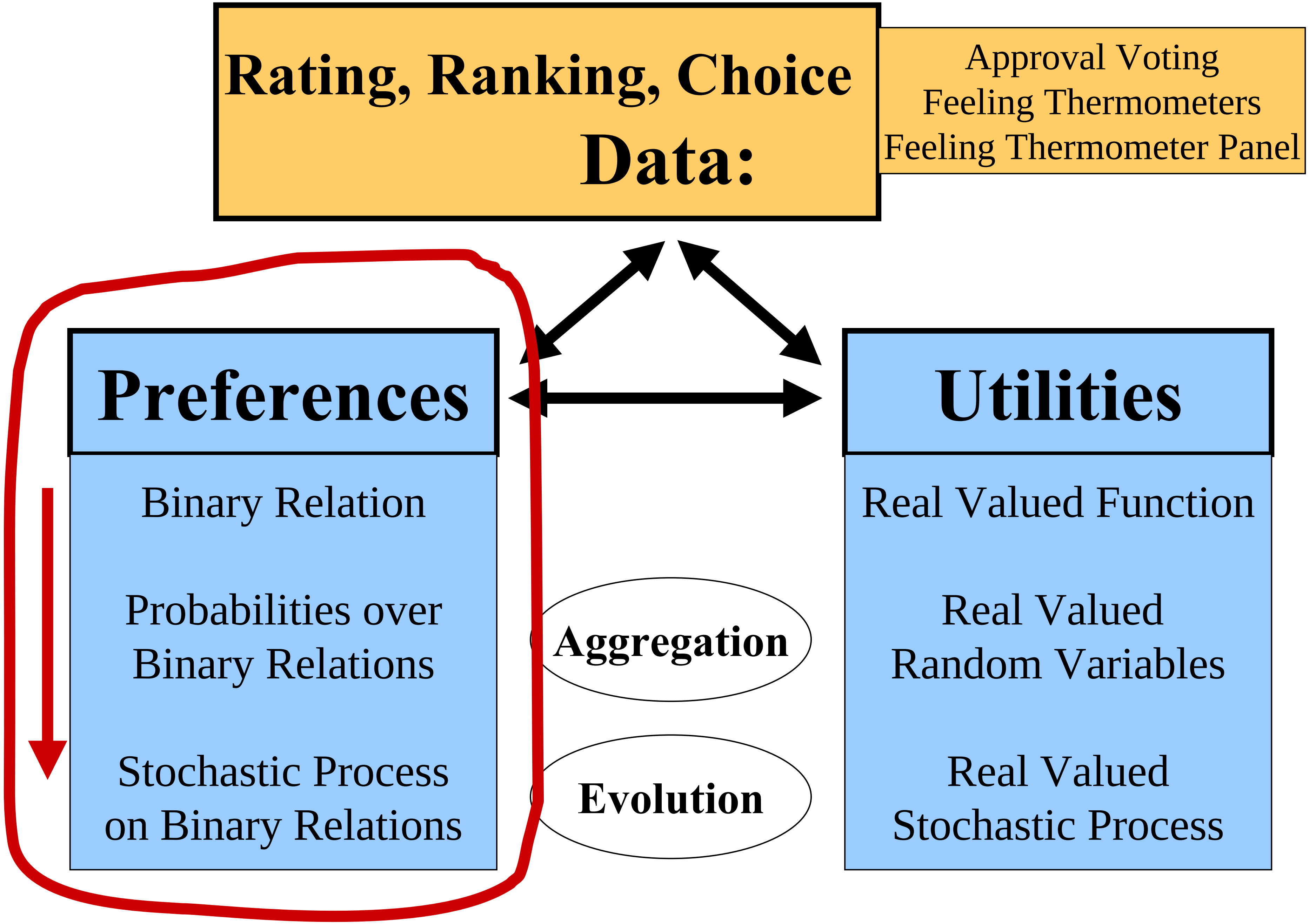
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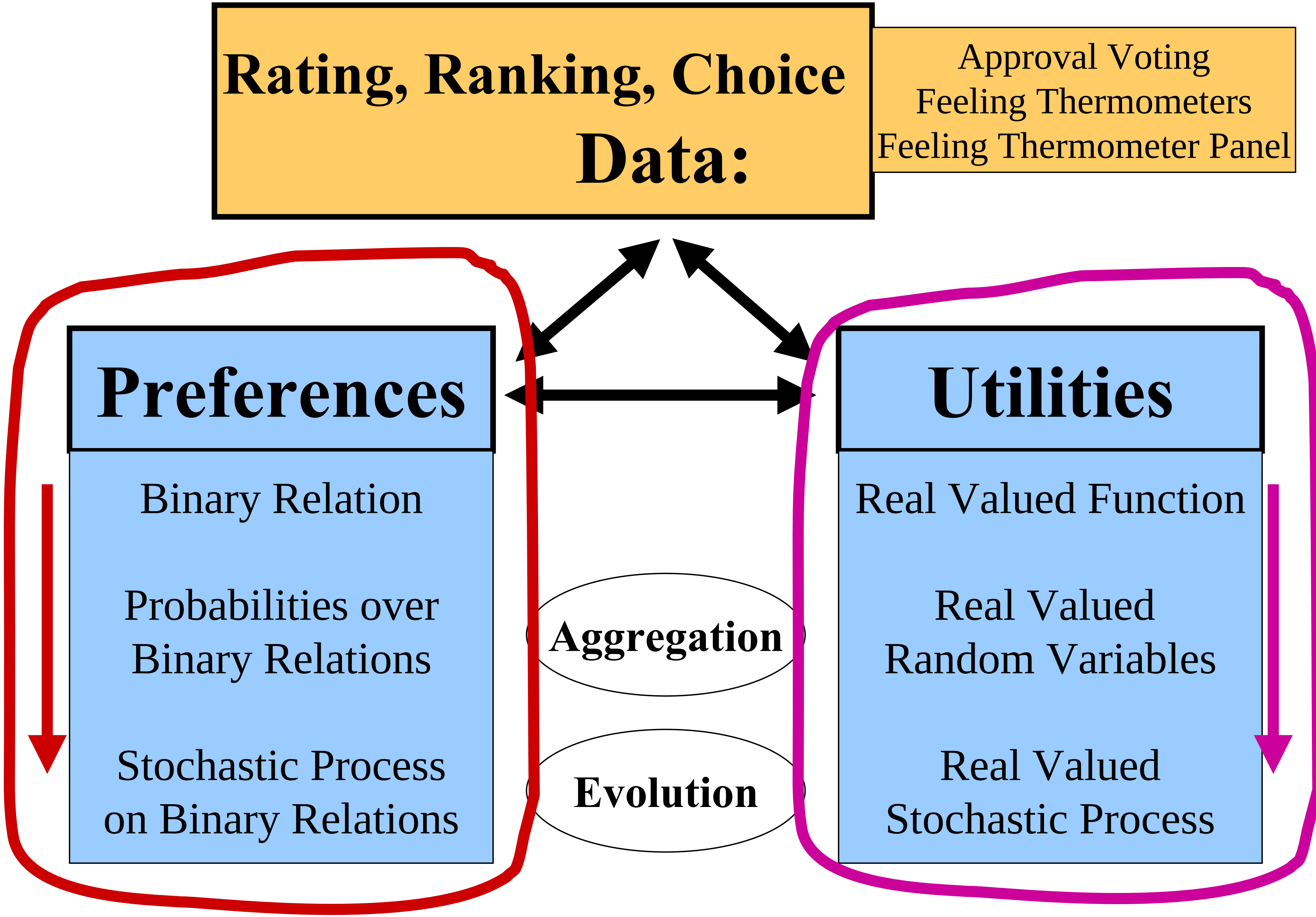
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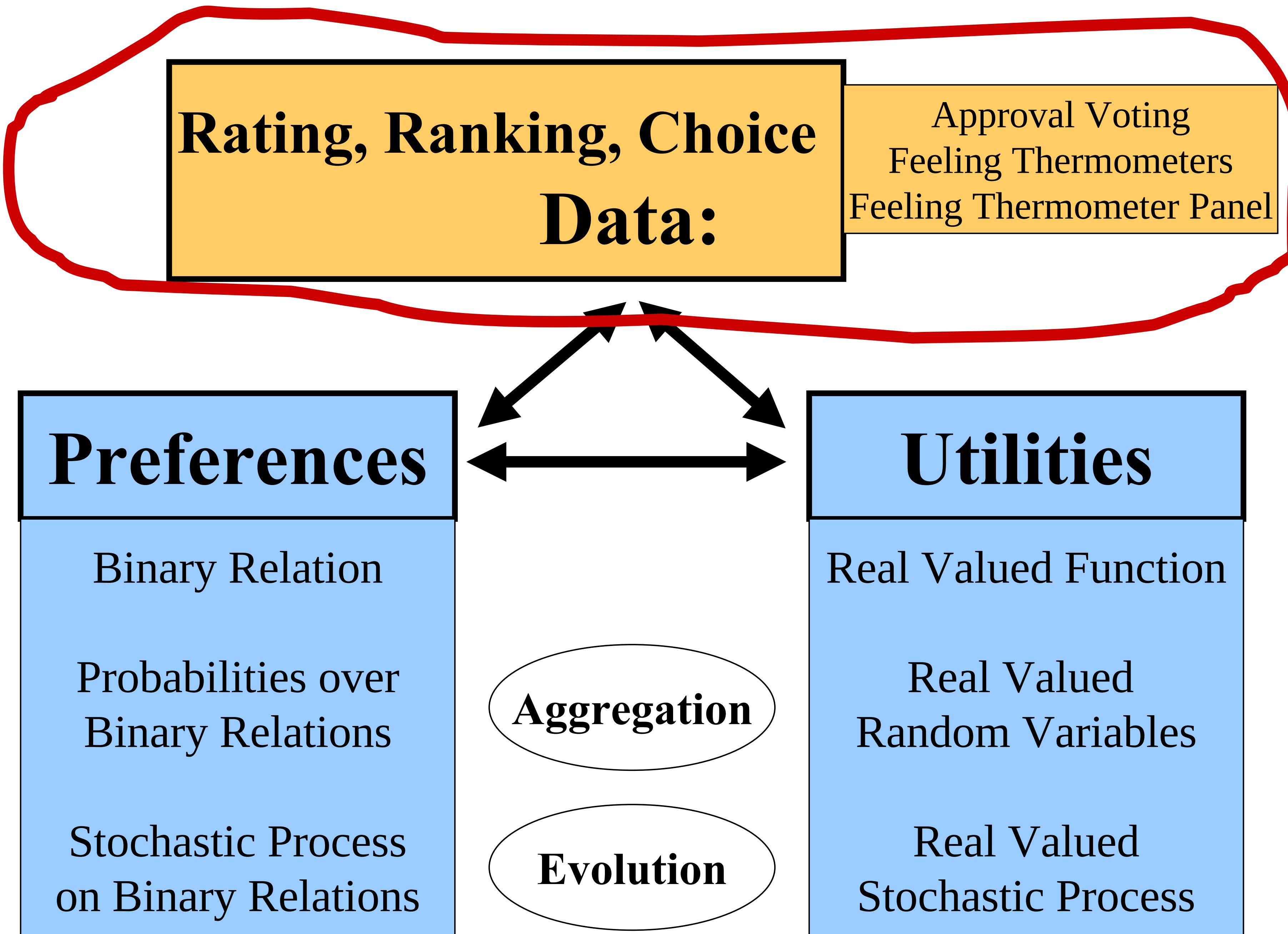

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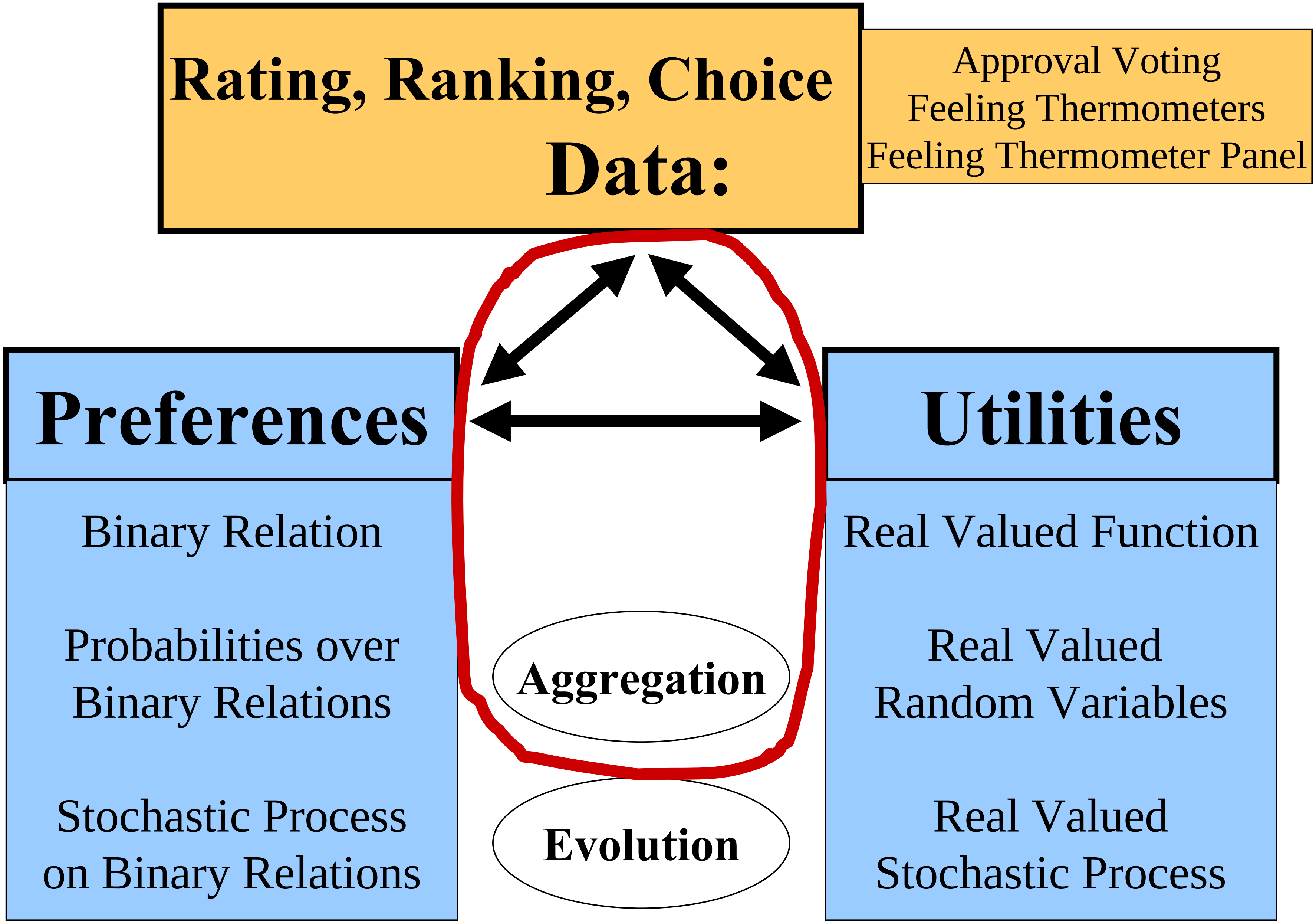
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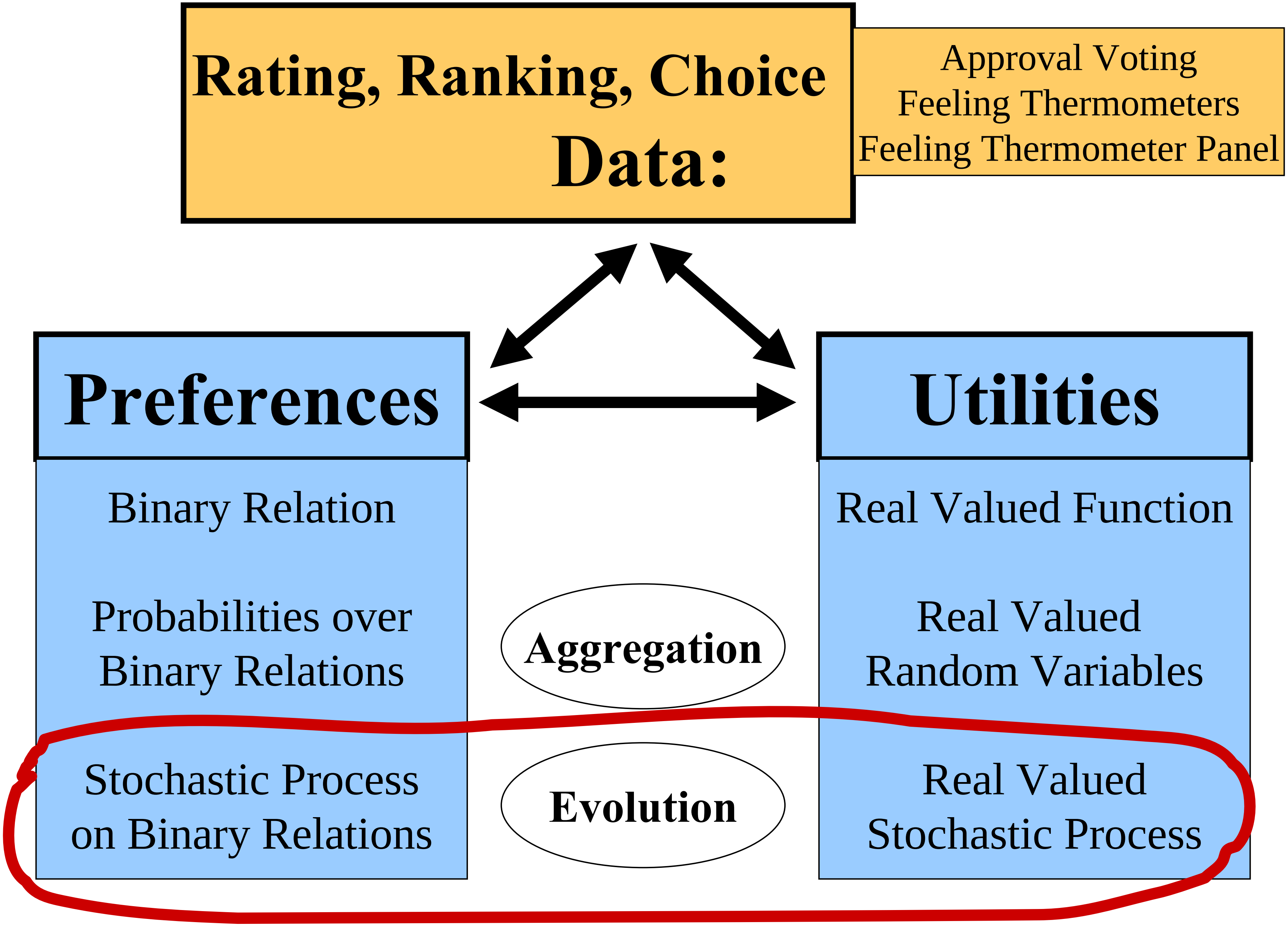
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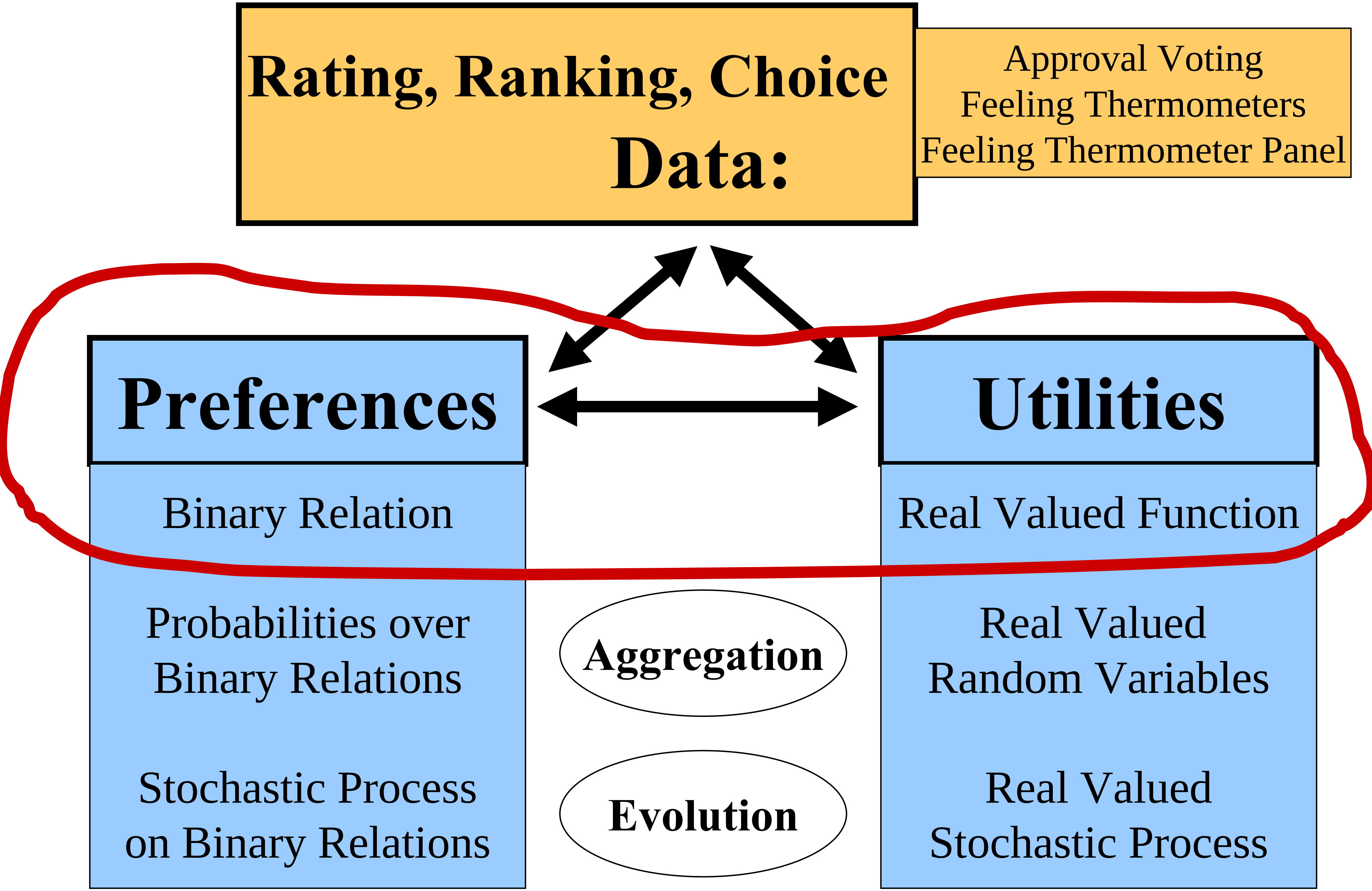
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Binary (Preference) Relations

For a standard reference with the definitions used here, see, e.g., Roberts [Rob79]. A binary relation on a fixed finite set $\mathcal{C}$ takes the form $B \subseteq \mathcal{C} \times \mathcal{C}$. For any binary relation B, its *reverse* is $B^{-1} = \{(b,a) | (a,b) \in B\}$ and let $\overline{B} = [\mathcal{C} \times \mathcal{C}] - B$. Given binary relations B, B', let $BB' = \{(a,c) \in \mathcal{C} \times \mathcal{C} \mid \exists b$ such that $(a,b) \in B$ and $(b,c) \in B'\}$. This is also commonly referred to as the *relative product* of B and B'. Let $Id = \{(c,c) | c \in \mathcal{C}\}$ be the identity relation on $\mathcal{C}$.

A binary relation is said to be

reflexive, if $Id \subseteq B$;

transitive, if $BB \subseteq B$;

asymmetric, if $B \cap B^{-1} = \emptyset$;

antisymmetric, if $B \cap B^{-1} \subseteq Id$;

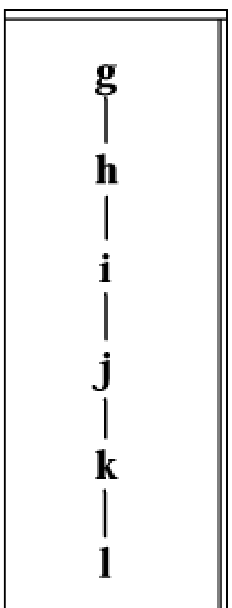
negatively transitive, if $\overline{B}\overline{B} \subseteq \overline{B}$;

strongly complete, if $B \cup B^{-1} = \mathcal{C} \times \mathcal{C}$;

complete, if $B \cup B^{-1} \cup Id = \mathcal{C} \times \mathcal{C}$.

Binary (Preference) Relations

A *linear order* is a transitive, antisymmetric, and strongly complete binary relation.
A *strict linear order* is a transitive, asymmetric, and complete binary relation.



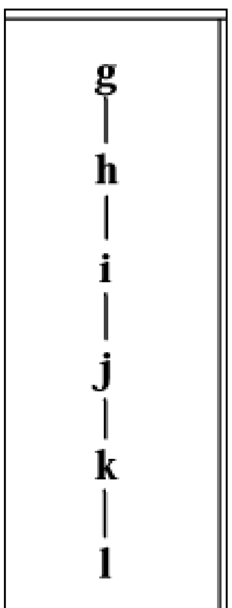
Binary (Preference) Relations

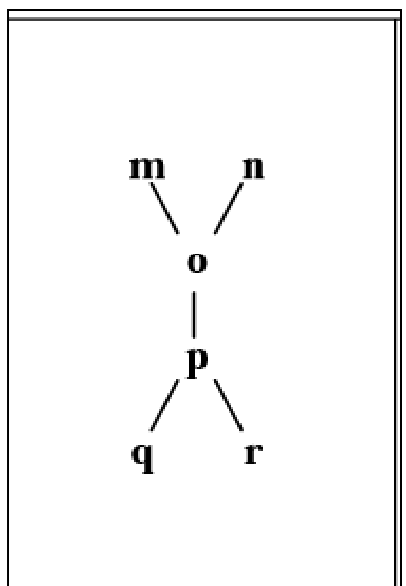
A *linear order* is a transitive, antisymmetric, and strongly complete binary relation.

A *strict linear order* is a transitive, asymmetric, and complete binary relation.

A *weak order* is a transitive and strongly complete binary relation.

A *strict weak order* is an asymmetric and negatively transitive binary relation.





Binary (Preference) Relations

A *linear order* is a transitive, antisymmetric, and strongly complete binary relation.

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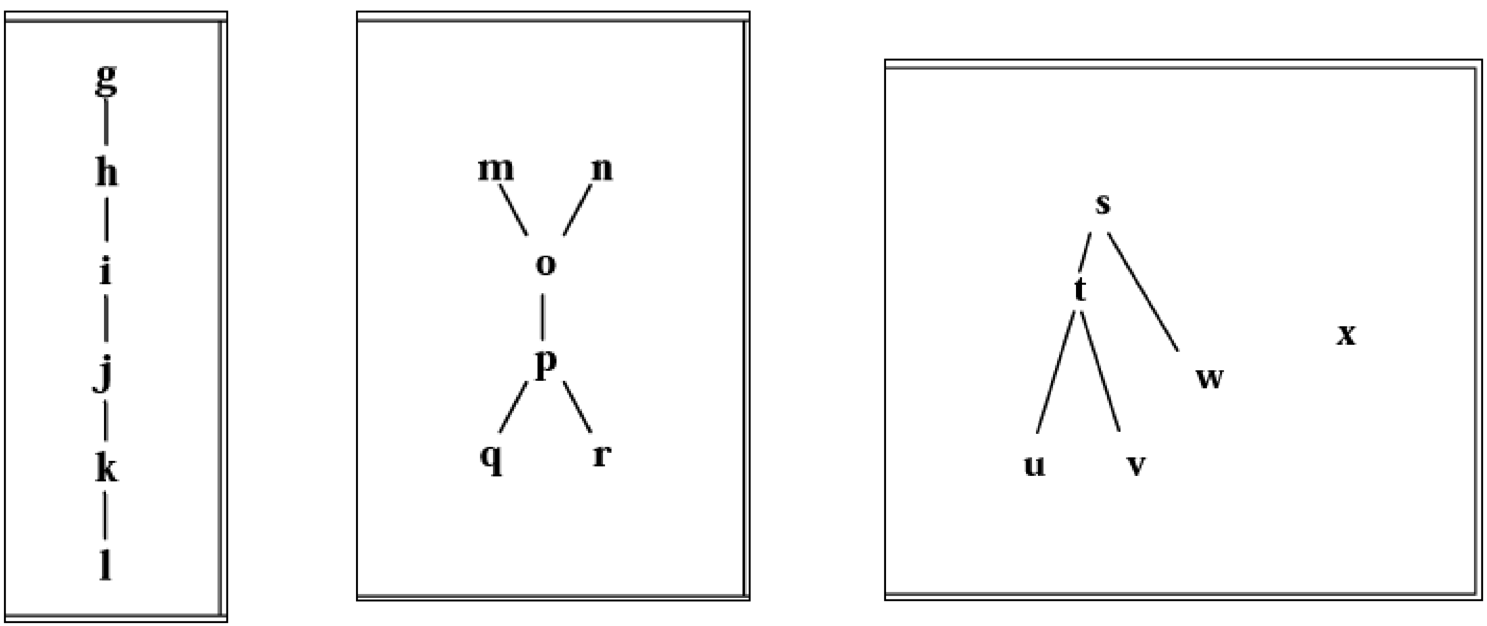
A *weak order* is a transitive and strongly complete binary relation.

A *strict weak order* is an asymmetric and negatively transitive binary relation.

A binary relation B is a *partial order* if it is reflexive, transitive, and antisymmetric.

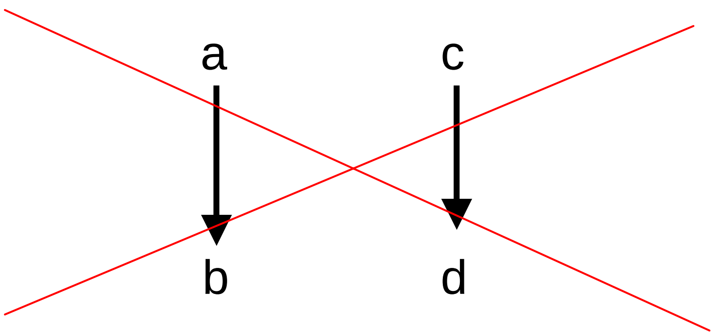
A *strict partial order* is a partial order B which is transitive and asymmetric

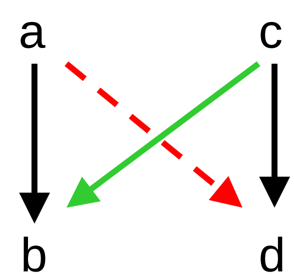


Binary (Preference) Relations

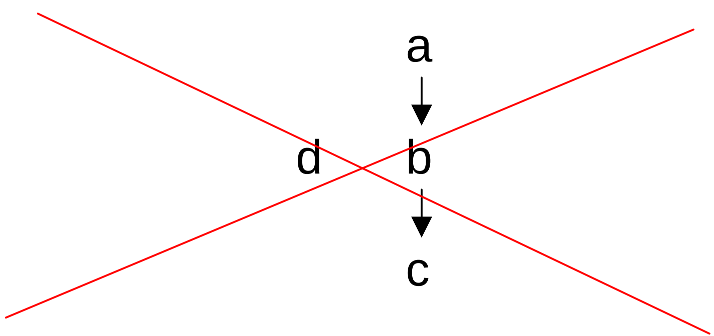
A binary relation B is a *partial order* if it is reflexive, transitive, and antisymmetric.
A *strict partial order* is a partial order B which transitive and asymmetric

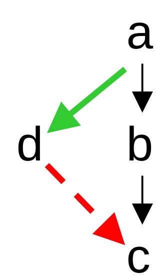
An *interval order* is a strict partial order B with the additional property $B\overline{B^{-1}}B \subseteq B$.



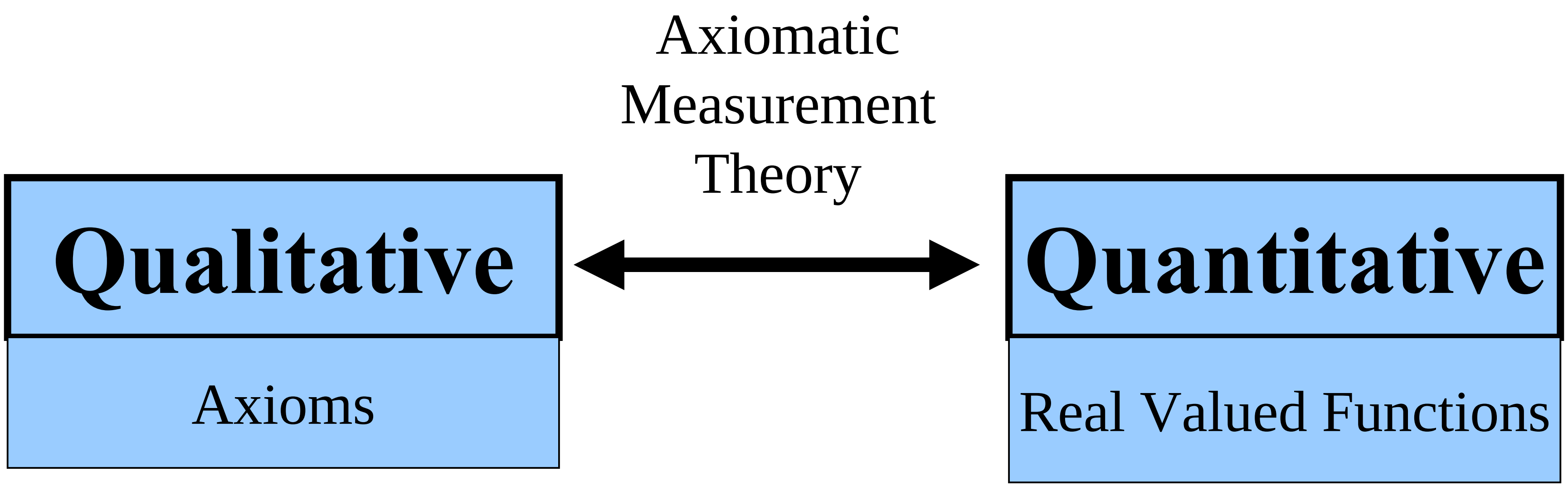


A *semiorder* is an interval order B with the additional property that $BB\overline{B^{-1}} \subseteq B$.





Deterministic Models: Real Representations

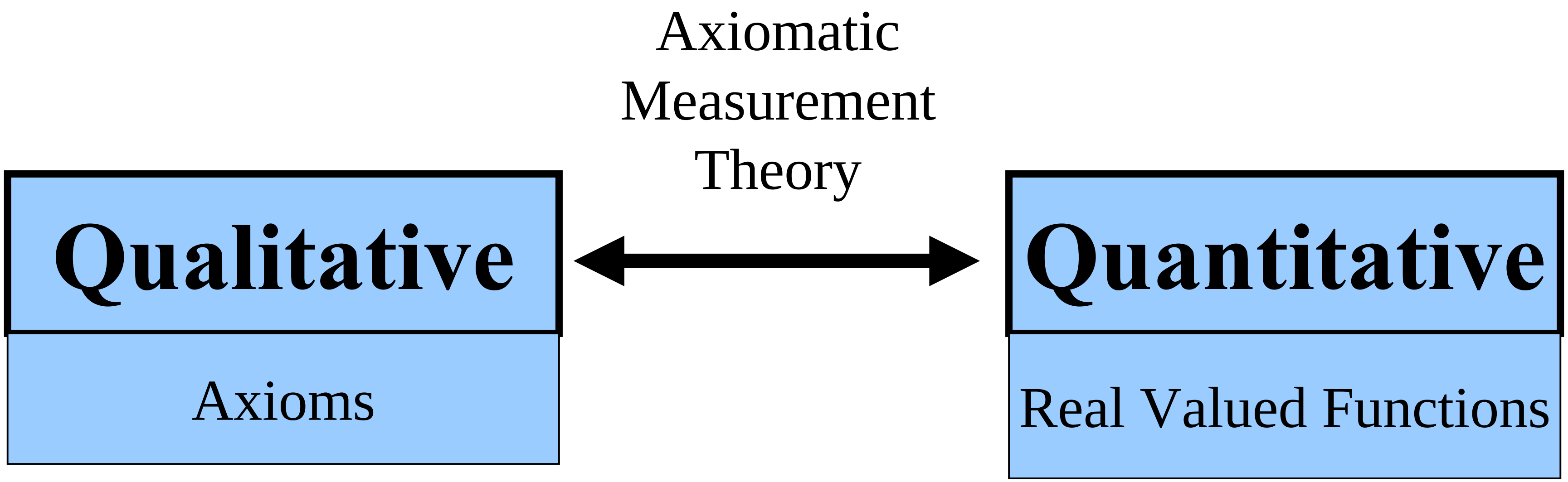


Theorem 2.1.8 *Let B be a binary relation on a finite set $\mathcal{C}$. B is a strict weak order if and only if it has a real representation $u : \mathcal{C} \longrightarrow \mathbb{R}$ of the following form:*

$$aBb \Leftrightarrow u(a) > u(b).$$

If B is a linear order, then it has the above representation, but the converse holds only if u is a one-to-one mapping.

Deterministic Models: Real Representations



B is a semiorder if and only if it has a real representation $u : \mathcal{C} \longrightarrow \mathbb{R}$ of the following form:

$$aBb \Leftrightarrow u(a) > u(b) + \epsilon,$$

where $\epsilon \in \mathbb{R}^{++}$ is a fixed strictly positive real valued (utility) threshold.

B is an interval order if and only if it has a real representation $l, u : \mathcal{C} \longrightarrow \mathbb{R}$, with $l(x) < u(x)$ (for all x), of the following form:

$$aBb \Leftrightarrow l(a) > u(b).$$

Deterministic Models: Real Representations

Axiomatic Measurement Theory

Qualitative

Axioms

Quantitative

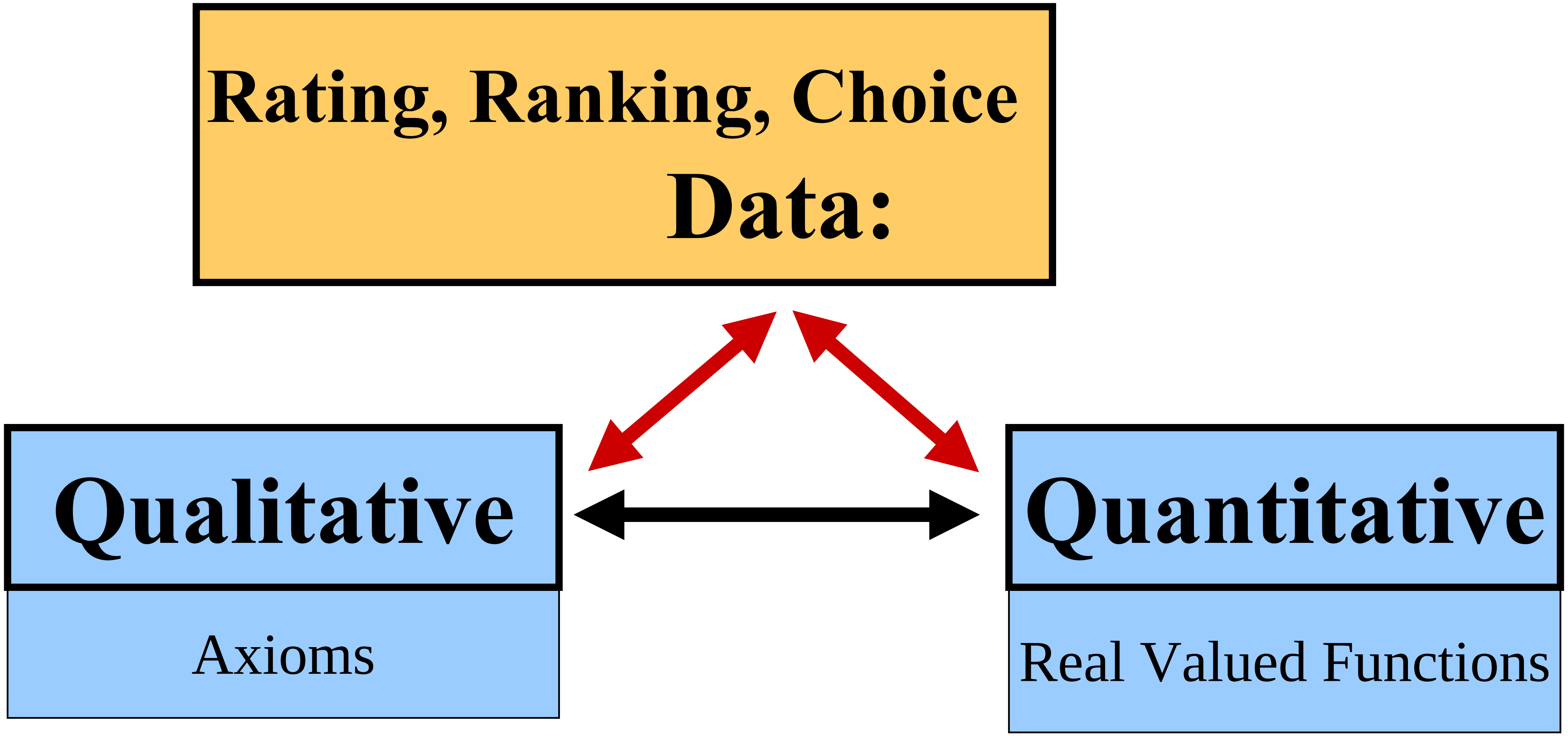
Real Valued Functions

(Normative) Utility Theory

Expected Utility Theory

…

Prospect Theory



Example:
Violations of Expected Utility Theory

Why Probabilistic Models?

Data: Result of Random Sampling

Preferences/Utilities Vary

Between Subjects:

Social Choice (e.g., Voting)

Between and Within Subjects:

Persuasion (e.g., Campaigns)

Deterministic Models: Real Representations

Preferences

Binary Relation

Utilities

Real Valued Function

Strict Weak Preference Order:

a b
c
d
e

if
and only
if

Utility Function:
u(a) = u(b) > … > u(e)

Probabilistic Models:
Random Utility Representations

Preferences		**Utilities**
Binary Relation	$\longleftrightarrow$	Real Valued Function
Probabilities over Binary Relations	$\longleftrightarrow$	Real Valued Random Variables

Probability of the strict weak order

a b
c
d
e

$$= \quad \mathrm{Prob}[\mathbf{U}(a) = \mathbf{U}(b) > \ldots > \mathbf{U}(e)]$$

Probabilistic Models:
Random Utility Representations

Theorem 2.1.10 *A family of jointly distributed real valued utility random variables* $\mathbf{U} = (\mathbf{U}_{i,c})_{i=1,\dots,k;c\in\mathcal{C}}$ *satisfies the following properties:*

Random Utility Representations of Linear Orders: *If* $k = 1$ *and noncoincidence holds, that is,* $\mathbb{P}(\mathbf{U}_c = \mathbf{U}_d) = 0, \forall c, d \in \mathcal{C}$, *then* $\mathbb{P}$ *induces a probability distribution* $\pi \mapsto P(\pi)$ *on the set* Π *of linear orders over* $\mathcal{C}$ *through, for any linear order* $\pi = c_1c_2\dots c_N$ *(*c_1 *is best,* $\dots, c_N$ *is worst),*

$$P(\pi) = \mathbb{P}(\mathbf{U}_{c_1} > \mathbf{U}_{c_2} \cdots > \mathbf{U}_{c_N}). \tag{2.12}$$

Random Utility Representations of Weak Orders: *If* $k = 1$*, then, regardless of the joint distribution of* $\mathbf{U}$, $\mathbb{P}$ *induces a probability distribution* $B \mapsto P(B)$ *on the set* $\mathcal{SWO}$ *of strict weak orders over* $\mathcal{C}$ *through*

$$P(B) = \mathbb{P}\left(\left[\bigcap_{(a,b)\in B} (\mathbf{U}_a > \mathbf{U}_b)\right] \bigcap \left[\bigcap_{(c,d)\in\mathcal{C}^2-B} (\mathbf{U}_c \le \mathbf{U}_d)\right] \right). \tag{2.13}$$

Probabilistic Models:
Random Utility Representations

Random Utility Representations of Semiorders: *If $k = 1$, then, regardless of the joint distribution of* $\mathbf{U}$, $\mathbb{P}$ *induces a probability distribution $B \mapsto P(B)$ on the set $\mathcal{SO}$ of semiorders over $\mathcal{C}$ through, given a strictly positive threshold $\epsilon \in \mathbb{R}^{++}$,*

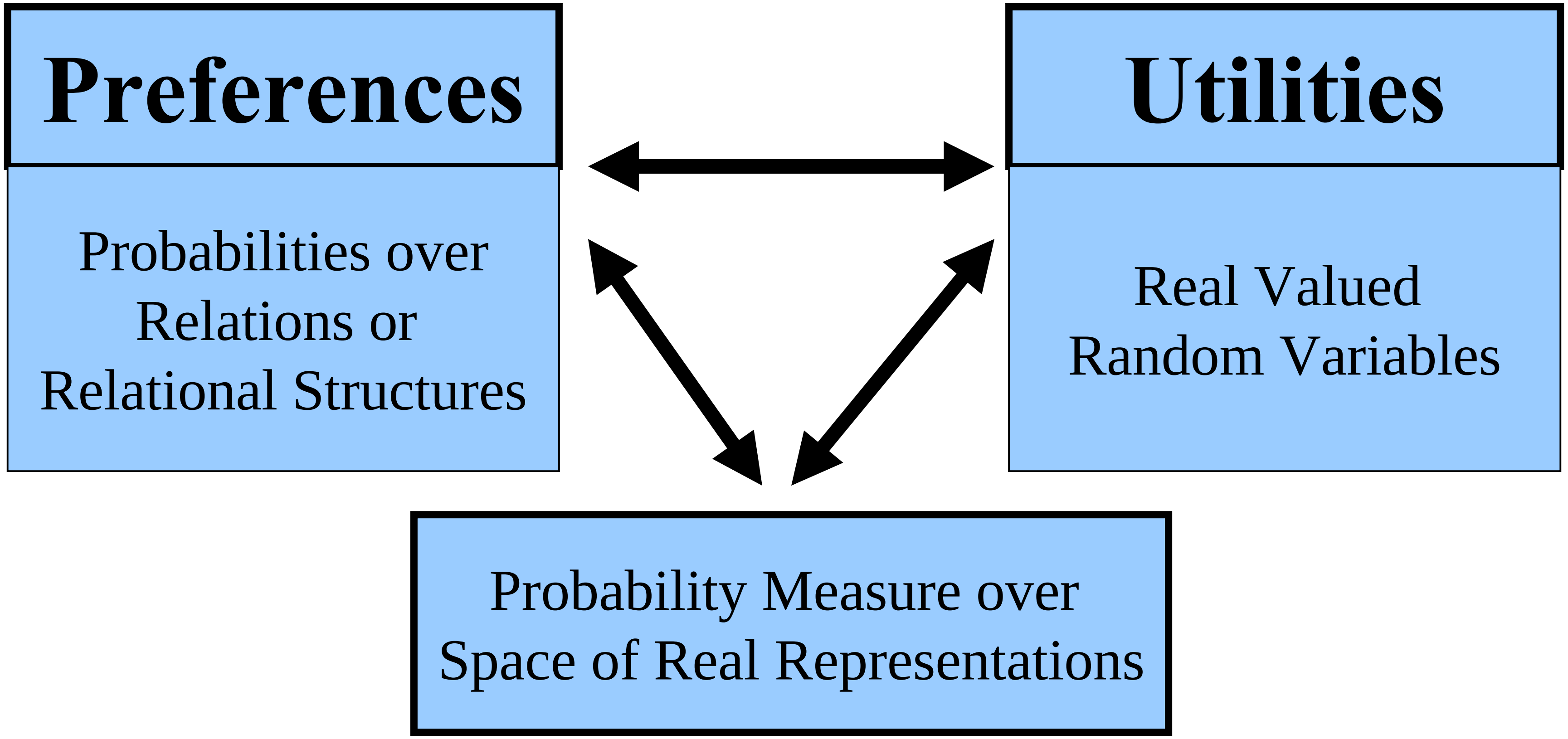
$$P(B) = \mathbb{P}\left(\left[\bigcap_{(a,b)\in B} (\mathbf{U}_a > \mathbf{U}_b + \epsilon) \right] \bigcap \left[\bigcap_{(c,d)\in \mathcal{C}^2 - B} (\mathbf{U}_c - \mathbf{U}_d \leq \epsilon) \right] \right). \tag{2.14}$$

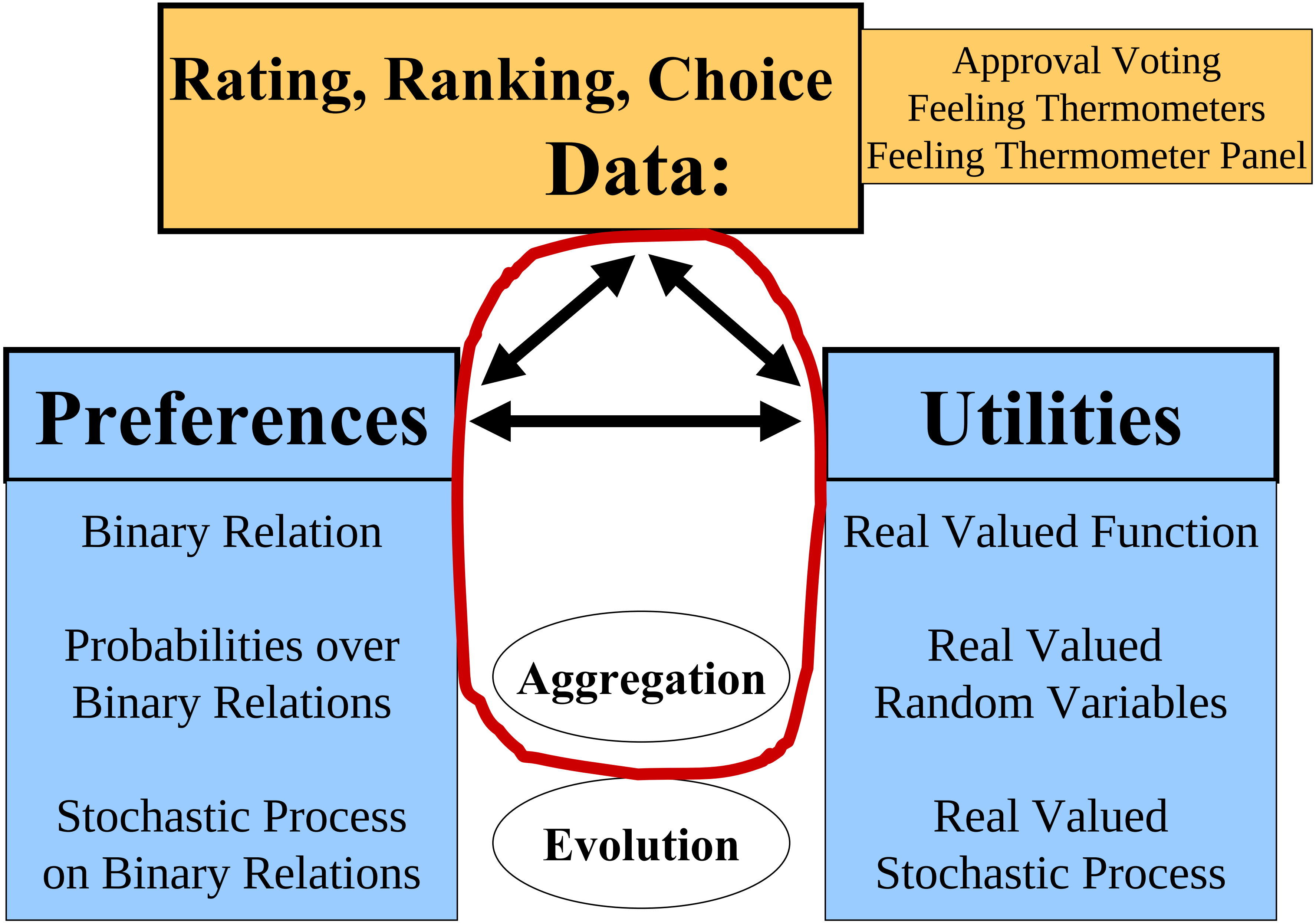
Random Utility Representations of Interval Orders: *If $k = 2$ and $\mathbb{P}(\mathbf{U}_{1,c} \leq \mathbf{U}_{2,c}) = 1$ for all choices of c, then, writing $\boldsymbol{L}_c$ for $\mathbf{U}_{1,c}$ (lower utility) and $\mathbf{U}_c$ for $\mathbf{U}_{2,c}$ (upper utility) we have the following result. In this case, $\mathbb{P}$ induces a probability distribution $B \mapsto P(B)$ on the set $\mathcal{IO}$ of interval orders over $\mathcal{C}$ through*

$$P(B) = \mathbb{P}\left(\left[\bigcap_{(a,b)\in B} (\boldsymbol{L}_a > \mathbf{U}_b) \right] \bigcap \left[\bigcap_{(c,d)\in \mathcal{C}^2 - B} (\boldsymbol{L}_c \leq \mathbf{U}_d) \right] \right). \tag{2.15}$$

General Results for Probabilistic Measurement

(Regenwetter, 1996, JMP)
(Regenwetter & Marley, 2001, JMP)

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Majority rule: (Condorcet Criterion)

Majority Winner

- Candidate who is ranked ahead of any other candidate by more than 50%
- Candidate who beats any other candidate in pairwise competition

Kenneth Arrow’s (1951) Nobel Prize winning *Impossibility Theorem*

- List of Axioms of Rationality
- Impossibility to simultaneously satisfy all Axioms
- Majority permits “cycles”.

The Obsession with Cycles

Majority Cycles

ABC	1 person
BCA	1 person
CAB	1 person

Majority Cycles

ABC	1 person
BCA	1 person
CAB	1 person

A beats B	2 times
B beats A	1 time

A is majority preferred to B

Majority Cycles

ABC	1 person
BCA	1 person
CAB	1 person

B beats C	2 times
C beats B	1 time

A is majority preferred to B

B is majority preferred to C

Majority Cycles

ABC	1 person
BCA	1 person
CAB	1 person

A beats C	1 time
C beats A	2 times

A is majority preferred to B

B is majority preferred to C

C is majority preferred to A

Majority Cycles

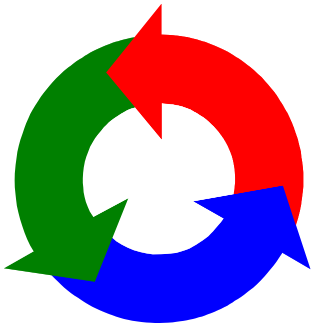
ABC	1 person
BCA	1 person
CAB	1 person

Democratic Decision Making at Risk!?!

A is majority preferred to B

B is majority preferred to C

C is majority preferred to A



$1,000,000 Question:

Where is the empirical evidence for voting paradoxes in practice?

Oops….

For instance, hardly any evidence that majority cycles have ever occurred among serious contenders of major elections.

Actually, evidence circumstantial at best.

Where is the evidence for cycles?

Majority Winner

- Candidate who is ranked ahead of any other candidate by more than 50%
  - Candidate who beats any other candidate in pairwise competition

- **Plurality**: *Choose one*
- **SNTV & Limited Vote**: *Choose k many*
- **Approval Voting**: *Choose any subset*
- **STV (Hare), AV (IRV)**: *Rank top k many*
- **Cumulative Voting**: *Give m pts to k many*
- **Survey Data**: *Thermometer, Likert Scales*

Data are incomplete!!

Example 1:
Probabilistic Models for Approval Voting
and
Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

Example 1:
Probabilistic Models for Approval Voting
and
Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

A: 50 votes

Example 1:
Probabilistic Models for Approval Voting
and
Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

A: 50 votes
B: 32 votes

Example 1: Probabilistic Models for Approval Voting and Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

A: 50 votes
B: 32 votes
C: 38 votes

A is the Approval Voting Winner!

Is there a Majority Winner? Who is it?

Sorry! Majority Winner not defined for Approval Voting

Majority Winner

- Candidate who is ranked ahead of any other candidate by more than 50%
- Candidate who beats any other candidate in pairwise competition

Majority Winner is Counterfactual

Example 1:
Probabilistic Models for Approval Voting
and
Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

A beats B	48 times
B beats A	30 times

A is majority preferred to B

Example 1:
Probabilistic Models for Approval Voting
and
Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

A beats C	42 times
C beats A	30 times

A is majority preferred to B

A is majority preferred to C

Example 1:
Probabilistic Models for Approval Voting
and
Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

B beats C	22 times
C beats B	28 times

A is majority preferred to B

A is majority preferred to C

C is majority preferred to B

A
C
B

Example 1:
Probabilistic Models for Approval Voting
and
Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

ABC	8		
ACB	32	ABC	2
BCA	20	ACB	8
CBA	20	BCA	5
		CBA	5

Example 1:
Probabilistic Models for Approval Voting
and
Majority Rule

A	40	AB	2
B	20	AC	8
C	20	BC	10

ABC	8		
ACB	32	ABC	2
BCA	20	ACB	8
CBA	20	BCA	5
		CBA	5

A is majority <u>tied</u> with B
A is majority <u>tied</u> with C
C is majority preferred to B

C
B

Majority Winner may be Model Dependent

First computation: ***Topset Voting Model***

(Regenwetter, 1997, MSS)
(Niederee & Heyer, 1997, Luce volume)

Second computation: ***Size-Independent Model***

(Falmagne & Regenwetter, 1996, JMP)
(Doignon & Regenwetter, 1997, JMP)
(Regenwetter & Grofman, 1998a,b; SCW, MS)
(Regenwetter & Doignon, 1998, JMP)
(Regenwetter, Marley & Joe, 1998, AJP)

	Order by AV scores	Majority Order Topset Model	Majority Order SIM Model
TIMS E1			
TIMS E2			
MAA1			
MAA2			
A25			
A72			
IEEE			

	Order by AV scores	Majority Order Topset Model	Majority Order SIM Model
TIMS E1	b c a	Same as AV order	c b a *or* b c a

	Order by AV scores	Majority Order Topset Model	Majority Order SIM Model
TIMS E1	b c a	Same as AV order	c b a *or* b c a
TIMS E2	c b a	Same as AV order	b c a *or* c b a

	Order by AV scores	Majority Order Topset Model	Majority Order SIM Model
TIMS E1	b c a	Same as AV order	c b a *or* b c a
TIMS E2	c b a	Same as AV order	b c a *or* c b a
MAA1	c a b	Same as AV order	a c b *or* c a b

	Order by AV scores	Majority Order Topset Model	Majority Order SIM Model
TIMS E1	b c a	Same as AV order	c b a *or* b c a
TIMS E2	c b a	Same as AV order	b c a *or* c b a
MAA1	c a b	Same as AV order	a c b *or* c a b
MAA2	b c a	Same as AV order	Same as AV order

	Order by AV scores	Majority Order Topset Model	Majority Order SIM Model
TIMS E1	b c a	Same as AV order	c b a *or* b c a
TIMS E2	c b a	Same as AV order	b c a *or* c b a
MAA1	c a b	Same as AV order	a c b *or* c a b
MAA2	b c a	Same as AV order	Same as AV order
A25	b c a	Same as AV order	Same as AV order

	Order by AV scores	Majority Order Topset Model	Majority Order SIM Model
TIMS E1	b c a	Same as AV order	c b a *or* b c a
TIMS E2	c b a	Same as AV order	b c a *or* c b a
MAA1	c a b	Same as AV order	a c b *or* c a b
MAA2	b c a	Same as AV order	Same as AV order
A25	b c a	Same as AV order	Same as AV order
A72	c a b	Same as AV order	Same as AV order

	Order by AV scores	Majority Order Topset Model	Majority Order SIM Model
TIMS E1	b c a	Same as AV order	c b a *or* b c a
TIMS E2	c b a	Same as AV order	b c a *or* c b a
MAA1	c a b	Same as AV order	a c b *or* c a b
MAA2	b c a	Same as AV order	Same as AV order
A25	b c a	Same as AV order	Same as AV order
A72	c a b	Same as AV order	Same as AV order
IEEE	a b c	Same as AV order	Cycle *or* one of a c b , a b c

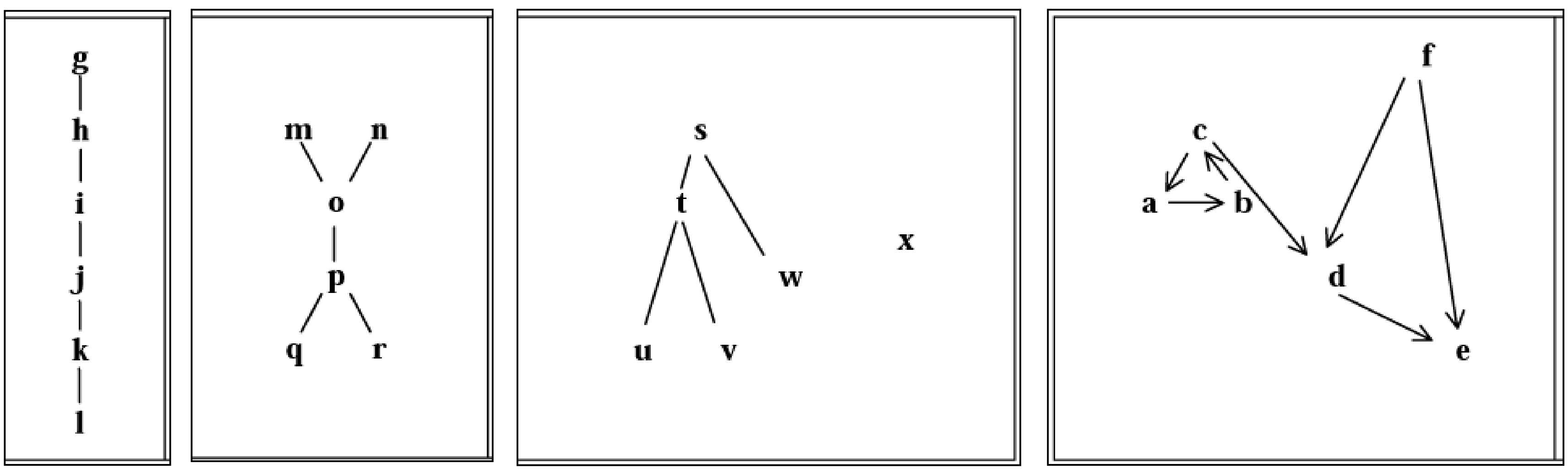
Preliminary Conclusions:

Majority Preference Relation

is *model dependent*
should be treated in an *inference framework*
may or may not be *robust*

A General Concept of Majority Rule

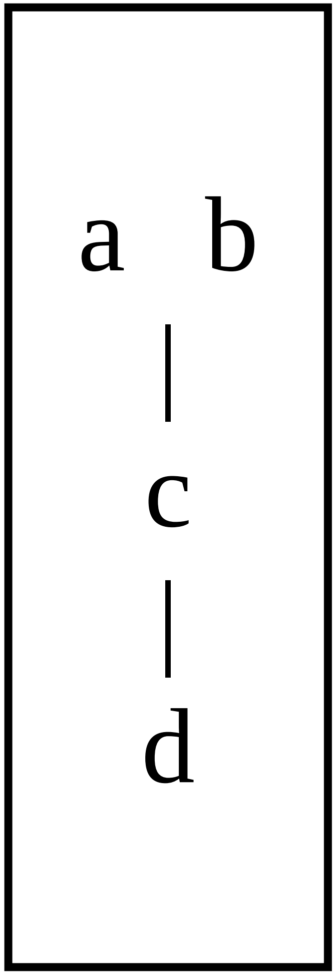
Linear Orders	“complete rankings”
Weak Orders	“rankings with possible ties”
Semiorders	“rankings with (fixed) threshold”
Interval Orders	“rankings with (variable) threshold”
Partial Orders	asymmetric, transitive
Asymmetric Binary Relations	

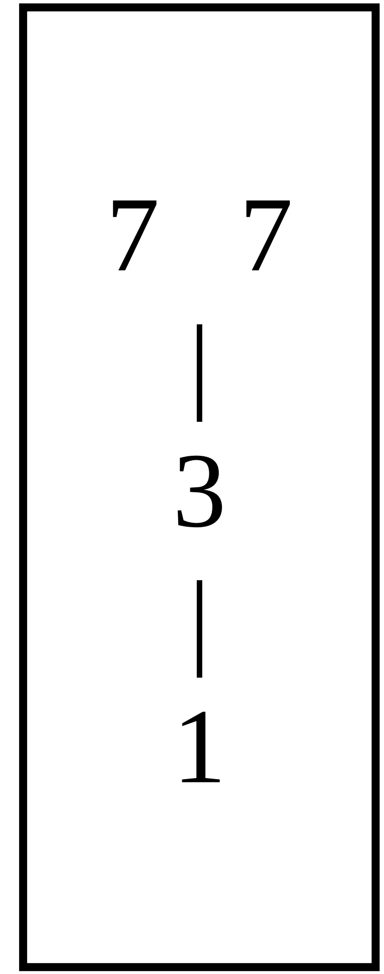


B

$>$

Real Representation of Weak Orders





$$(a,b) \in B \iff u(a) > u(b)$$

Variable Preferences:
Probability Distribution
on Binary Relations

Variable Utilities:
Jointly Distributed Family of
Utility Random Variables
(Random Utilities)
(parametric or nonparametric)

Random Utility Representations

Semiorders

Interval Orders

$$P(B) = P\begin{pmatrix} \mathbf{L}_i > \mathbf{U}_j \mid (i,j) \in B \\ \text{and} \\ \mathbf{L}_i \leq \mathbf{U}_j \mid (i,j) \notin B \end{pmatrix}$$

With $\mathbf{U}_i(\omega) = \mathbf{L}_i(\omega) + \varepsilon$

$\forall \omega$

A General Definition of Majority Rule

Given a probability distribution

$$P : B \to [0,1]$$

$$B \mapsto P(B)$$

on any set B of binary relations,

a is strictly majority preferred to b

if and only if

$$\sum_{(a,b)\in B} P(B) > \sum_{(b,a)\in B'} P(B')$$

A General Definition of Majority Rule

Given a probability distribution

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on any set B of binary relations,

a is strictly majority preferred to b

if and only if

$$\sum_{(a,b)\in B} P(B) > \sum_{(b,a)\in B'} P(B')$$

For Utility Functions or Random Utility Models choose a Random Utility Representation and obtain a *consistent* Definition

Examples:

$$i \text{ majority preferred to } j$$

$$\Leftrightarrow$$

$$\text{Proportion}\left(u(i) > u(j)\right) > \text{Proportion}\left(u(j) > u(i)\right)$$

$$i \text{ majority preferred to } j$$

$$\Leftrightarrow$$

$$P(\mathbf{U}_i > \mathbf{U}_j + 54) > P(\mathbf{U}_j > \mathbf{U}_i + 54)$$

Weak Utility Model
Weak Stochastic Transitivity
Transitivity of Majority Preferences

Definition 1.2.1 A *weak utility model* is a set of binary choice probabilities for which there exists a real-valued function w over $\mathcal{C}$ such that

$$p_{cd} \geq \frac{1}{2} \Leftrightarrow w(c) \geq w(d).$$

When $\mathcal{C}$ is finite, then the weak utility model is equivalent to weak stochastic transitivity of the binary choice probabilities, which we define next [LS65].

Definition 1.2.2 *Weak stochastic transitivity of binary choice probabilities* holds when

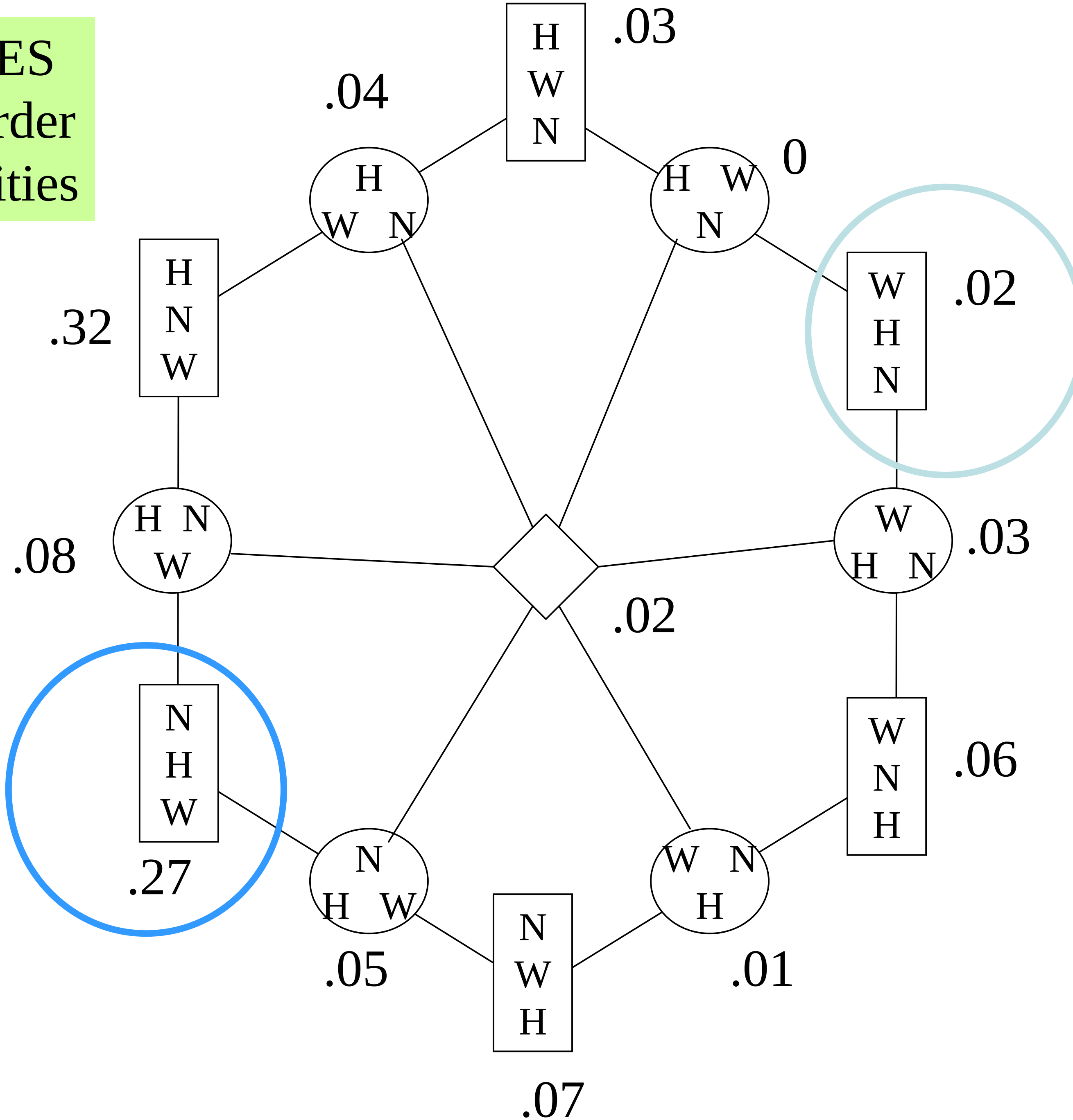
$$p_{cd} \geq \frac{1}{2} \quad \& \quad p_{de} \geq \frac{1}{2} \quad \Longrightarrow \quad p_{ce} \geq \frac{1}{2}.$$

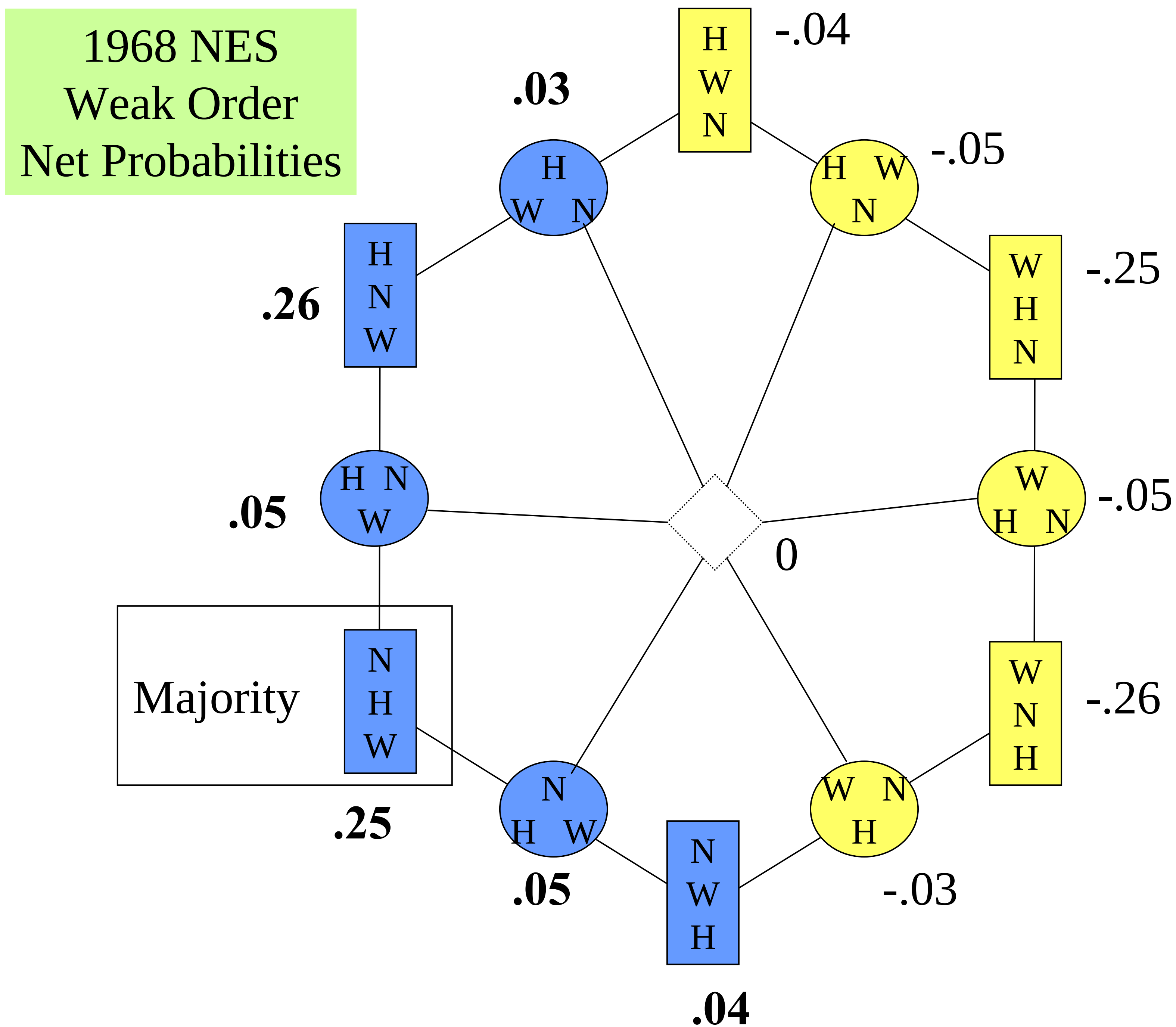
Remember: No Cycles in 7 Approval Voting Data Sets
(1 analysis ambiguous)

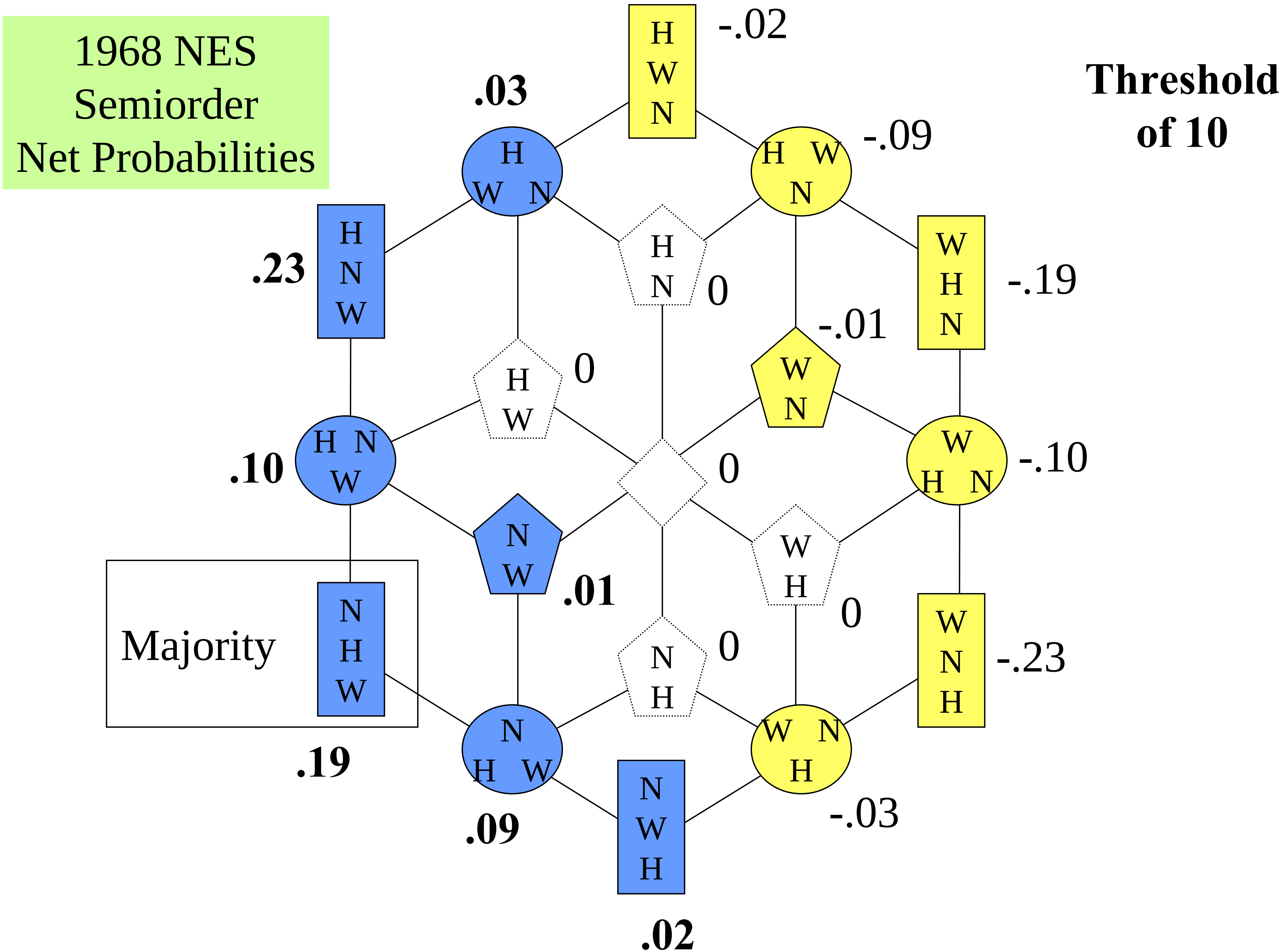
Let's analyze National Survey Data!
1968, 1980, 1992, 1996 ANES

Feeling Thermometer Ratings
translated into
Weak Orders or Semiorders

1968 NES
Weak Order
Probabilities

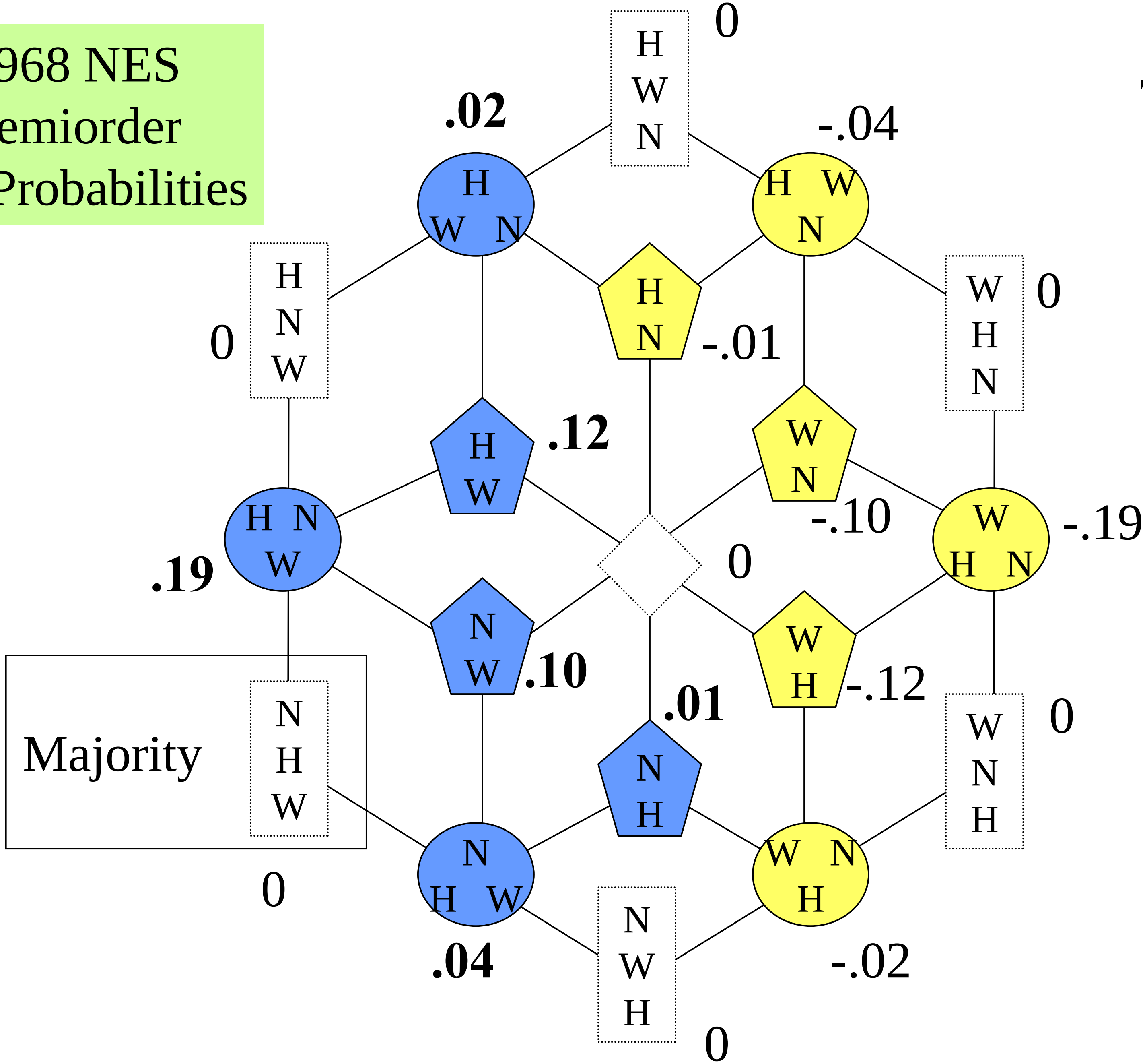



1968 NES
Weak Order
Net Probabilities
H W N
-.04
.03
H W N
H W N
-.05
W H N
-.25
.26
H N W
.05
H N W
0
W H N
-.05
Majority
N H W
W N H
-.26
.25
N H W
.05
N W H
W N H
-.03
.04


1968 NES
Semiorder
Net Probabilities
Threshold
of 10
H W N
-.02
.03
H W N
H W N
-.09
.23
H N W
H N
0
W H N
-.19
-.01
W N
H W
0
.10
H N W
0
W H N
-.10
N W
.01
W H
0
Majority
N H W
N H
0
W N H
-.23
.19
N H W
N W H
W N H
-.03
.09
.02

1968 NES
Semiorder
Net Probabilities

Threshold of 54



ANES Strict Majority Social Welfare Orders

Year	**Threshold**	**SWO**
1968	0, …, 96	Nixon Humphrey Wallace

ANES Strict Majority Social Welfare Orders

Year	**Threshold**	**SWO**
1992	0, …, 99	Clinton Bush Perot

However:
There is no Theory-Free Majority Preference Relation

ANES Strict Majority Social Welfare Orders

Year	Threshold	SWO
1980	0, …, 29	Carter Reagan Anderson
	30, …, 99	Reagan Carter Anderson

ANES Strict Majority Social Welfare Orders

Year	Threshold	SWO
1996	0, …, 49 85, …, 99	Clinton Dole Perot
	50, …,84	Dole Clinton Perot

Preliminary Conclusions:

Majority Preference Relation

is *model dependent*

We did not see any indication of cycles!

Borda Scoring rule:

- 1st ranked candidate gets 2 points,
- 2nd ranked candidate gets 1 point,
- 3rd ranked candidate gets 0 point.

In general, the i^{th} ranked among n candidates gets $n-i$ points.

Scoring rule:

- 1st ranked candidate gets x points,
- 2nd ranked candidate gets $y < x$ points,
- 3rd ranked candidate gets $z < y$ points.

In general, the i^{th} ranked among n candidates gets $f(n-i)$ many points with f increasing.

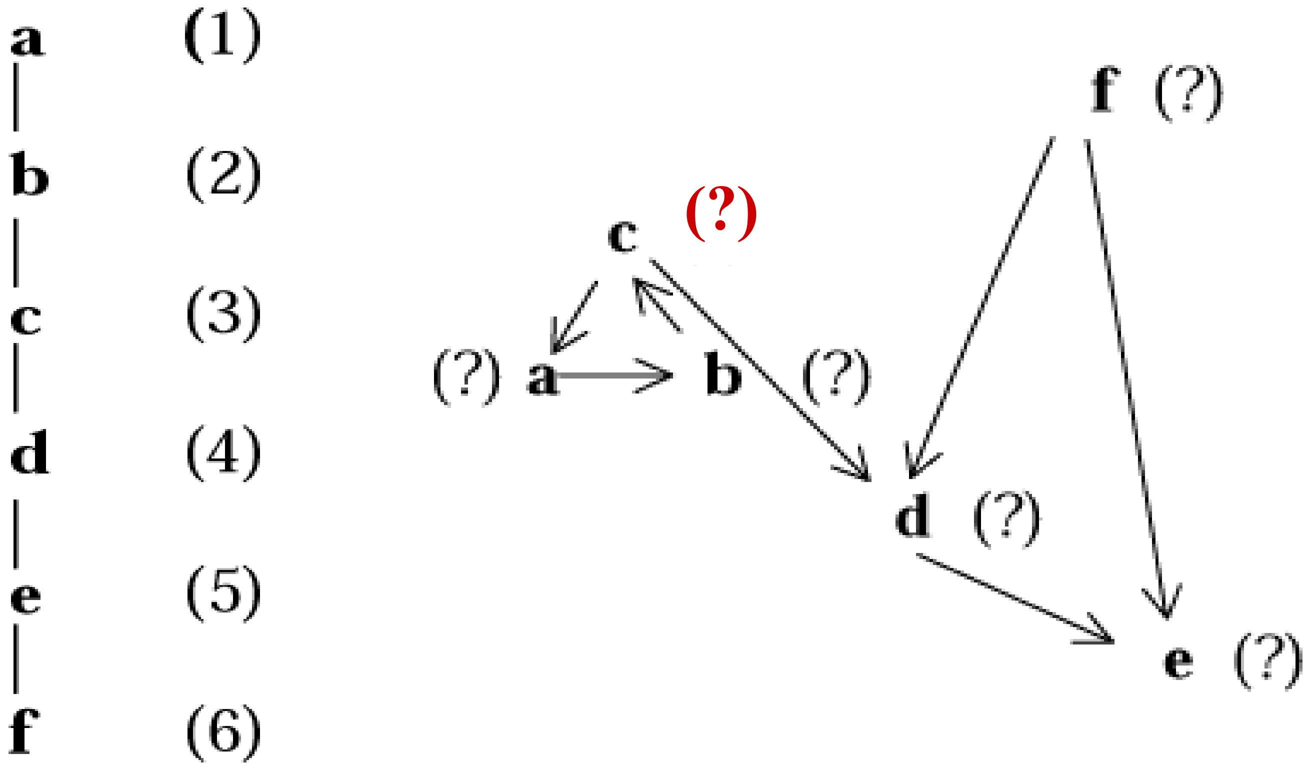
Plurality Scoring rule:

- 1^{st} ranked candidate gets 1 point,
- other candidates get 0 points.

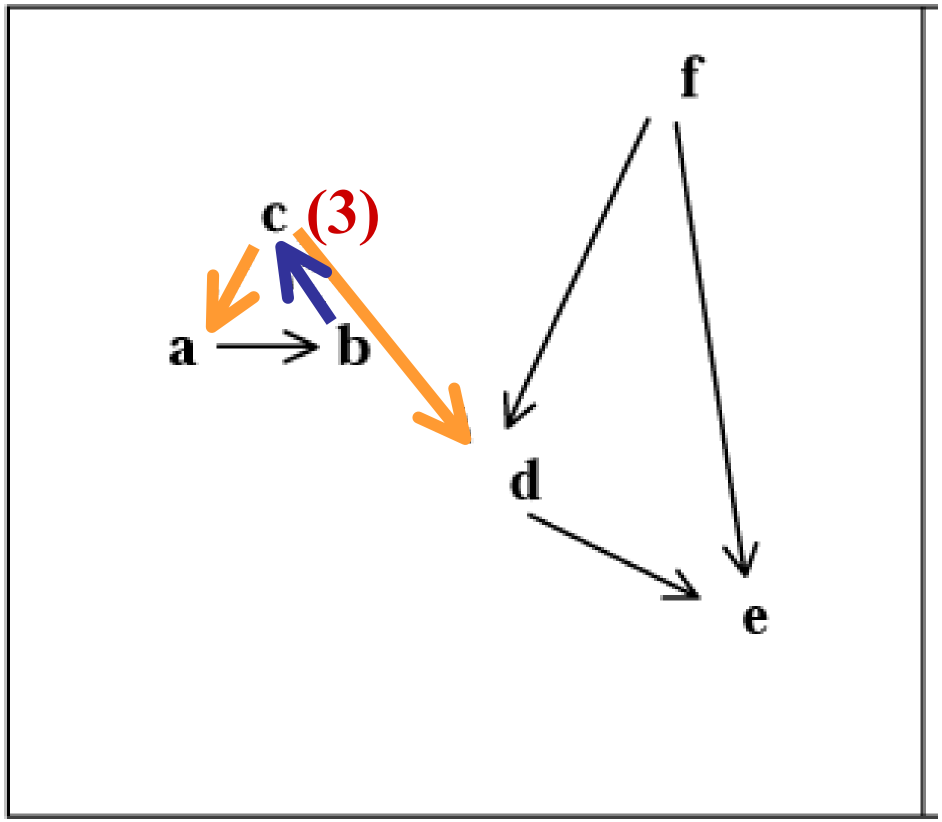
How about a General Concept of Scoring Rules?

Let’s generalize the concept of Ranks from Linear Orders to Arbitrary Finite Binary Relations

Generalizing ranks beyond linear orders



In-degree, Out-degree and Differential of an object



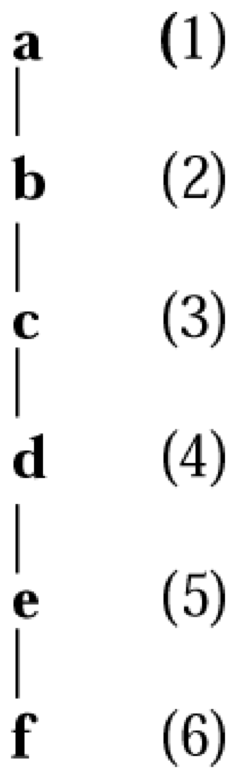
In-degree (c) = 1

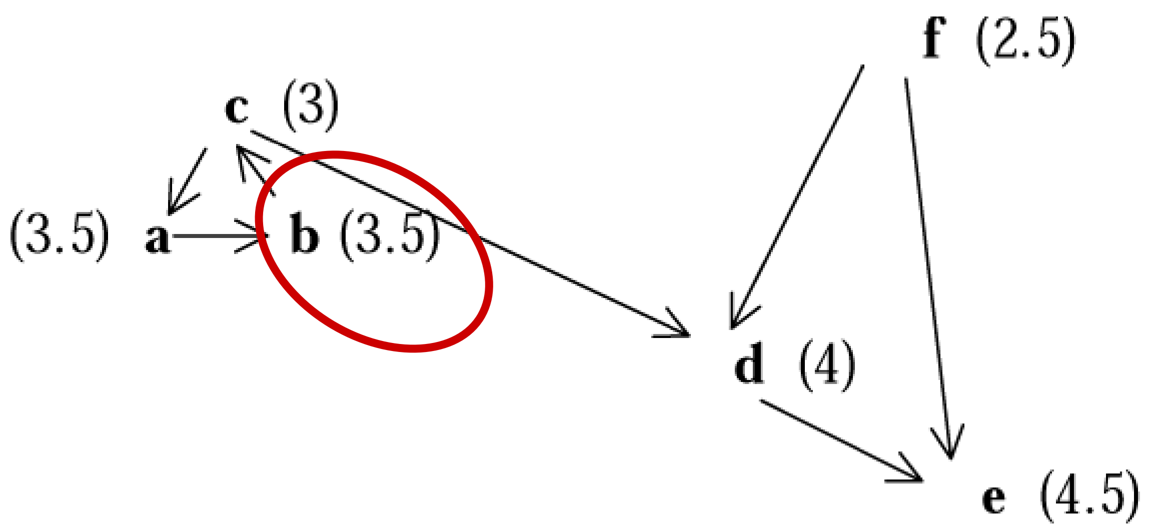
Out-degree (c) = **2**

Δ(c) = Differential (c) =
In-degree (c) - **Out-degree** (c) = **-1**

$$\textbf{Rank (c)} = \frac{n+1+\Delta(c)}{2}$$

Generalizing ranks beyond linear orders





$$\text{Rank}(c) = \frac{n+1+\Delta(c)}{2}$$

Some properties of generalized rank

- Average generalized rank is $\frac{\mathbf{n+1}}{\mathbf{2}}$
- Minimal possible rank is **1**
- Maximal possible generalized rank is **n**

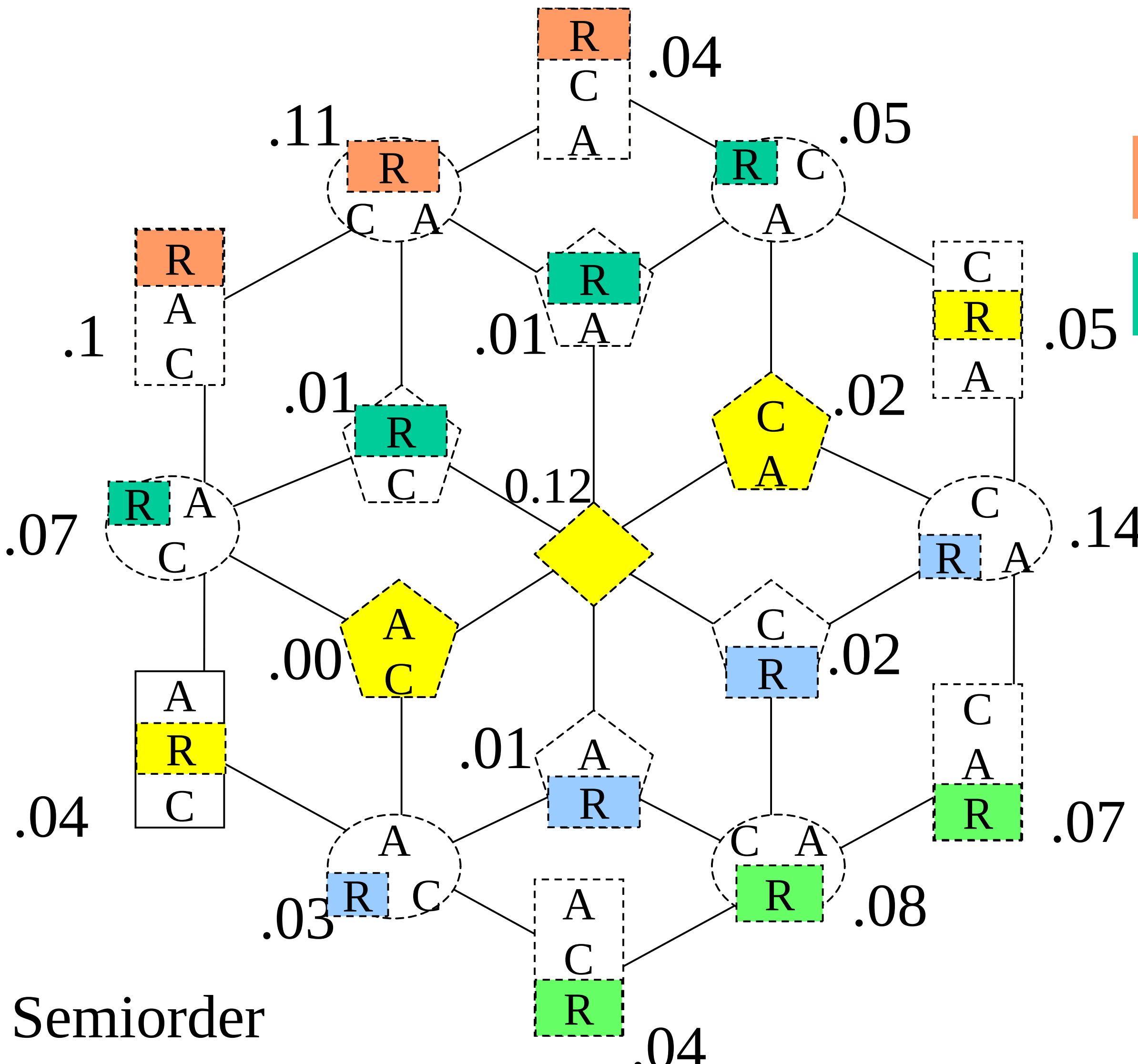
Borda Scoring rule:

(for n=3 candidates)

- 1^{st} ranked candidate gets 2 points,
- candidate with rank = 1.5 gets 1.5 points,
- 2^{nd} ranked candidate gets 1 point,
- candidate with rank = 2.5 gets 0.5 points,
- 3^{rd} ranked candidate gets 0 point.

In general, the i^{th} ranked among n candidates gets $n-i$ points.

Borda scores derived from semiorder probabilities



Borda (**R**) =

2*(.1+.11+.04) +

1.5*(.07+.01+.01+.05)+

1*(.04+.12+.02+.05) +

.5*(.03+.01+.02+.1) +

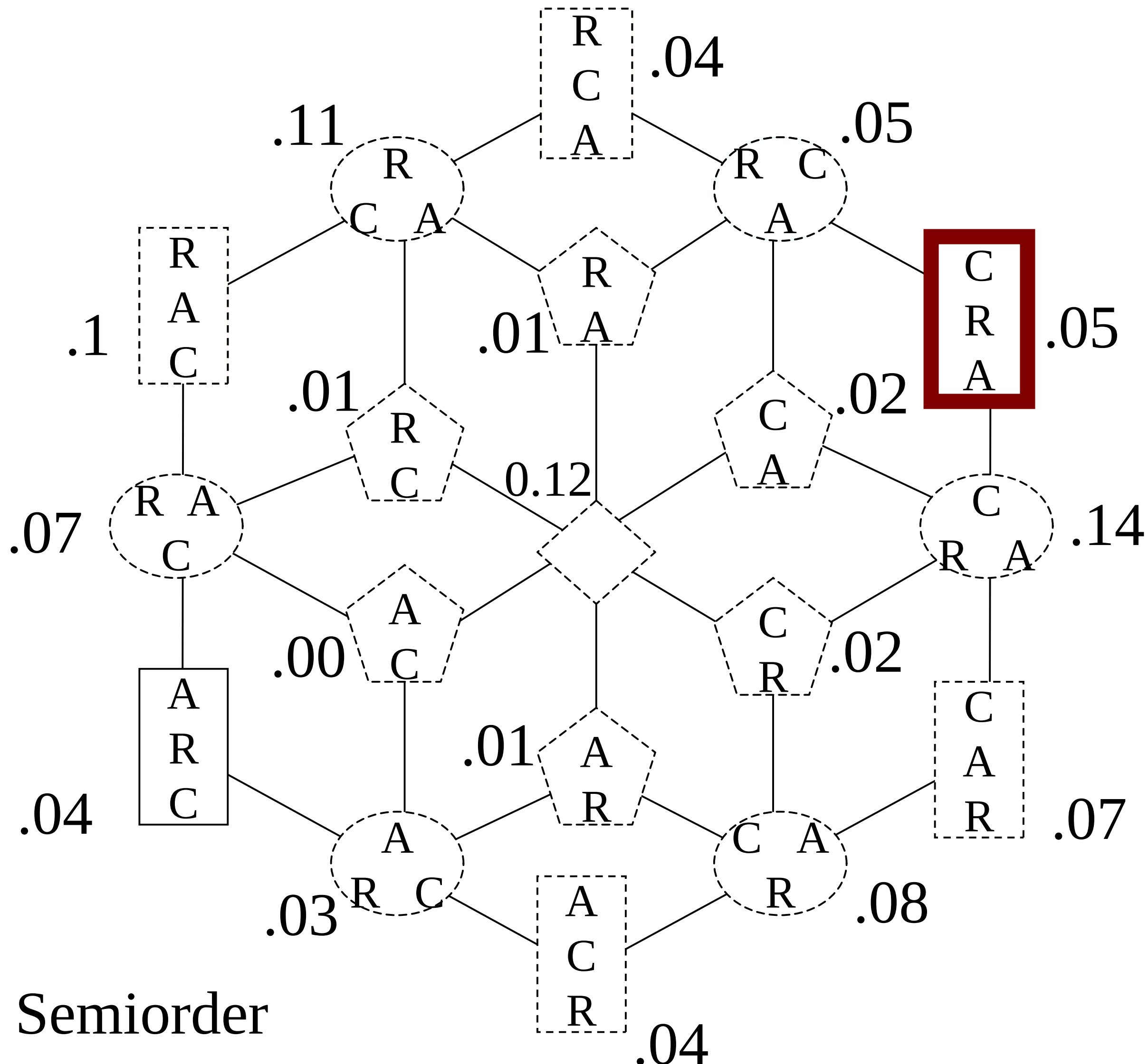
0*(.04+.08+.07) =

= **1.02**

Semiorder
Threshold=10

1980 NES

Borda scores derived from semiorder probabilities



Borda (R) =
1.02

Borda (A) =
0.92

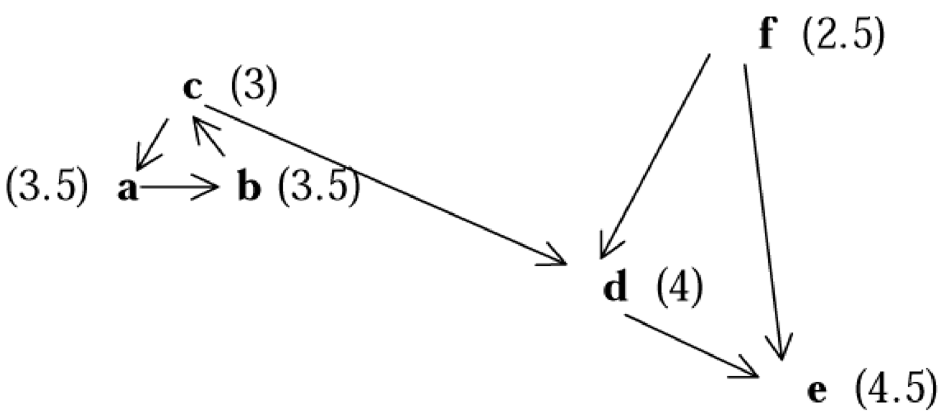
Borda (C) =
1.07

Semiorder
Threshold=10

1980 NES

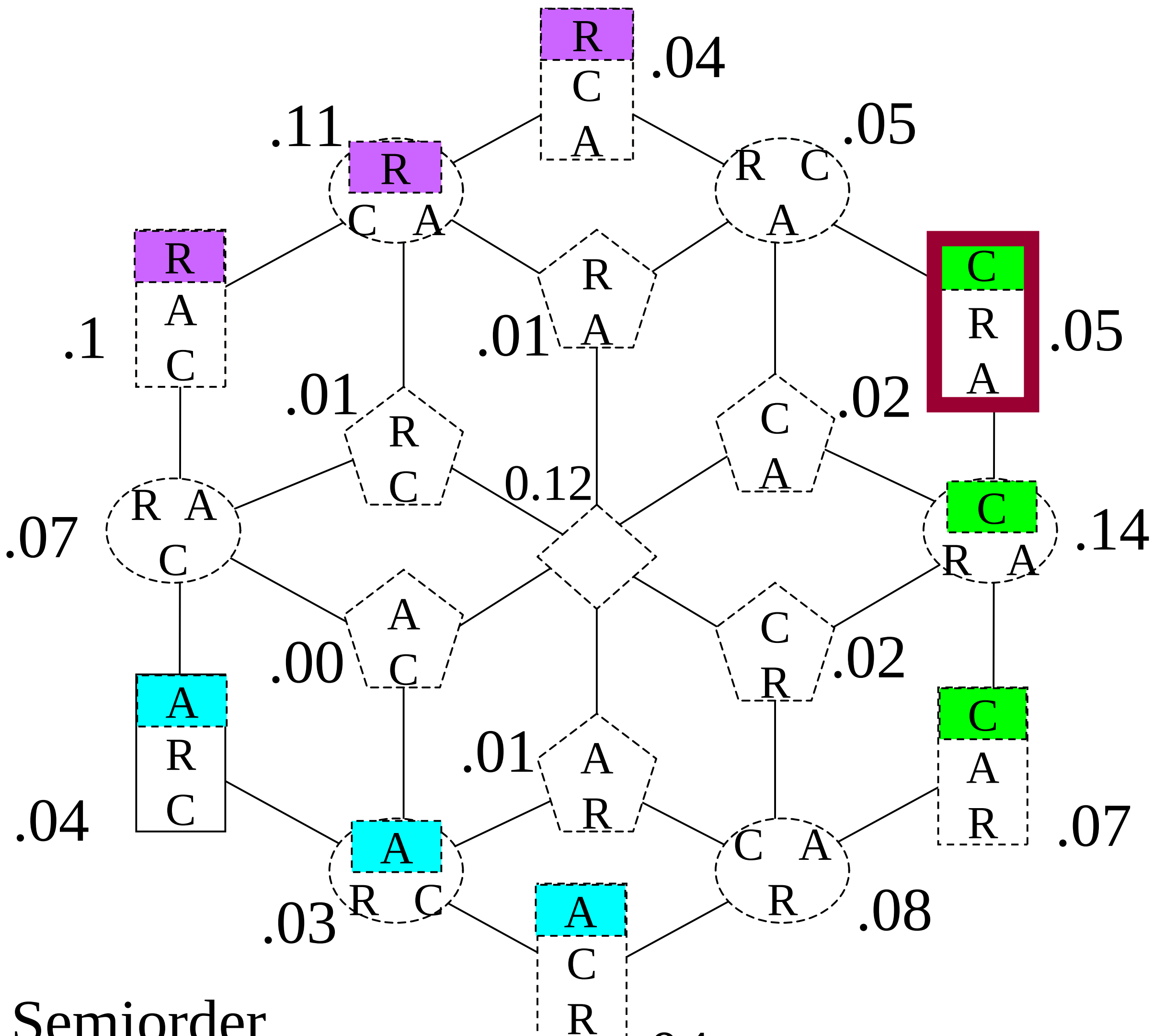
Plurality Scoring rule:
(for n candidates)

- 1st ranked candidate gets 1 point,
- other candidates get 0 points.



Note: If no (single) candidate has rank equal to 1, a given ballot is effectively ignored

Plurality scores derived from semiorder probabilities



Plurality (R)=

1*(.1+.11+.04) =

= 0.25

Plurality (A)=

= 0.11

Plurality (C)=

= .26

Semiorder
Threshold=10

1980 NES

Empirical example: NES thermometer scores

Social ordering depends on:

- model of preferences

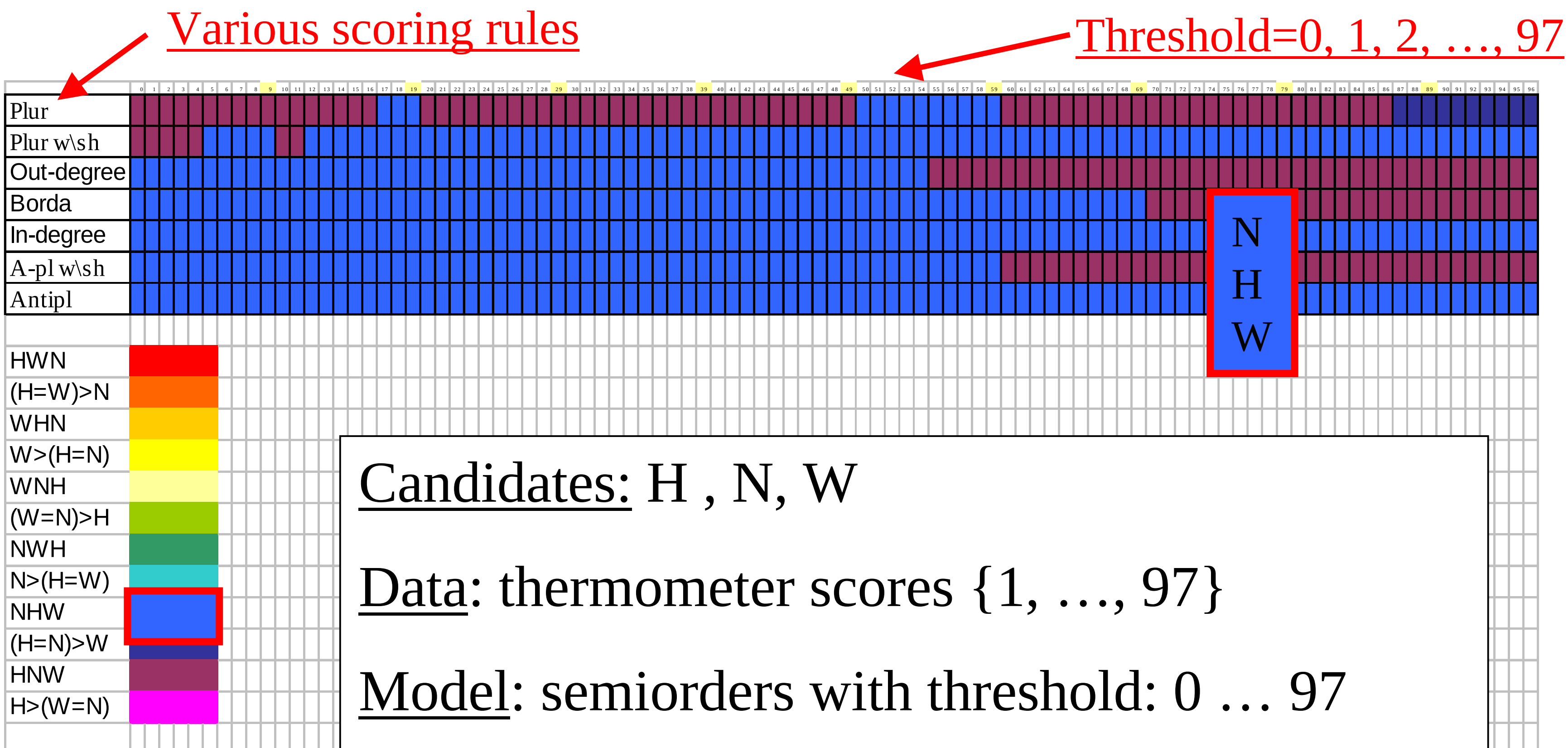
[translation of raw data into binary relations]

- social choice function

[Majority, Borda, Plurality, others]

- data

Empirical example: 1968 NES



Candidates: H , N, W

Data: thermometer scores {1, …, 97}

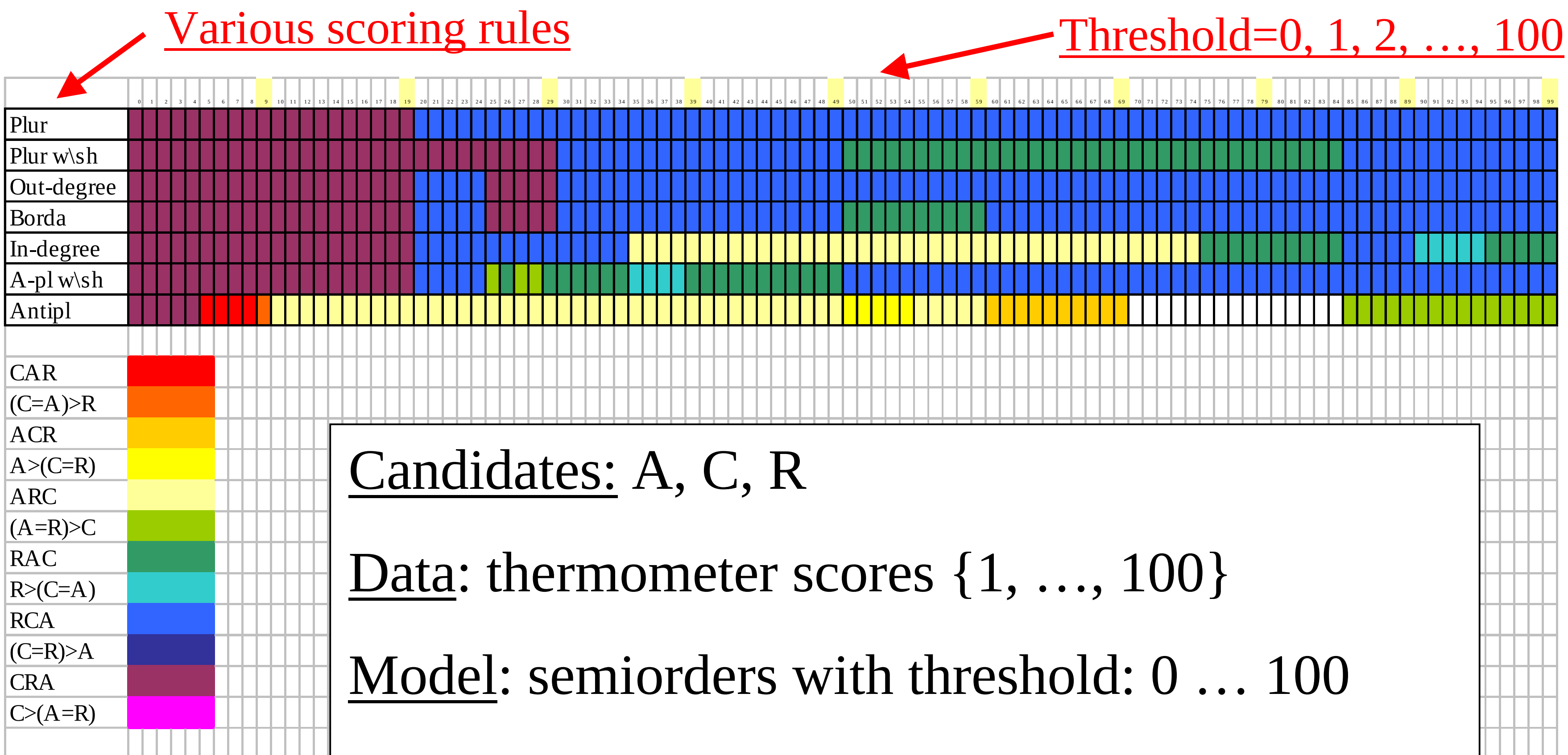
Model: semiorders with threshold: 0 … 97

Scoring rules: Plurality, Antiplurality (with or without sharing), Borda, In-degree, Out-degree

ANES Strict **Majority** Social Welfare Orders

Year	**Threshold**	**SWO**
1968	0, …, 96	Nixon Humphrey Wallace

Empirical example: 1980 NES



Candidates: A, C, R

Data: thermometer scores {1, …, 100}

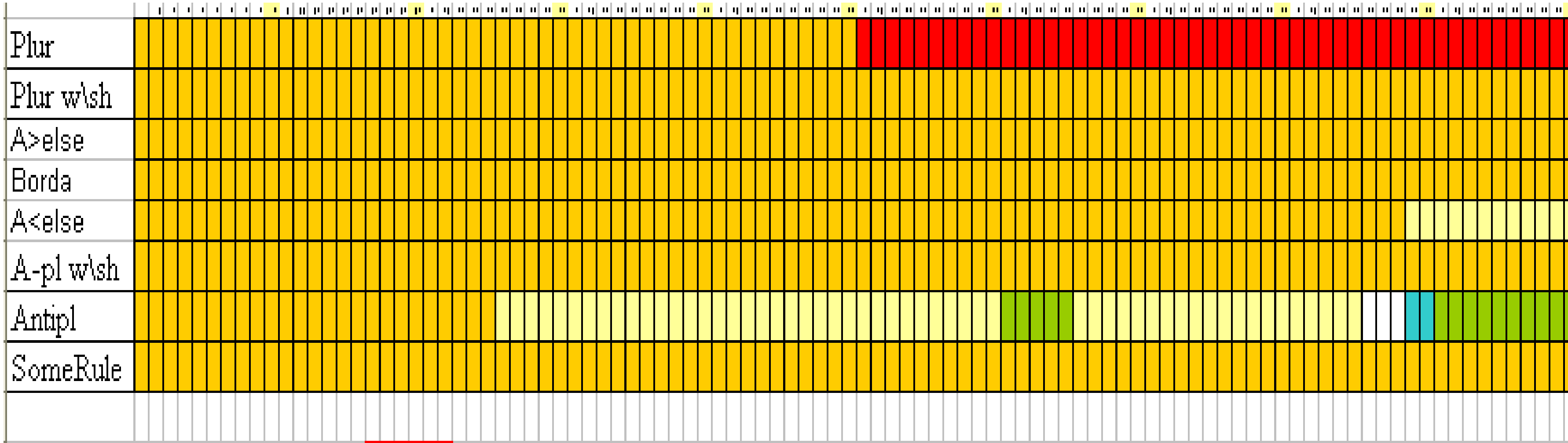
Model: semiorders with threshold: 0 … 100

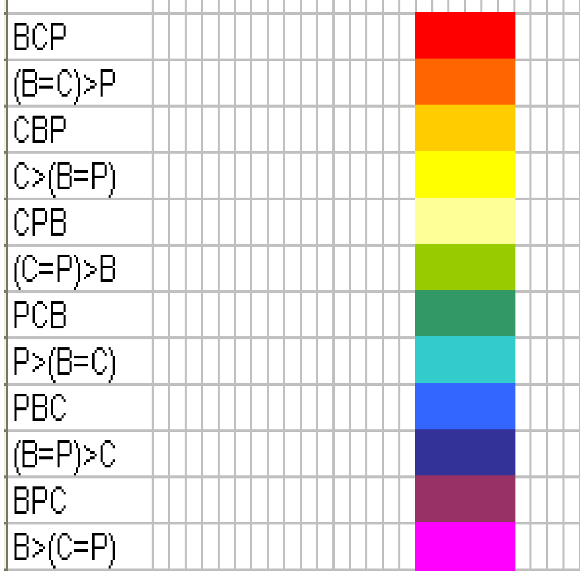
Scoring rules: Plurality, Antiplurality (with or without sharing), Borda, In-degree, Out-degree

ANES Strict **Majority** Social Welfare Orders

Year	Threshold	SWO
1980	0, …, 29	Carter Reagan Anderson
	30, …, 99	Reagan Carter Anderson

Empirical example: 1992 NES





Candidates: B, C, P

Data: thermometer scores {1, …, 100}

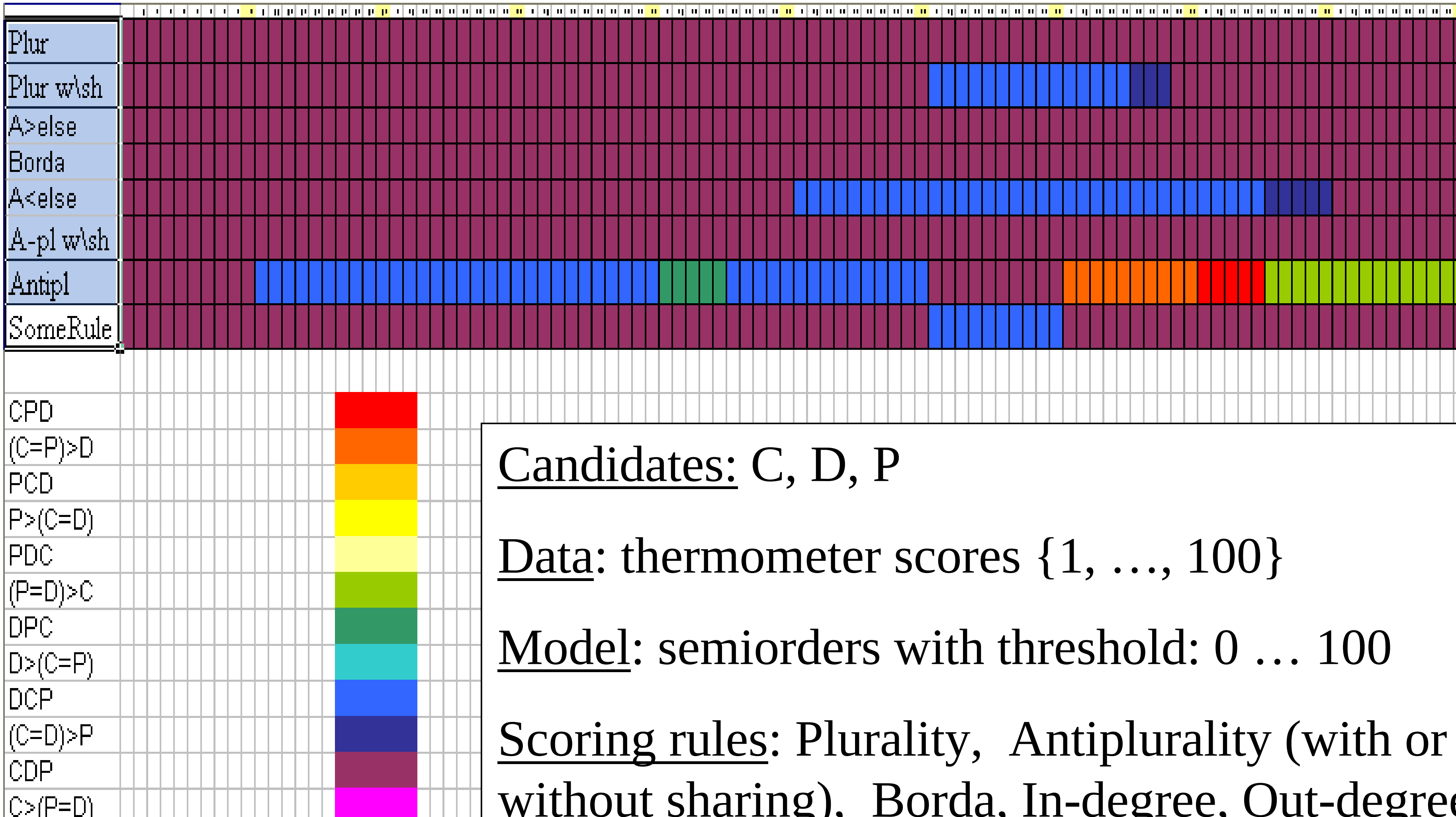
Model: semiorders with threshold: 0 … 100

Scoring rules: Plurality, Antiplurality (with or without sharing), Borda, In-degree, Out-degree

ANES Strict **Majority** Social Welfare Orders

Year	**Threshold**	**SWO**
1992	0, …, 99	Clinton Bush Perot

Empirical example: 1996 NES



Candidates: C, D, P

Data: thermometer scores {1, …, 100}

Model: semiorders with threshold: 0 … 100

Scoring rules: Plurality, Antiplurality (with or without sharing), Borda, In-degree, Out-degree

ANES Strict **Majority** Social Welfare Orders

Year	**Threshold**	**SWO**
1996	0, …, 49 85, …, 99	Clinton Dole Perot
	50, …,84	Dole Clinton Perot

Feeling Thermometer Data:

NES Polling Data

Preferences

Probabilities over Binary Relations

Aggregation

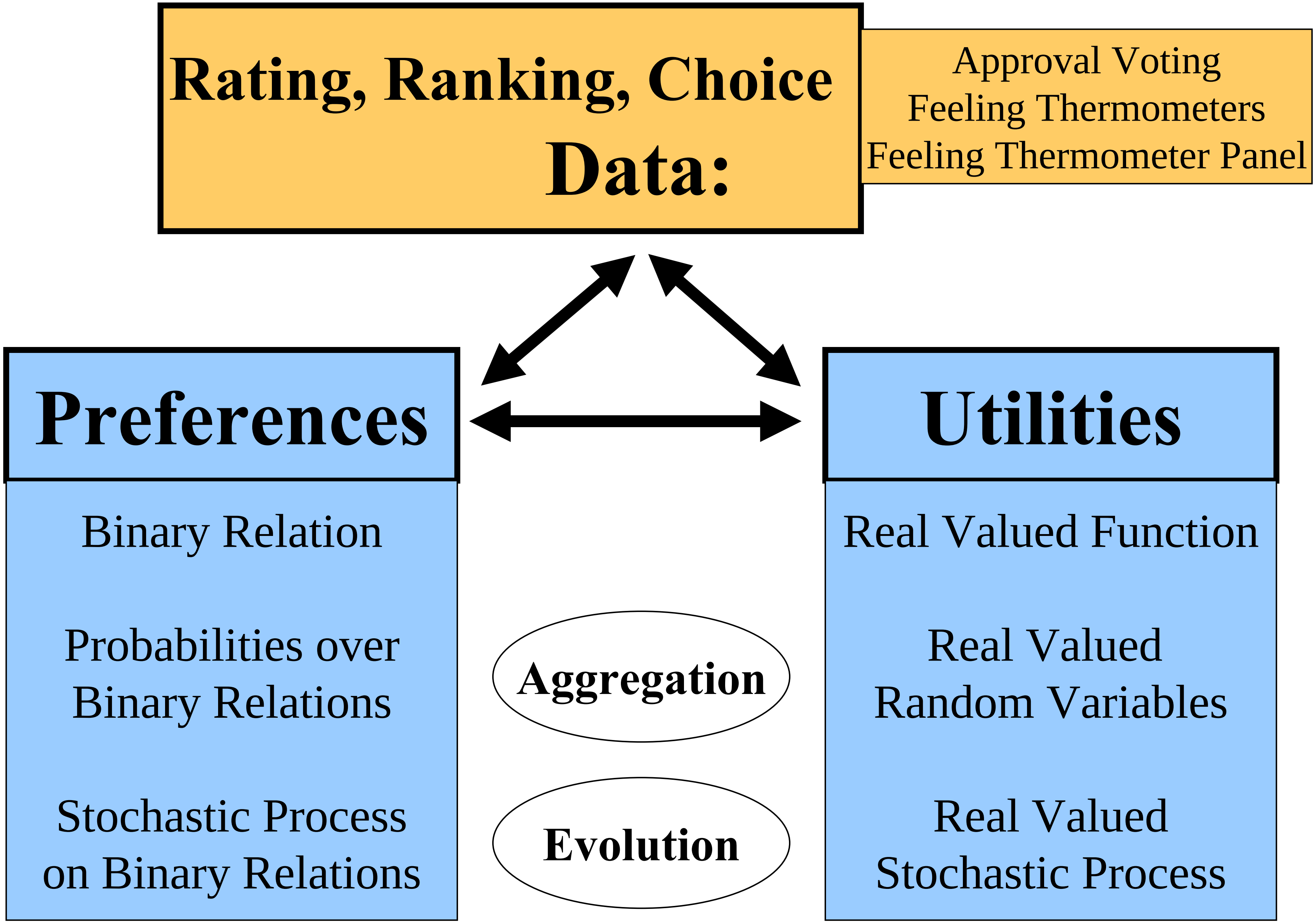
Majority (Condorcet) Winner:

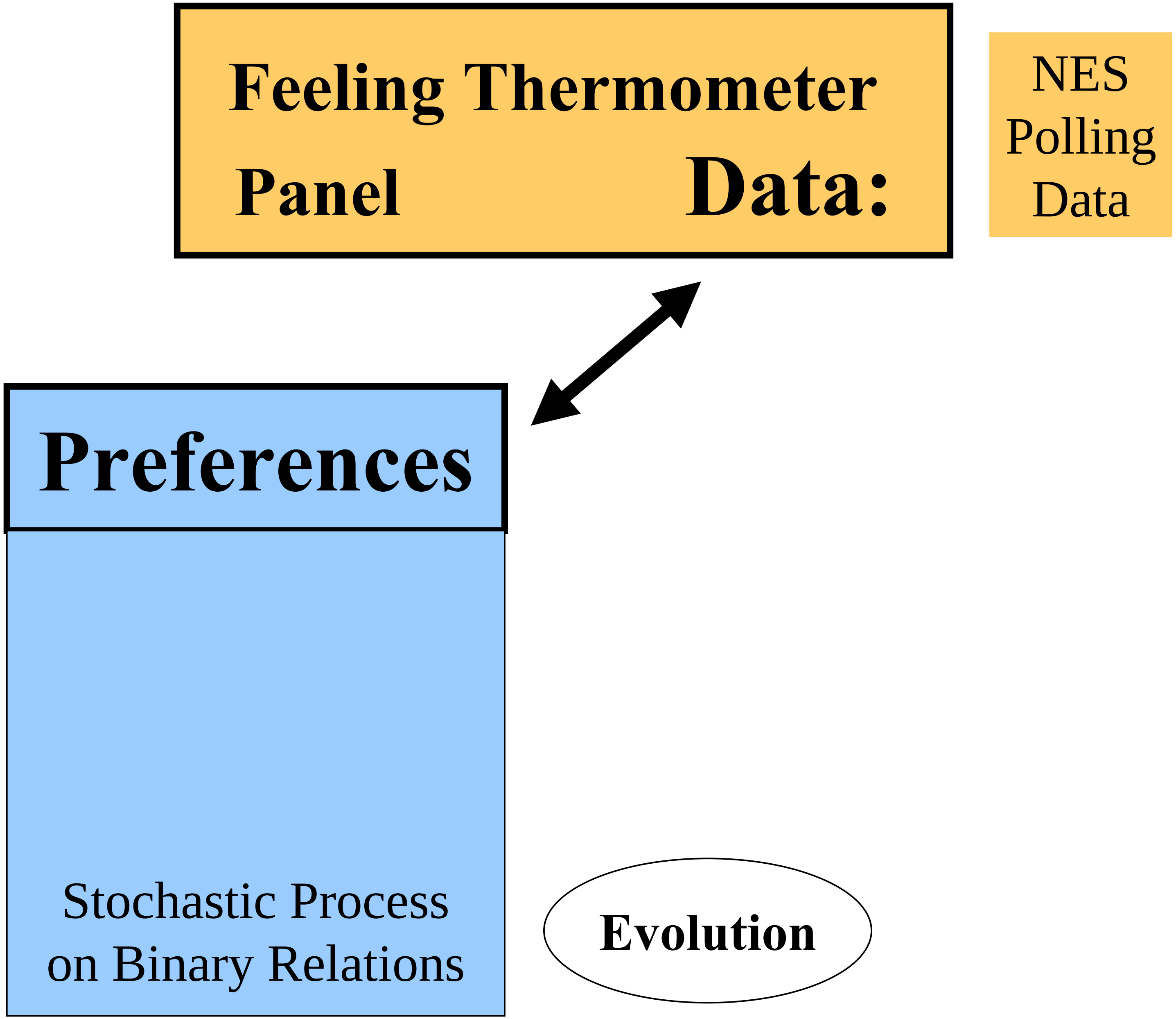
Exists

Model Dependence

Often the same as

Borda Winner and Winner by other Scoring Rules (Congruence)


Rating, Ranking, Choice
Data:
Approval Voting
Feeling Thermometers
Feeling Thermometer Panel
Preferences
Binary Relation
Probabilities over
Binary Relations
Stochastic Process
on Binary Relations
Aggregation
Evolution
Utilities
Real Valued Function
Real Valued
Random Variables
Real Valued
Stochastic Process


Feeling Thermometer
Panel
Data:
NES
Polling
Data
Preferences
Stochastic Process
on Binary Relations
Evolution

Camaro

=

most power for the buck

Camaro

=

most power for the buck

President Bush
sent troups to Iraq

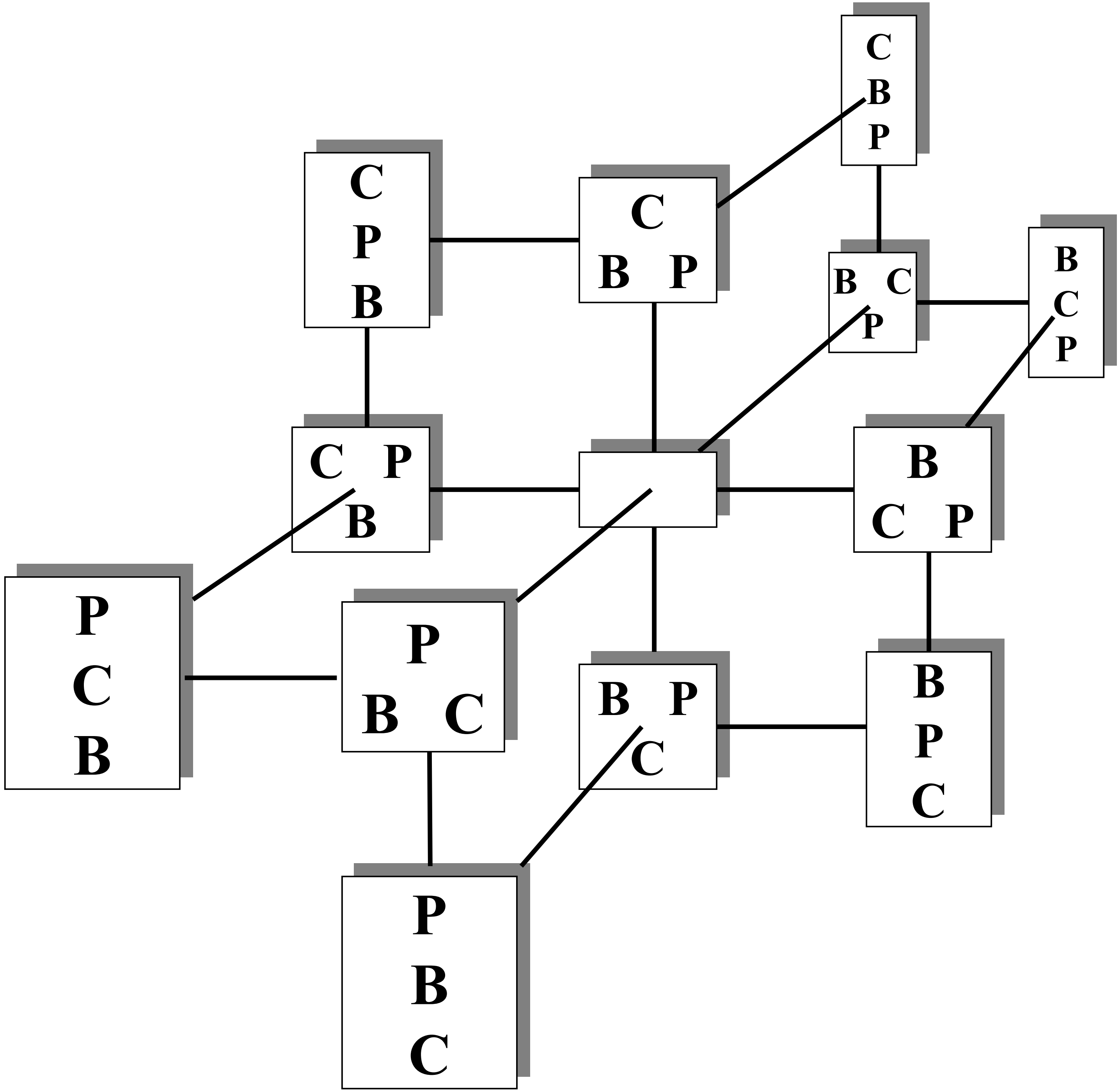
Question:

Can we infer the perceived properties of the information environment without looking at the physical information flow?

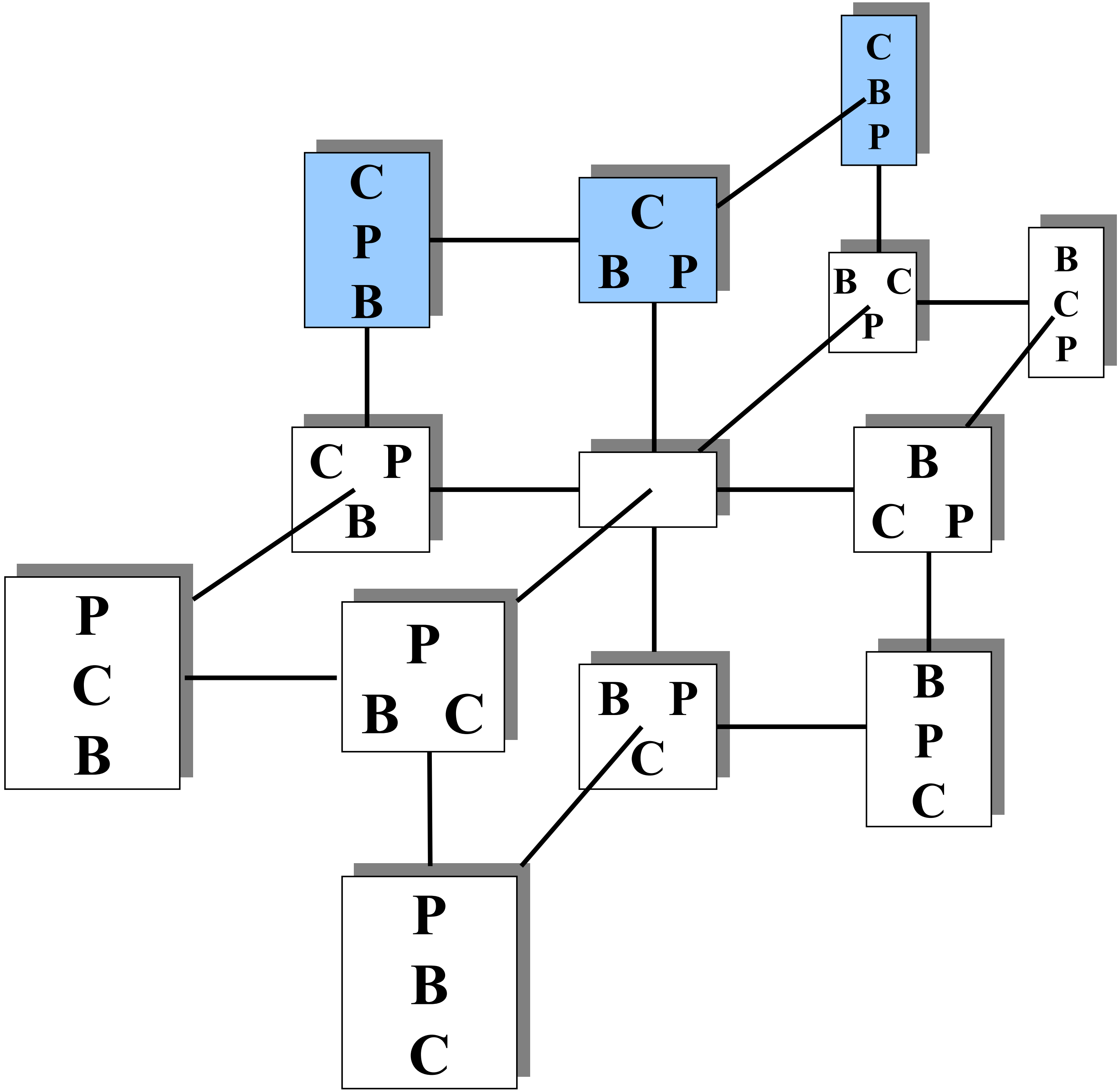
Can we analyze a Presidential Campaign without content analysis of the mass media?

Model Primitives:

- Preferences: **W**eak **O**rders



C B P
C P B
C B P
B C P
B C P
C P B
B C P
P C B
P B C
B P C
B P C
P B C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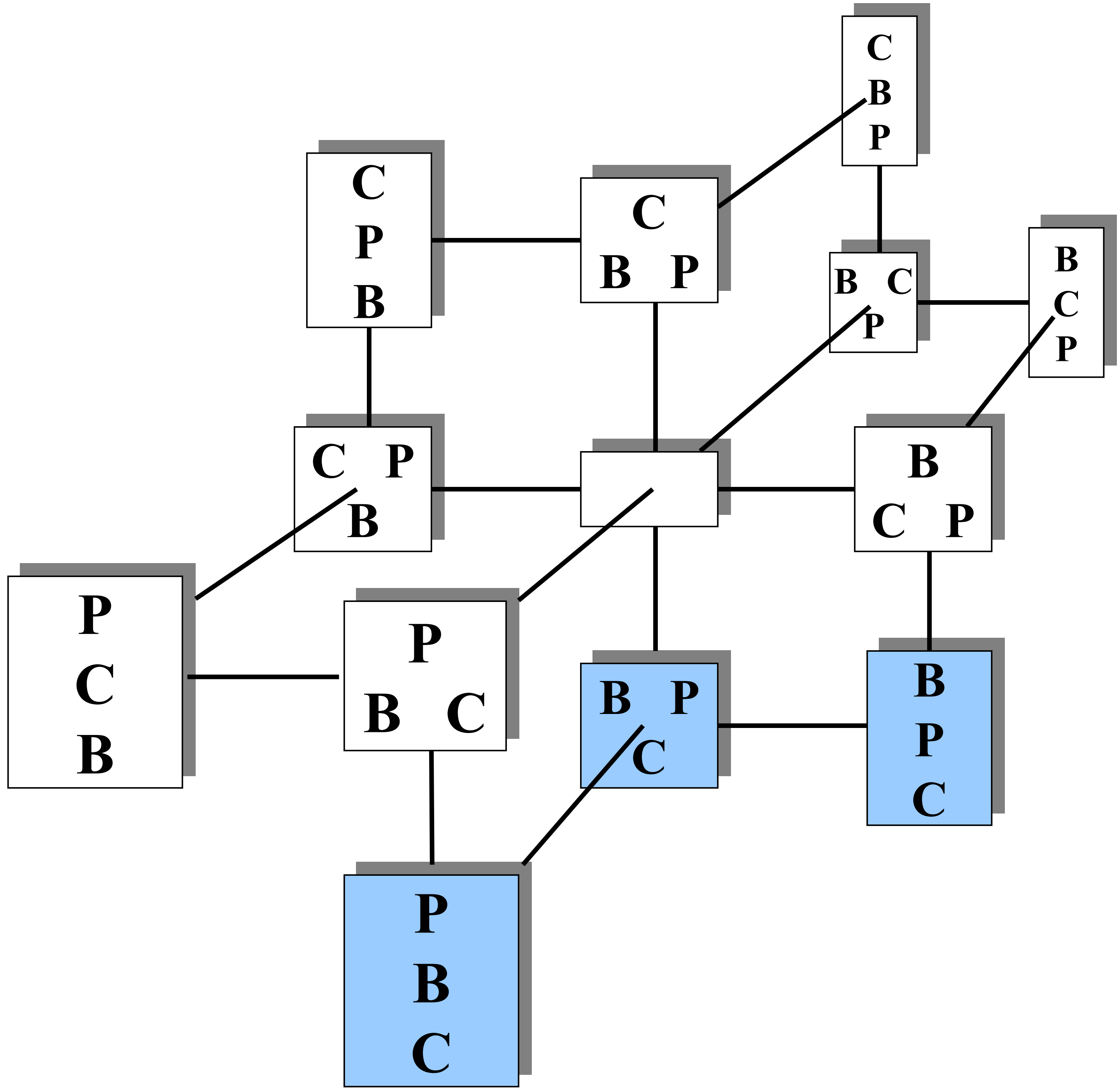

C B P
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B C P
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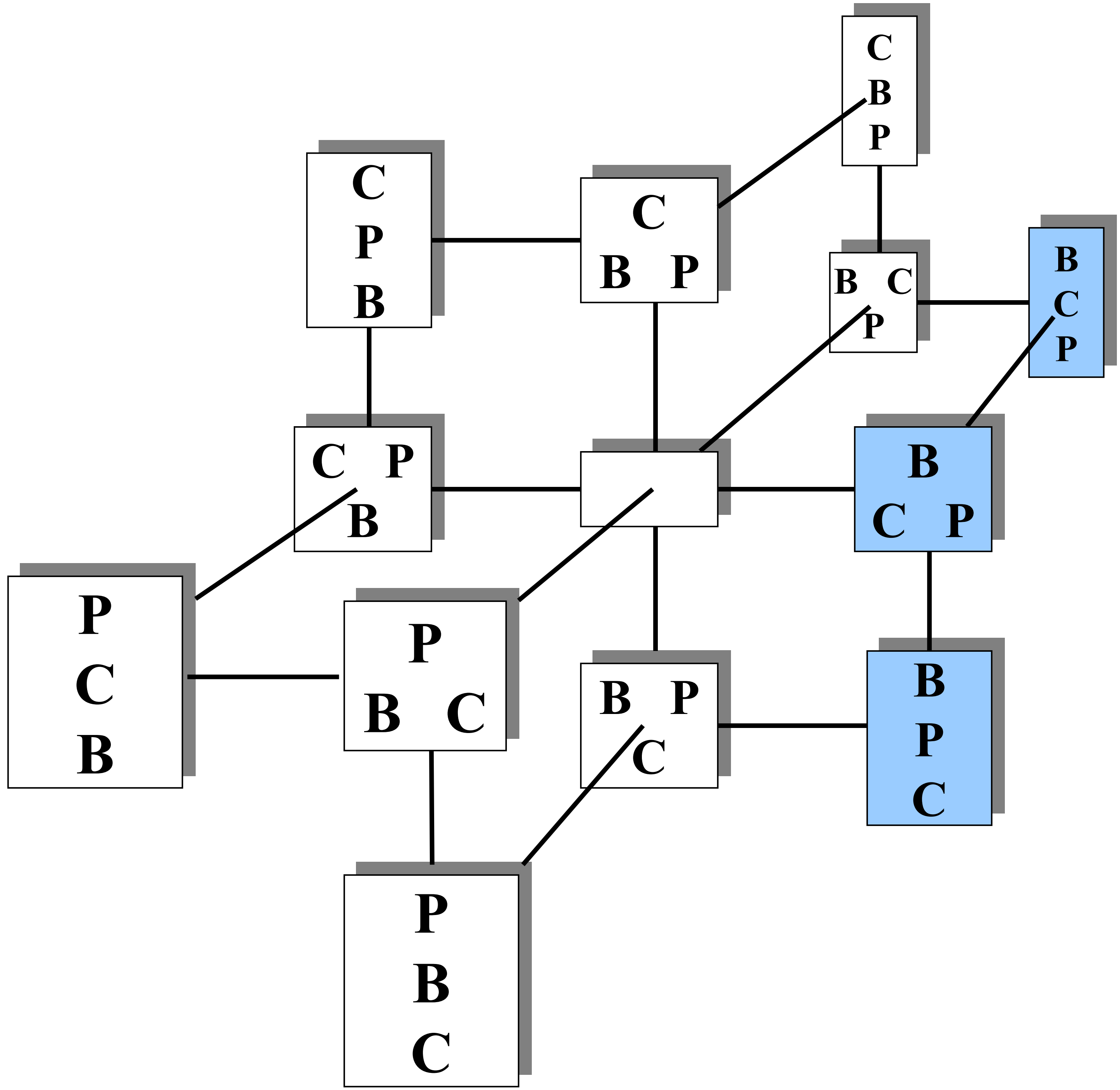
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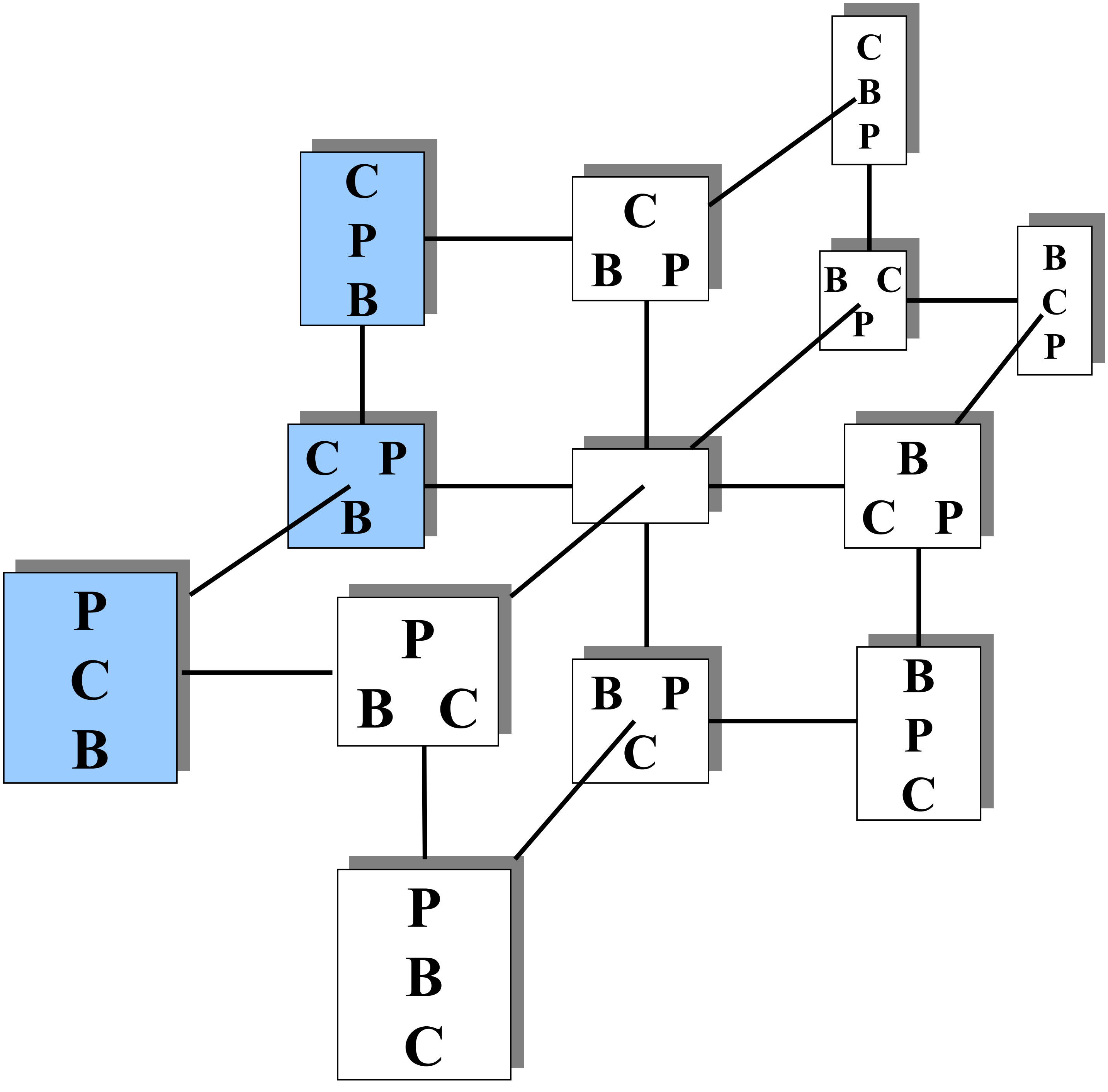

C P B
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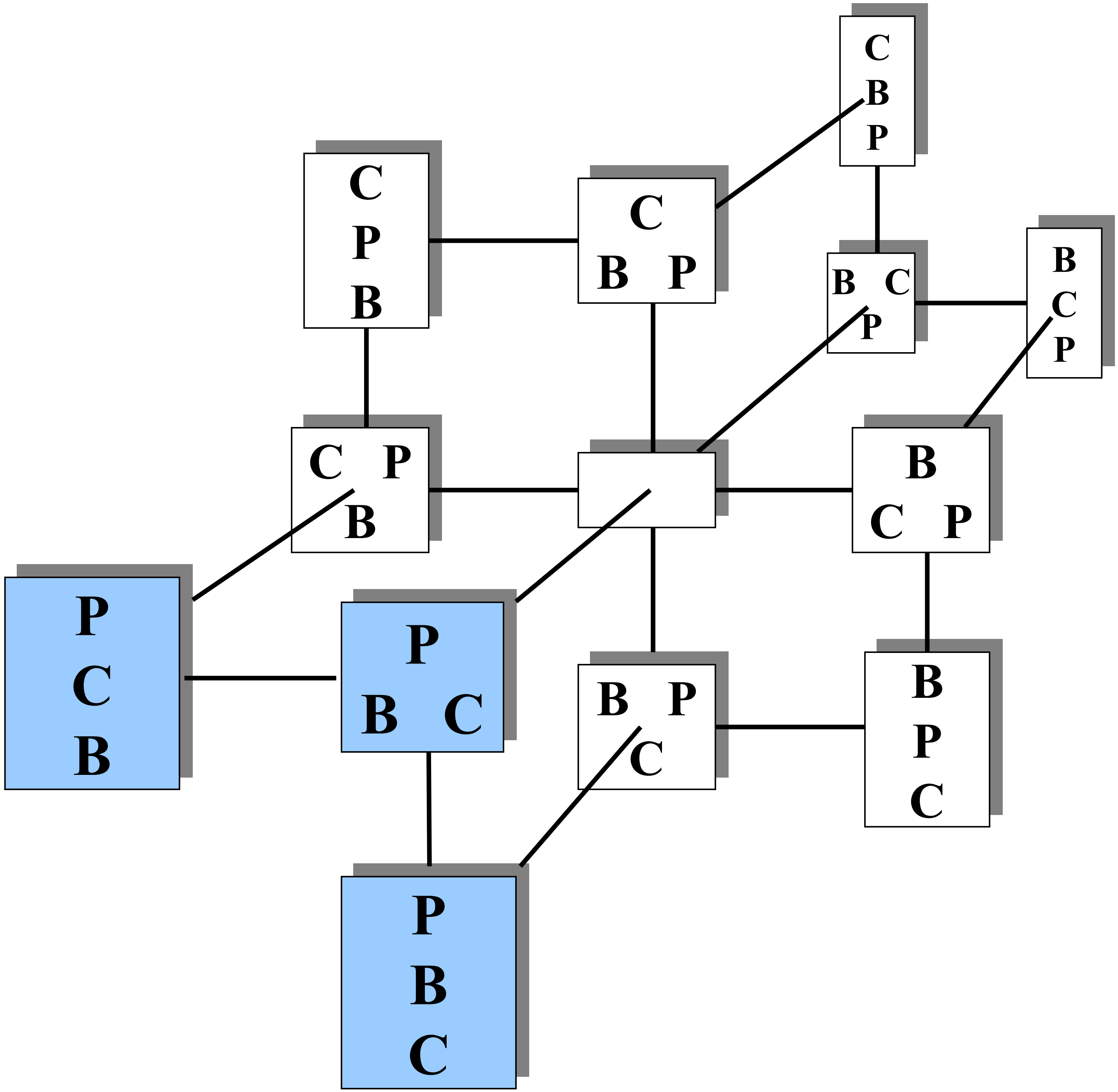
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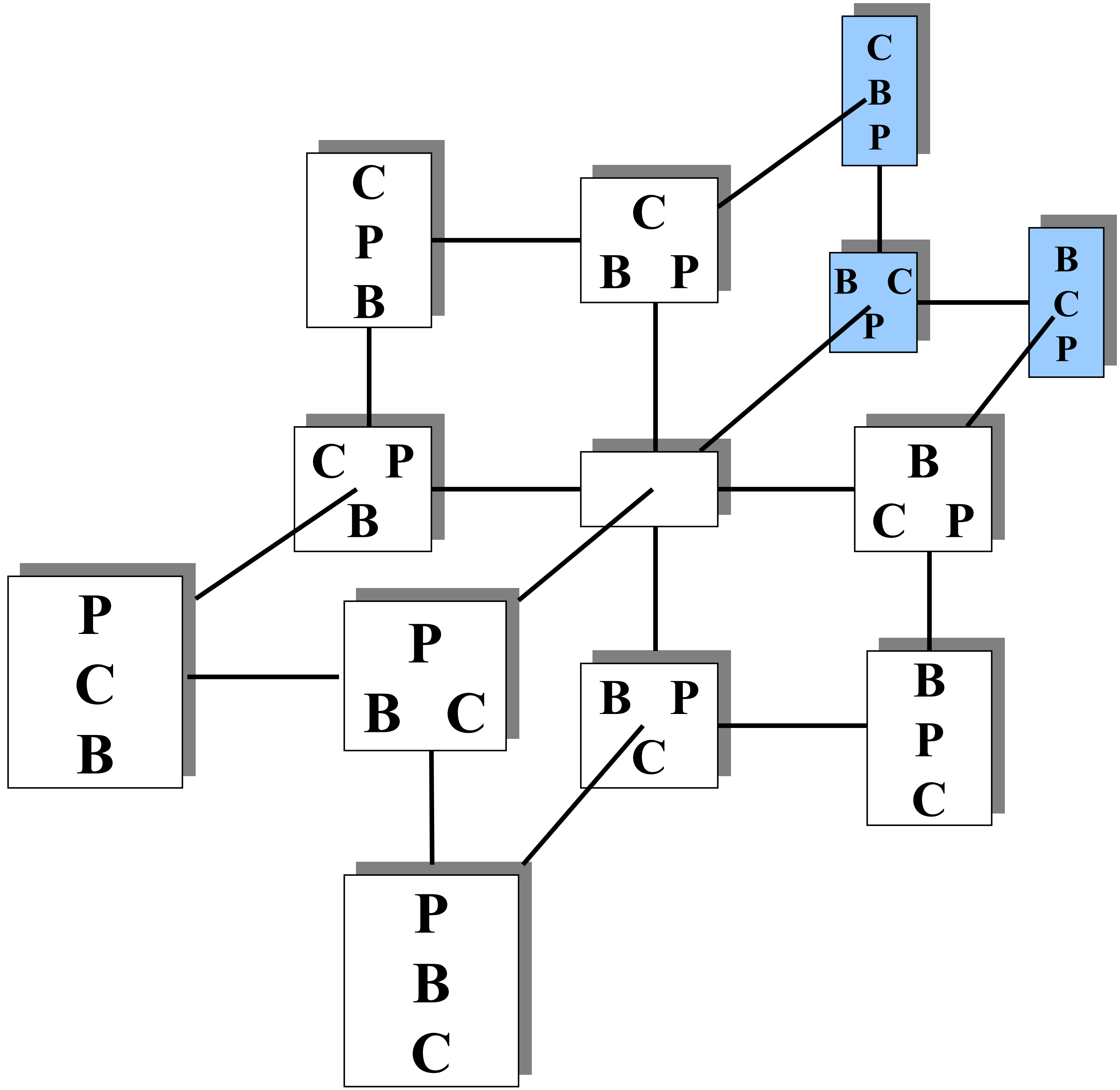
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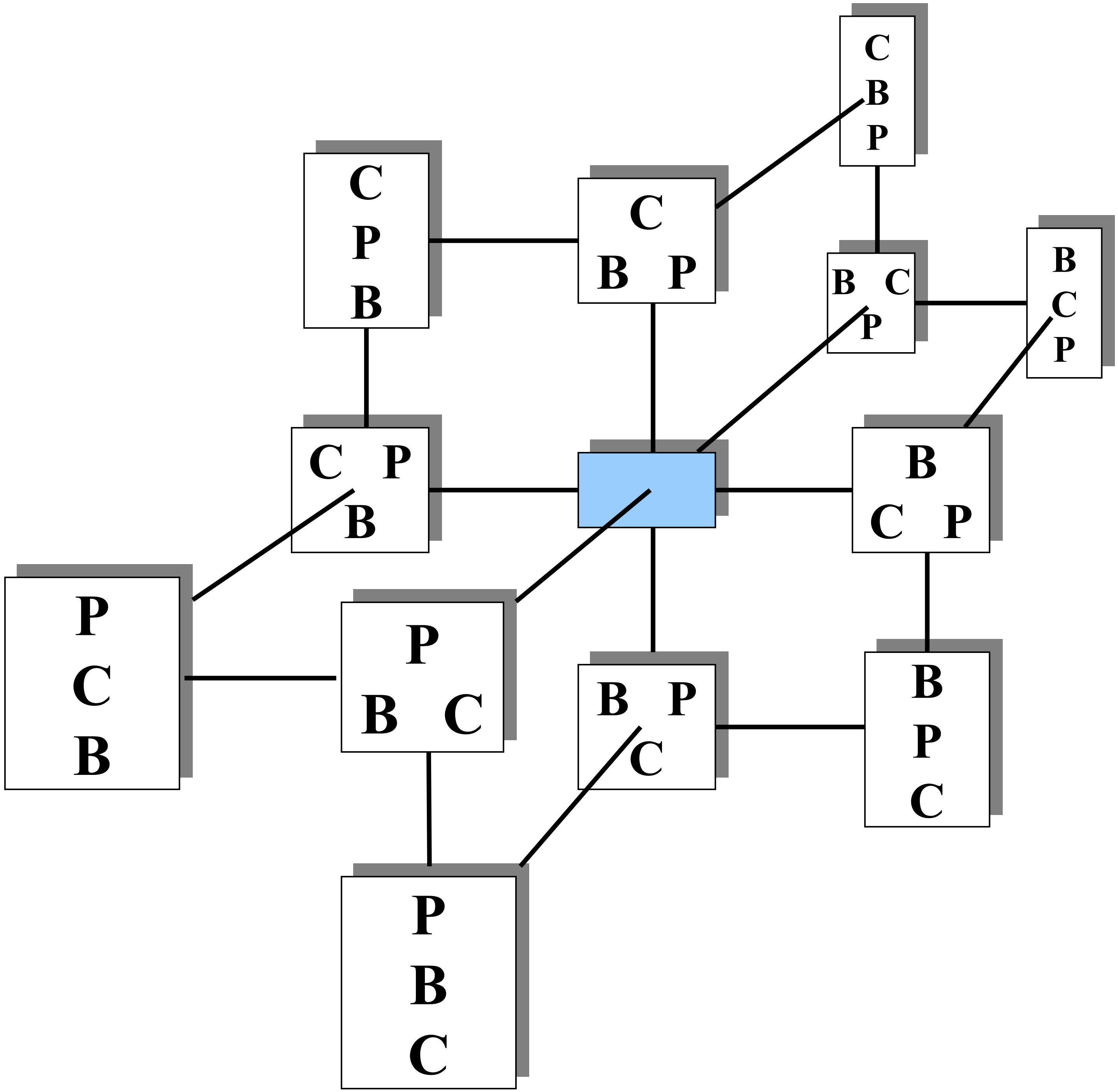
P B C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P
B C
B P
C
B
P
C
P
B
C

Model Primitives:

- Preferences: **W**eak **O**rders

    - Preference Distribution: Probability on **WO**
    - Preference Change: Transitions between **WO**

- Information: Tokens of information
- Continuous time: Stochastic process (Poisson)
- Time zero: Beginning of campaign

Information Environment:

EXTREMELY POSITIVE

moderately positive

moderately negative

EXTREMELY NEGATIVE

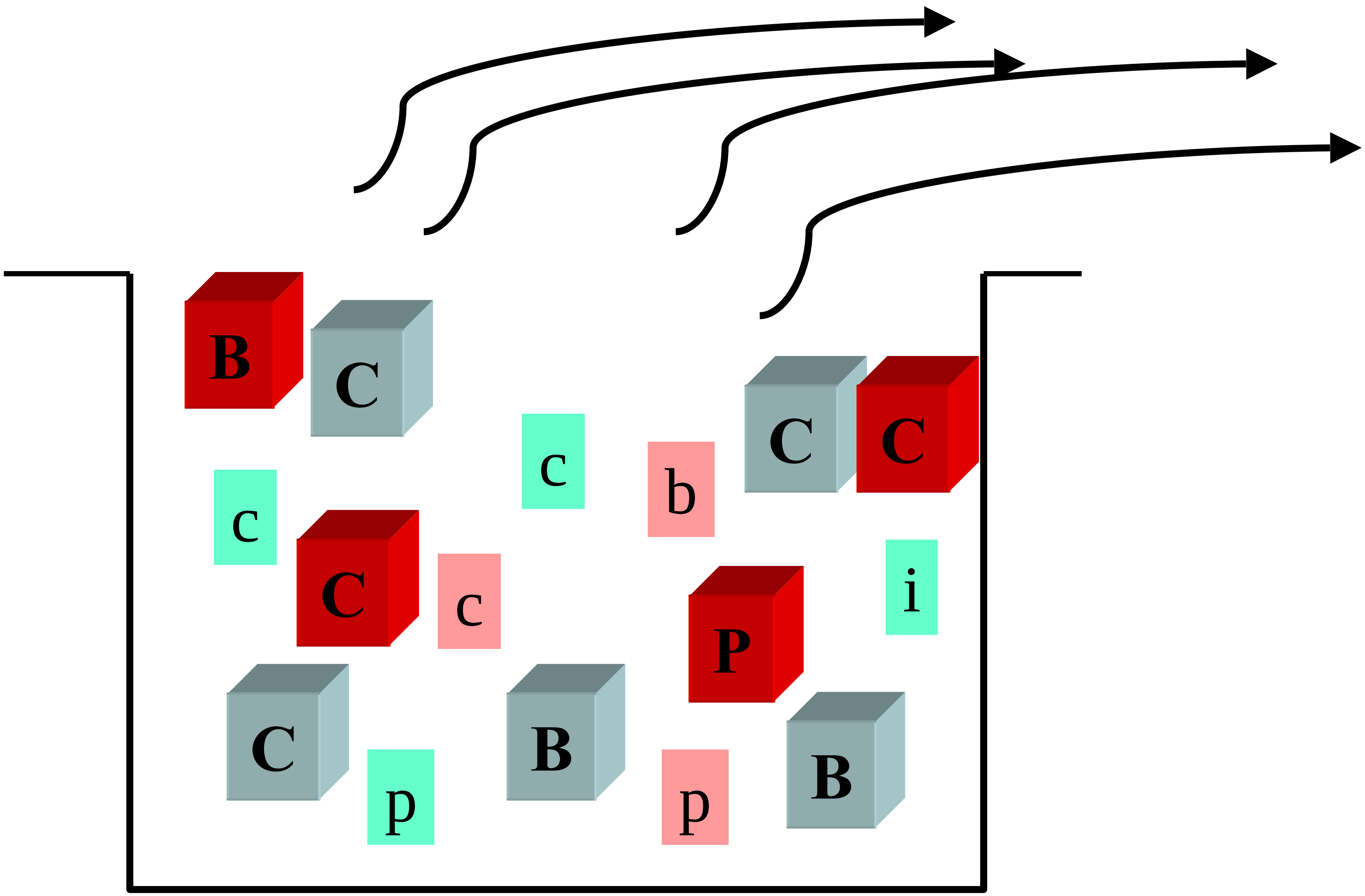
Tokens of Information:

Alternative A is the **best:** **A**

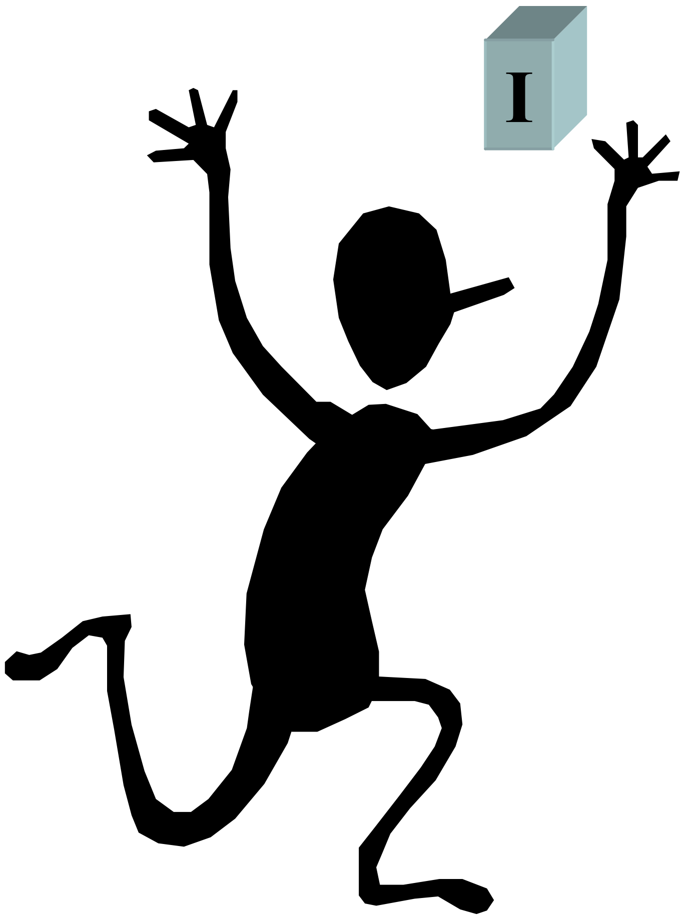
Alternative A is **not bad:** a

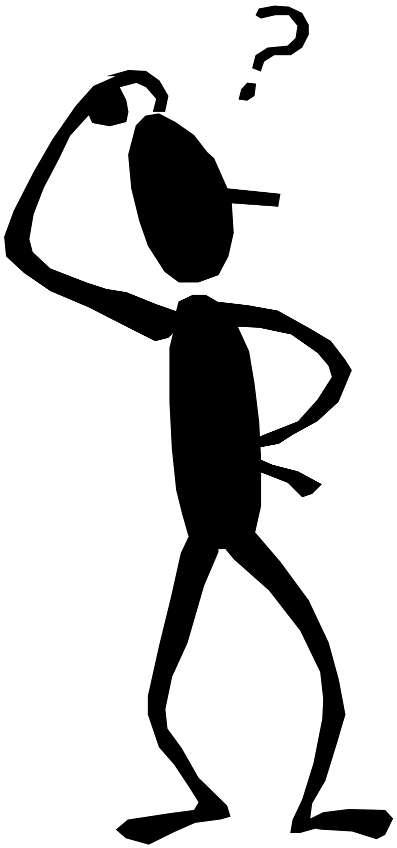
Alternative A is **not great:** a

Alternative A is the **worst:** **A**

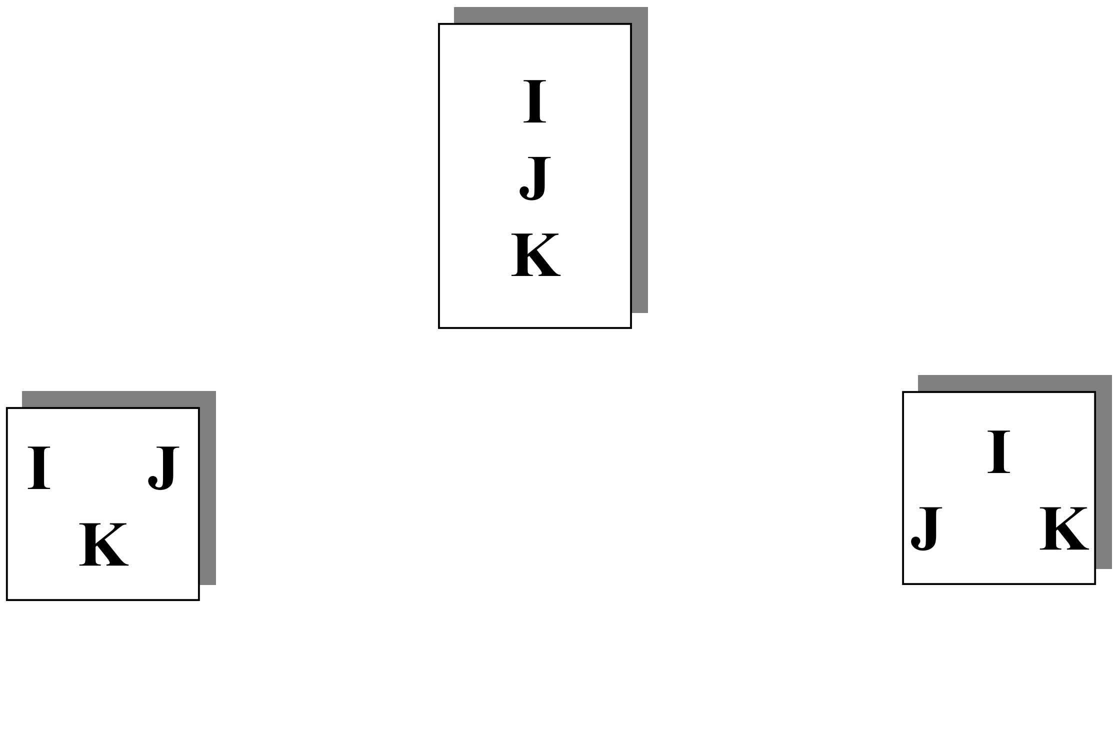


Poisson Process

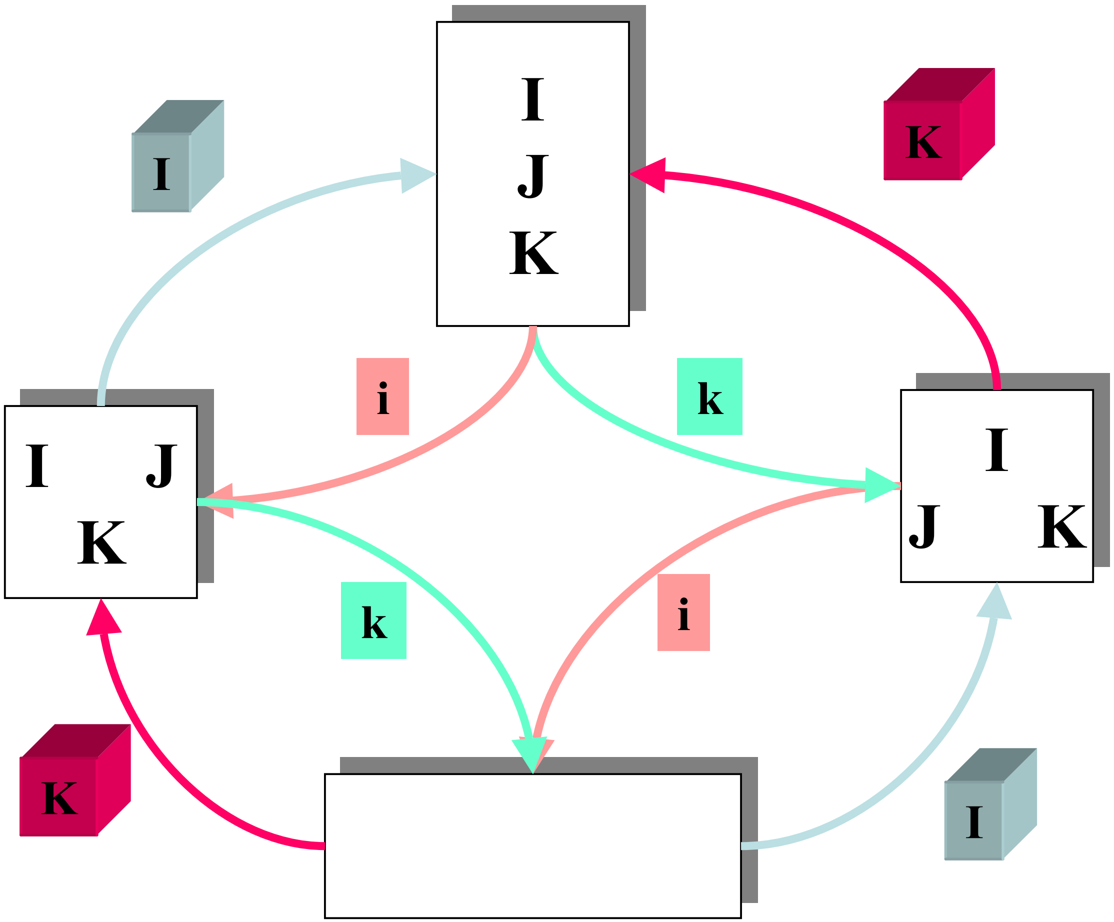

I



Operation of the Tokens:



Operation of the Tokens:



Main psychological features:

- **Extreme Information tends to move you towards an extreme state**
- **Moderate Information tends to move you towards the indifferent state**
- **Extreme information is discarded when incompatible with current extreme belief**
- **Need several steps to move from one extreme to the opposite extreme**
- **Current model has no reinforcement feature**

Let’s look into the black box

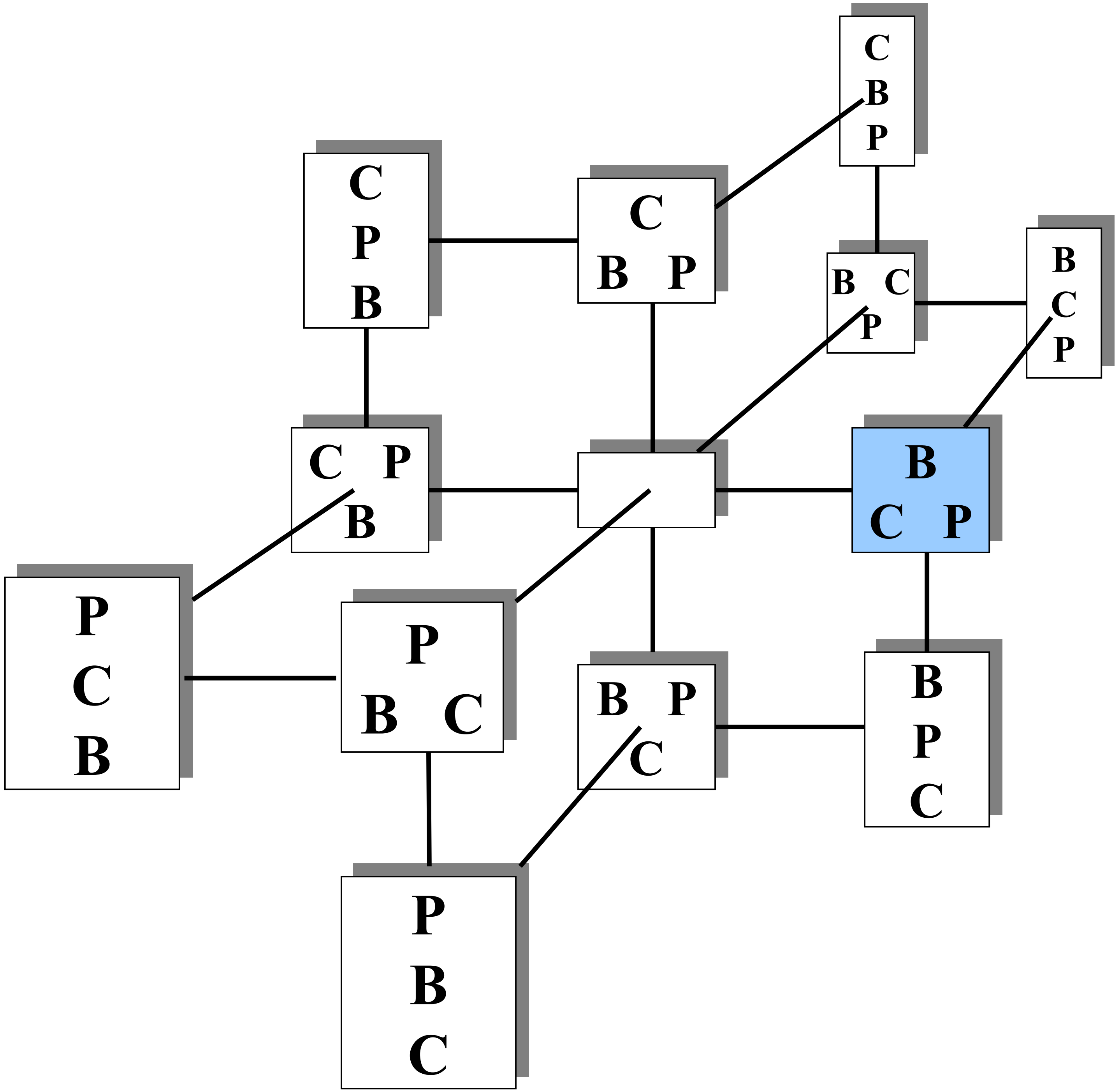
Beginning of the campaign

Republican voter

Initial Preference:

Bush is single best

Indifferent between Clinton & Perot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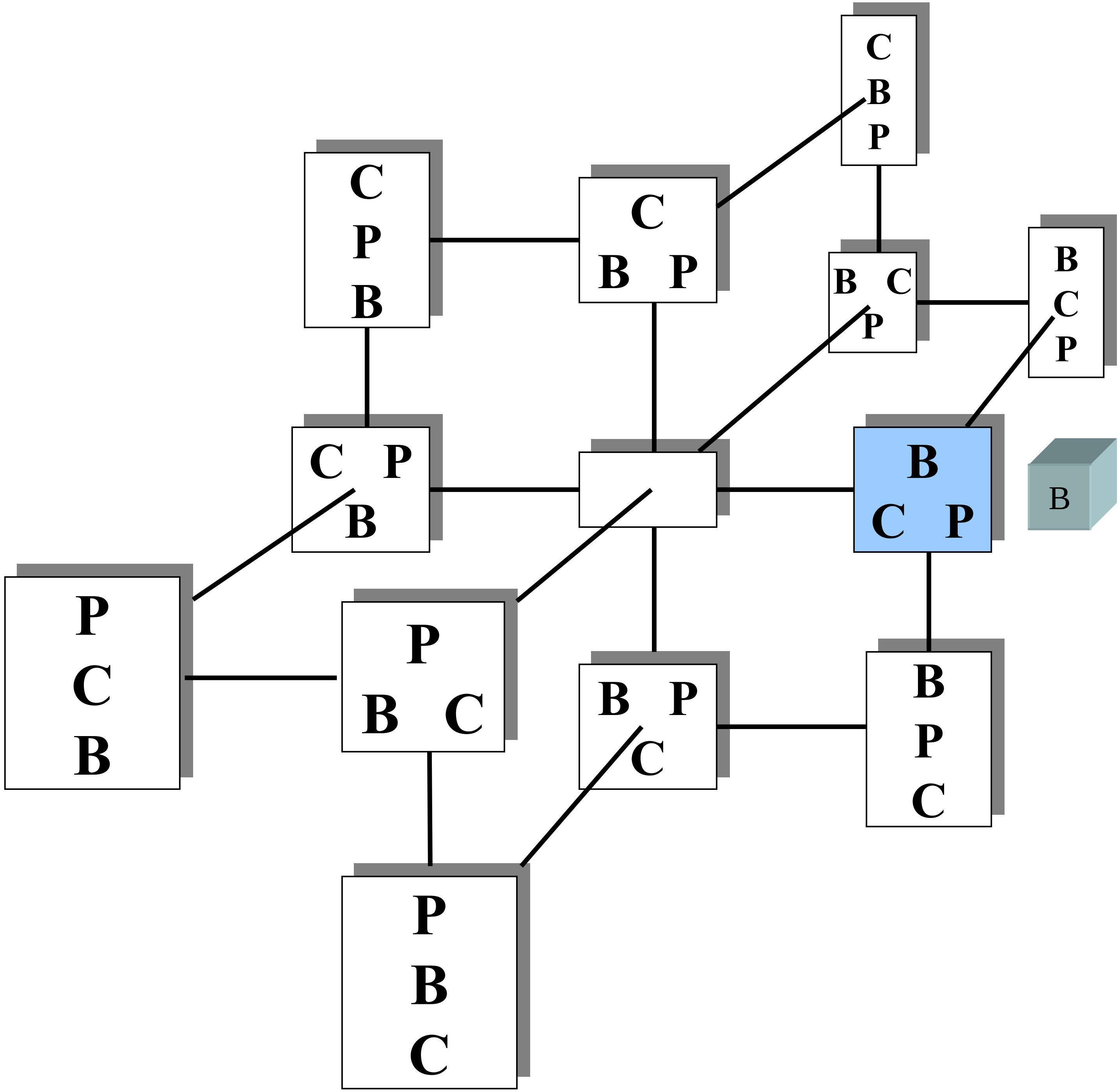
C
P
B
C

Conversation with a neighbor:

Bush is a true Republican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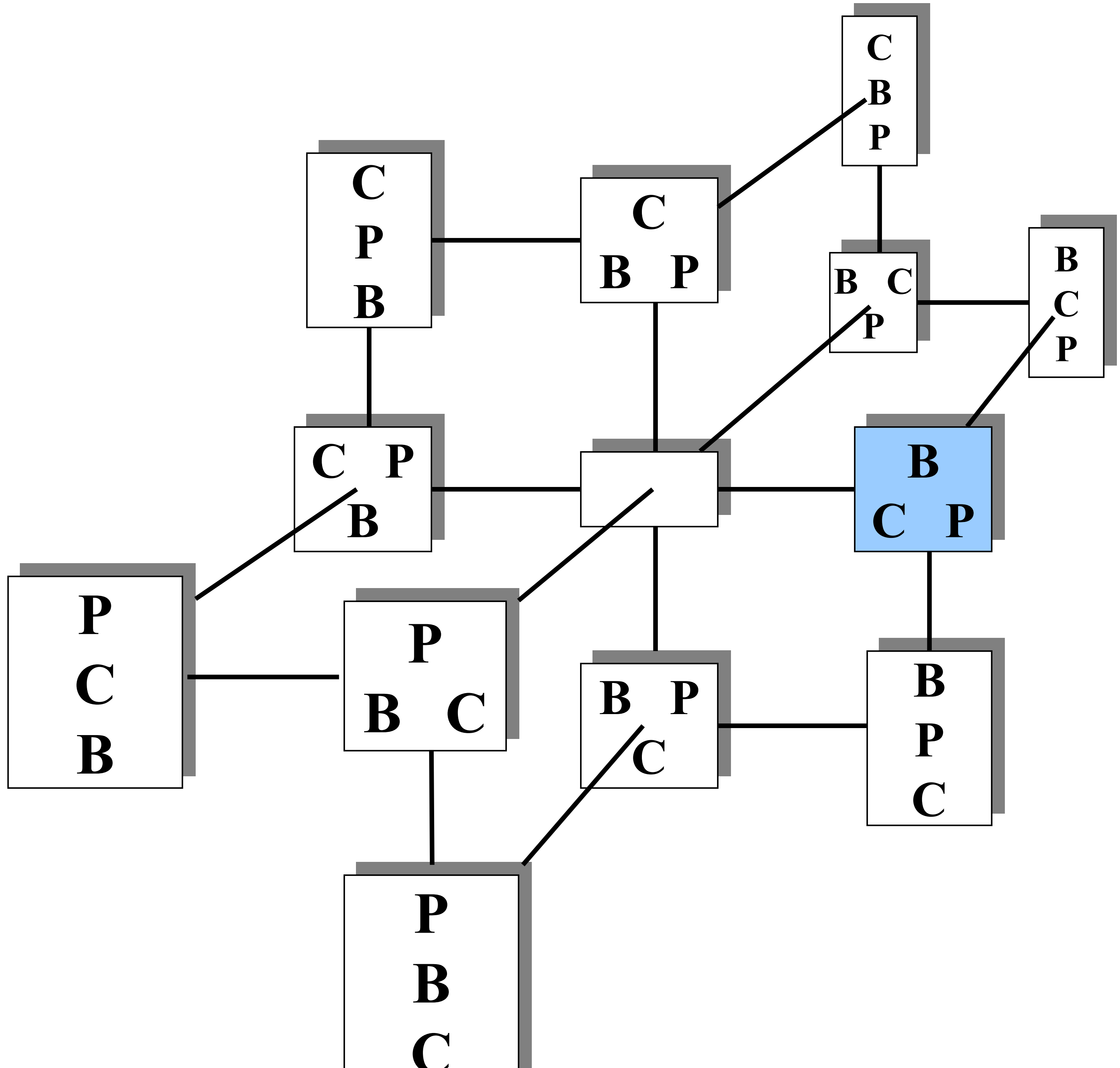
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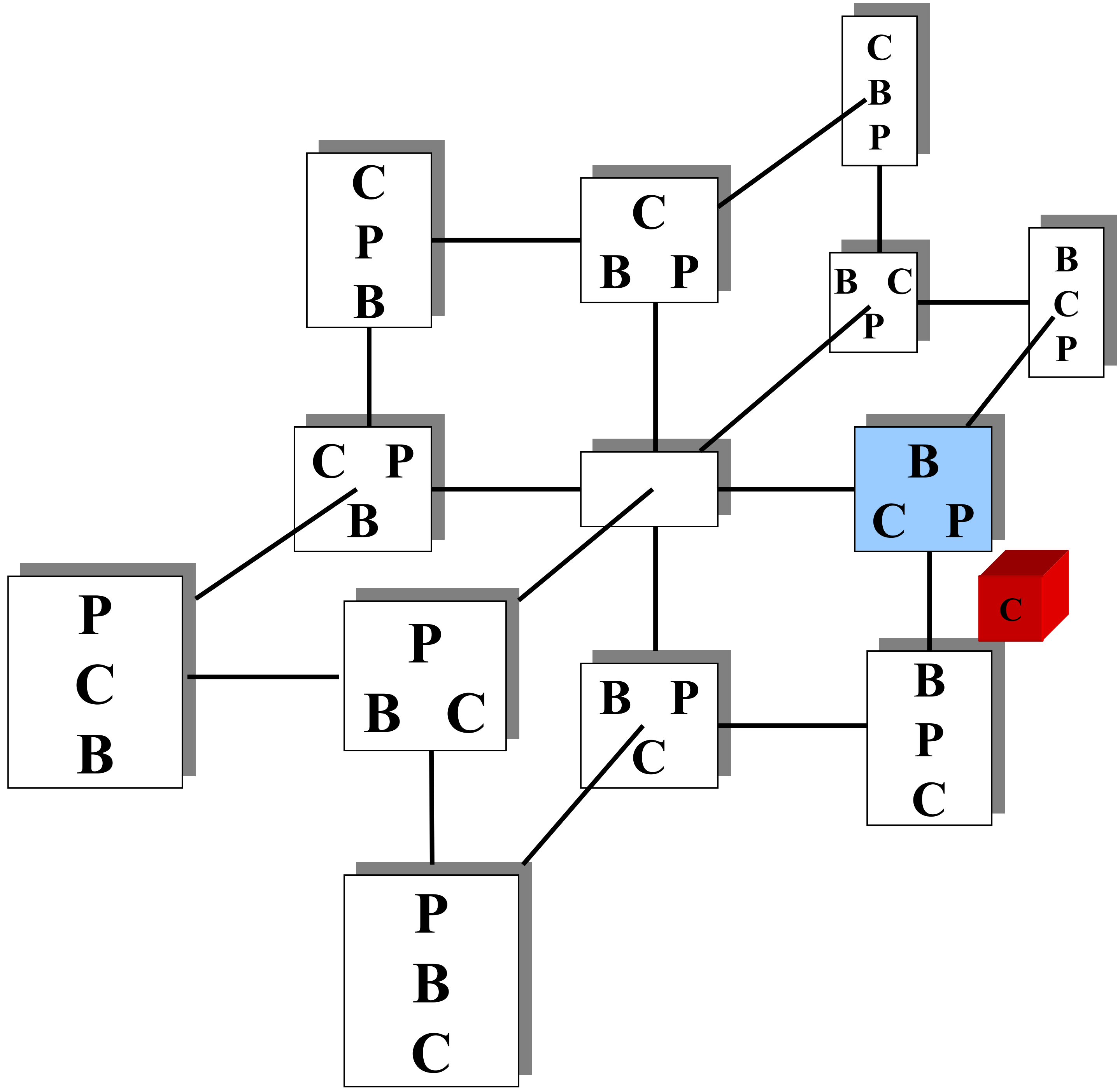
Television Interview:

- Clinton talks about Medicare



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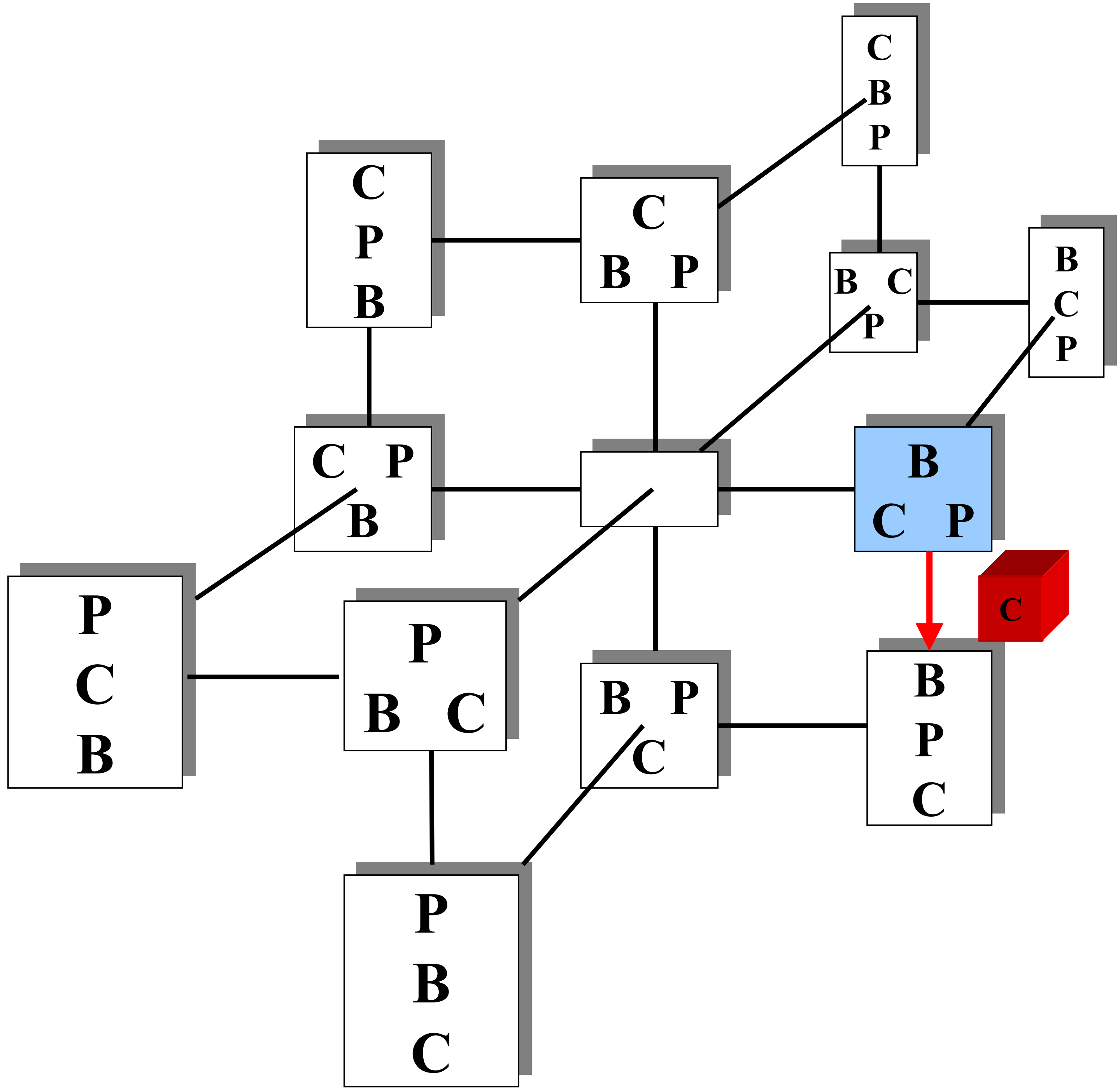

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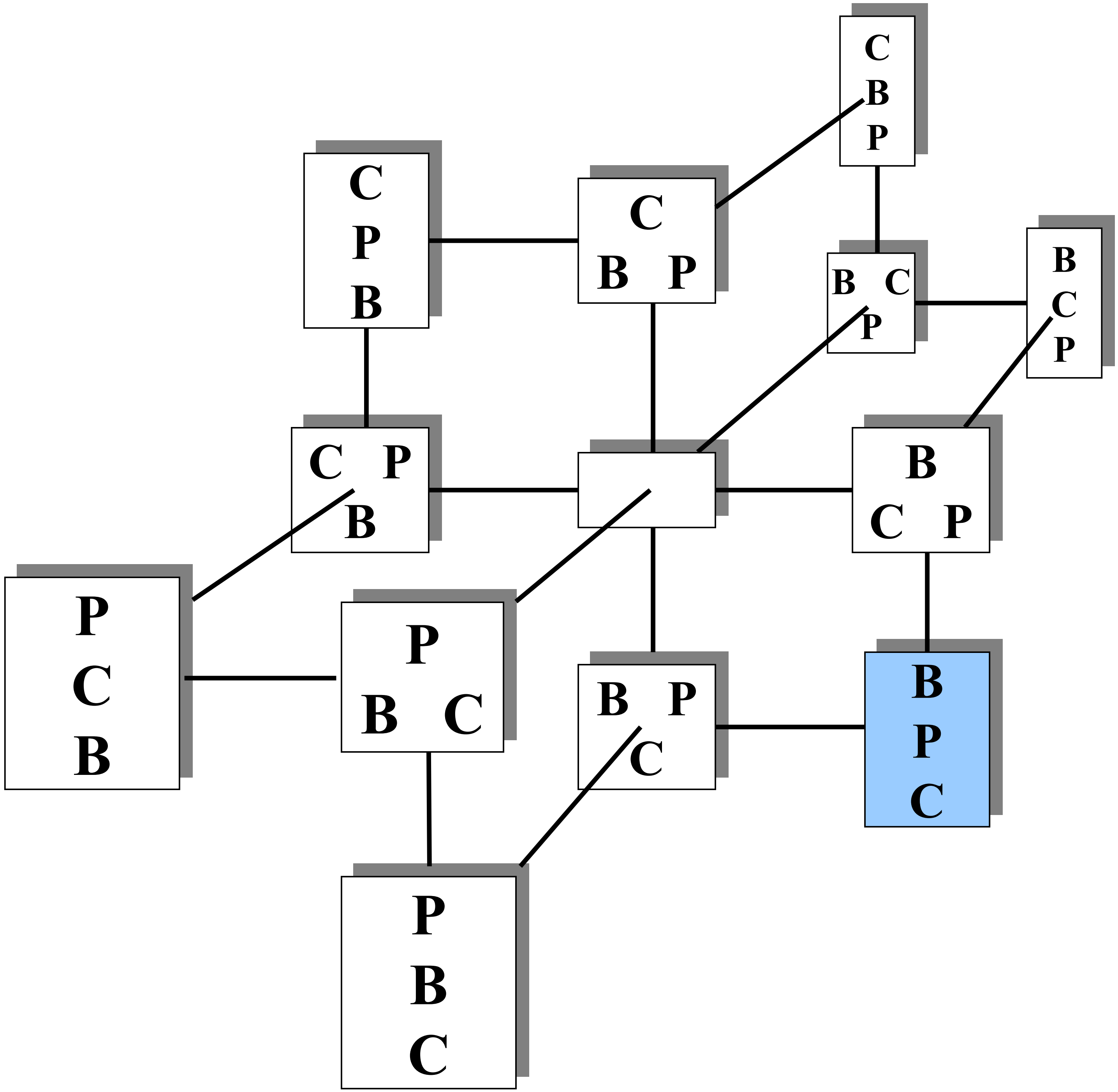
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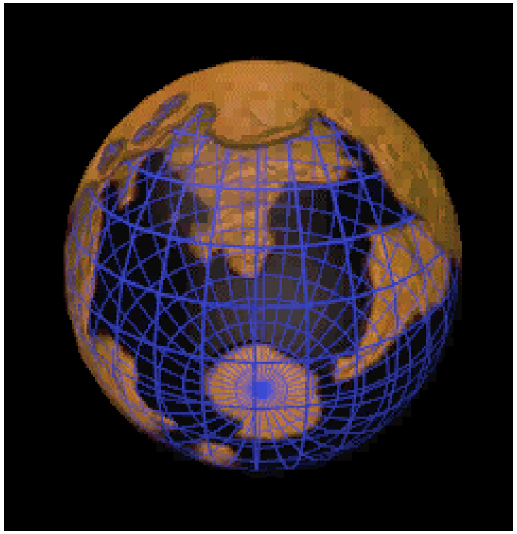
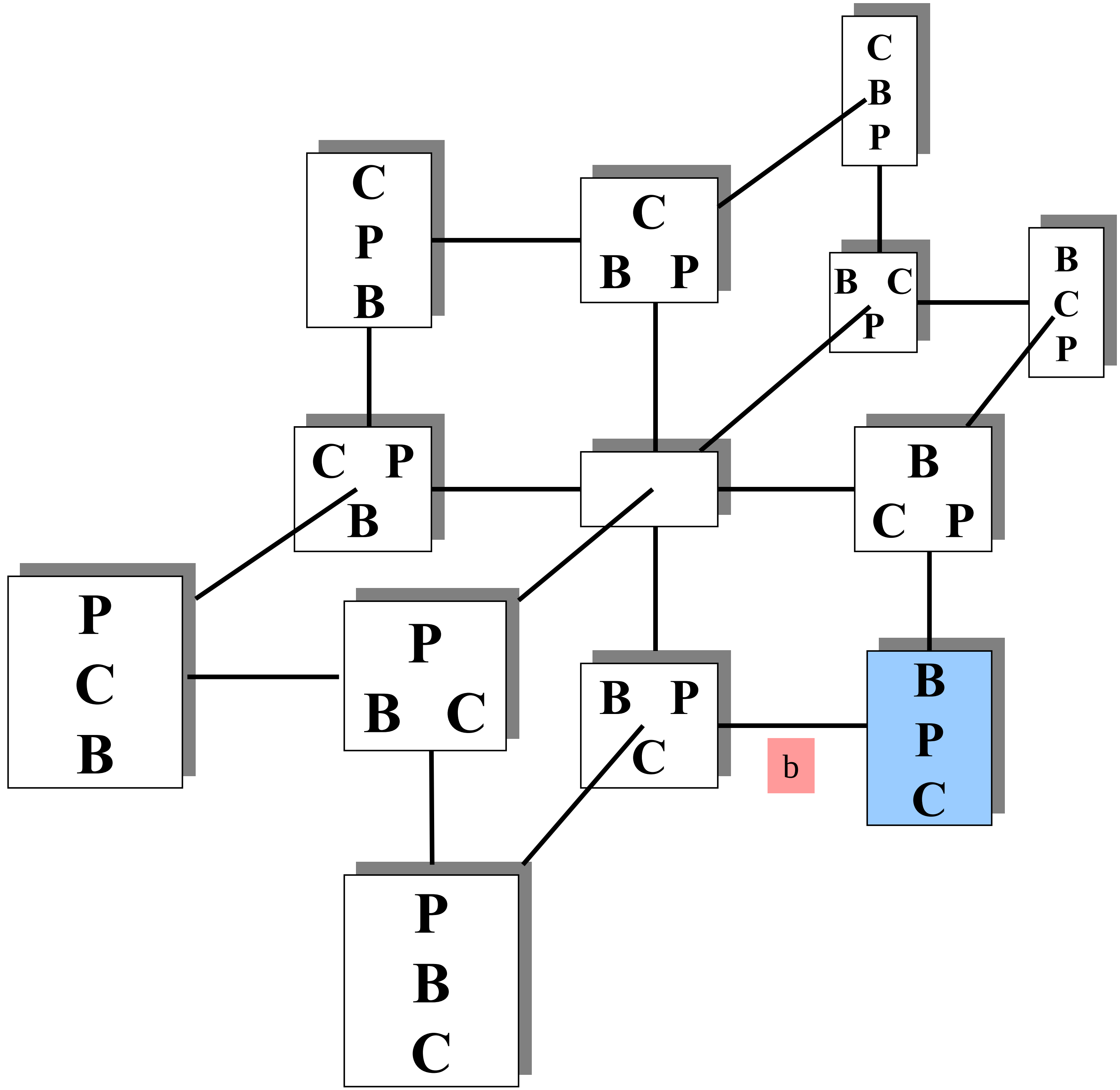
Evening Headlines:



Bush disagrees with fellow Republicans about Foreign Policy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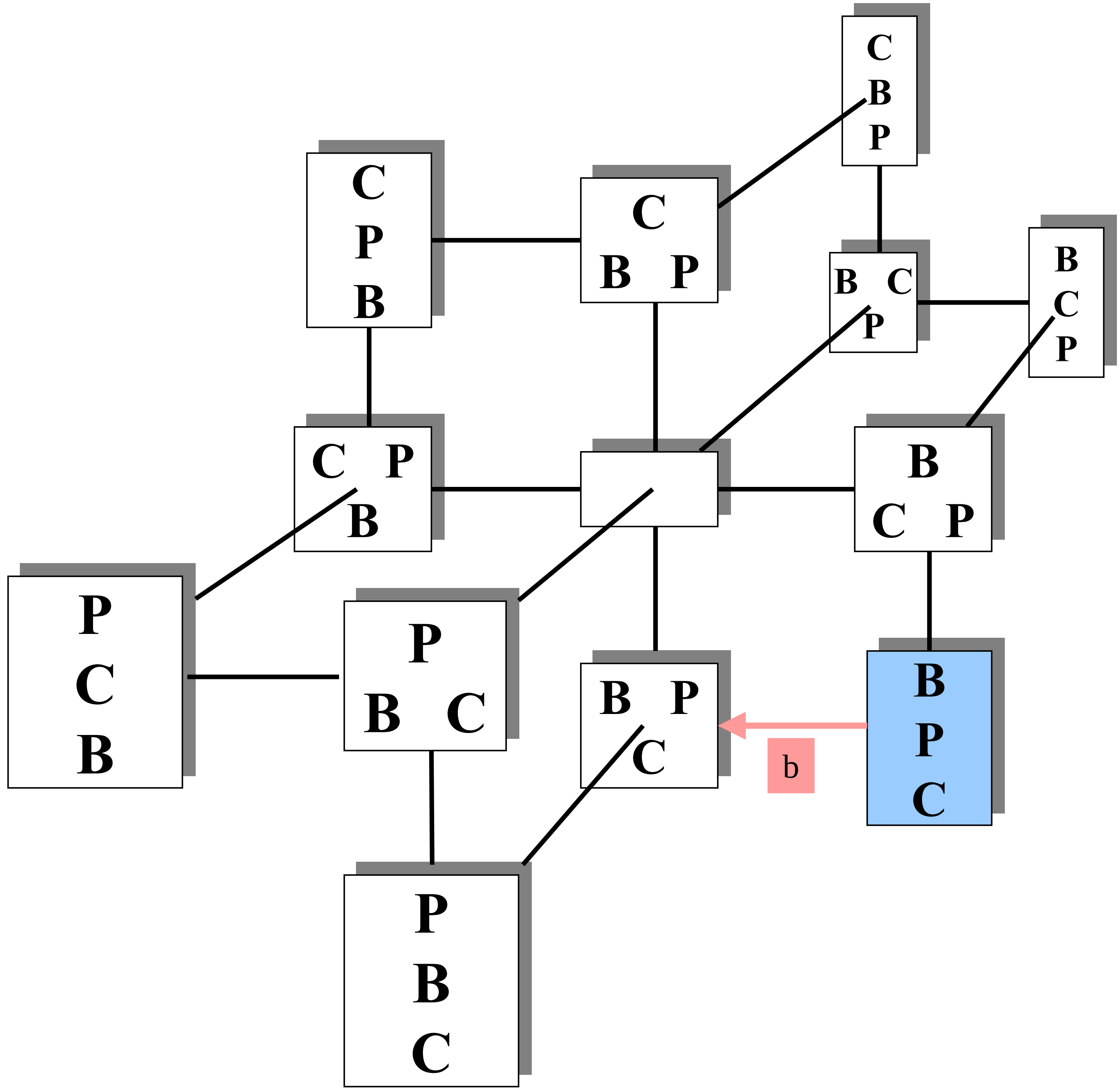

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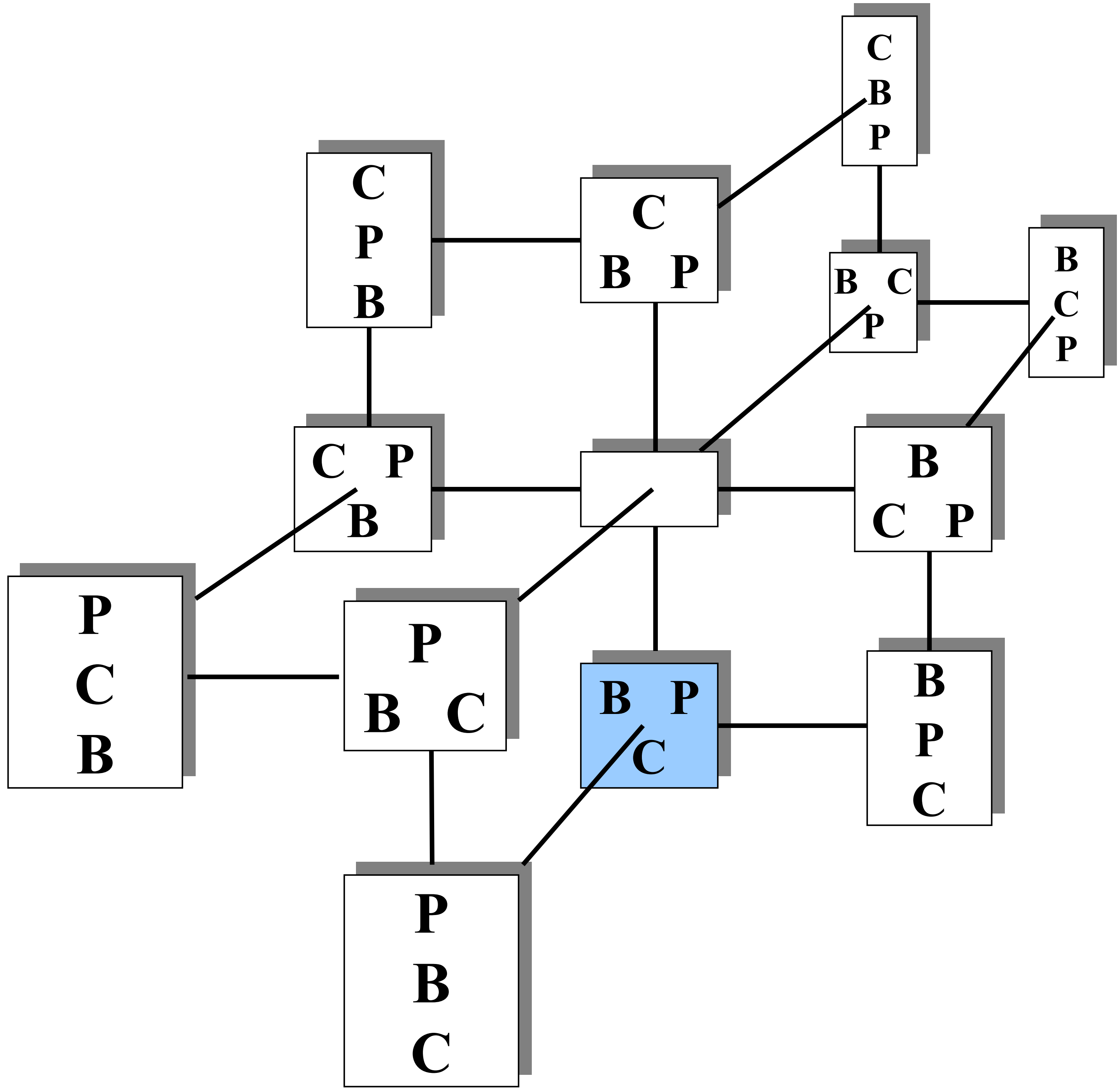

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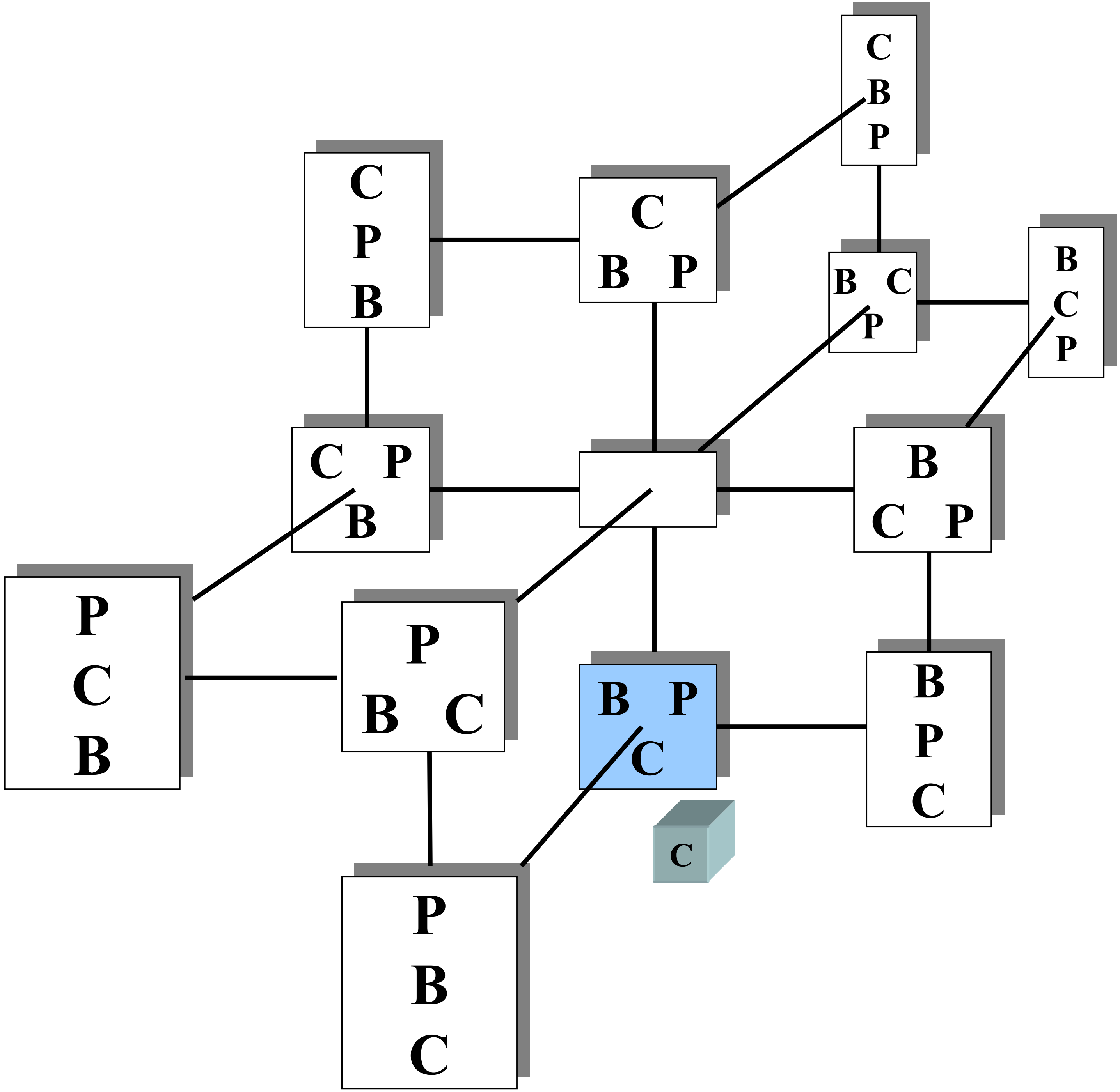
Party Time:

Clinton will

save America

Improve Economy

Rescue Environment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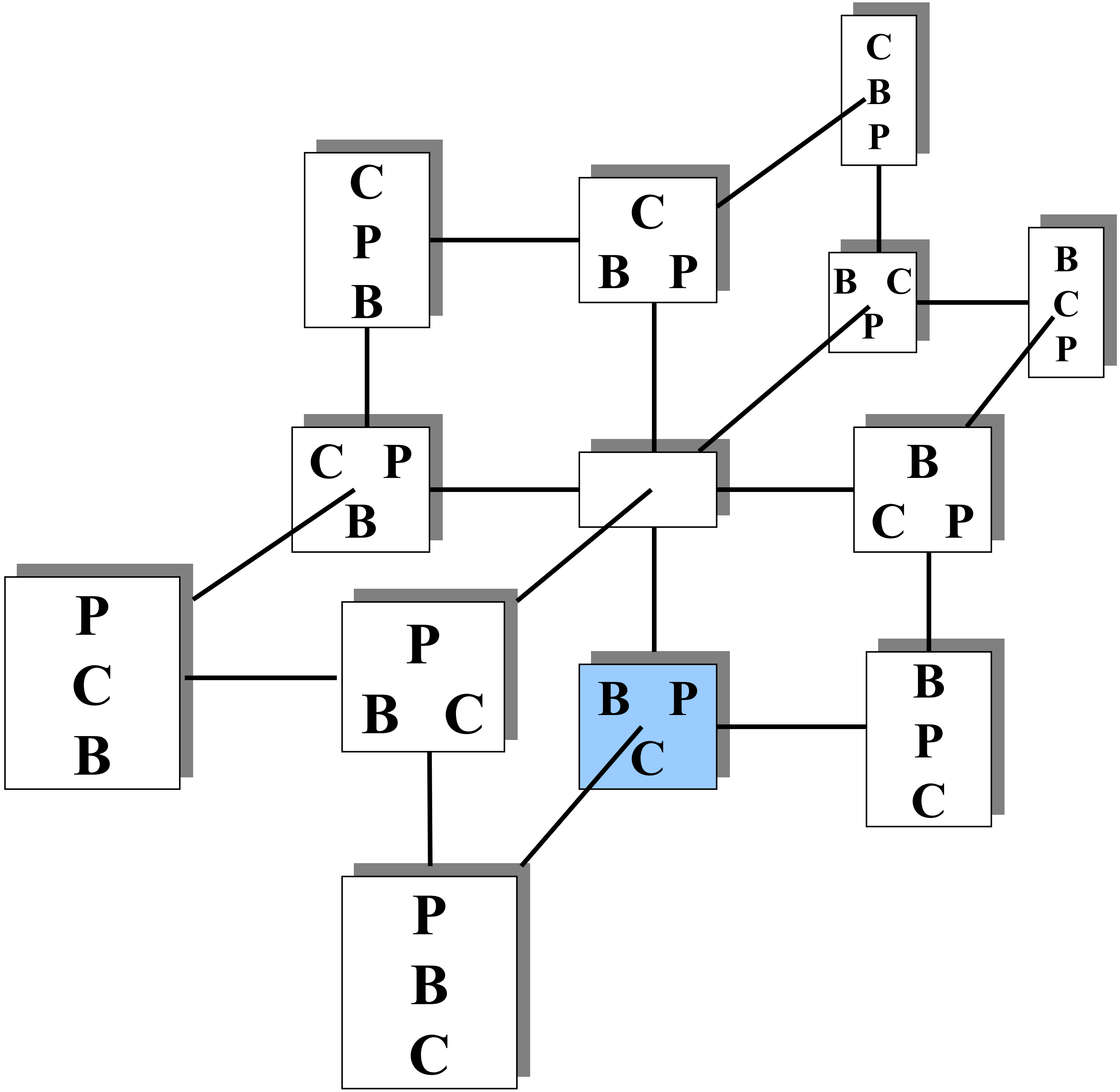
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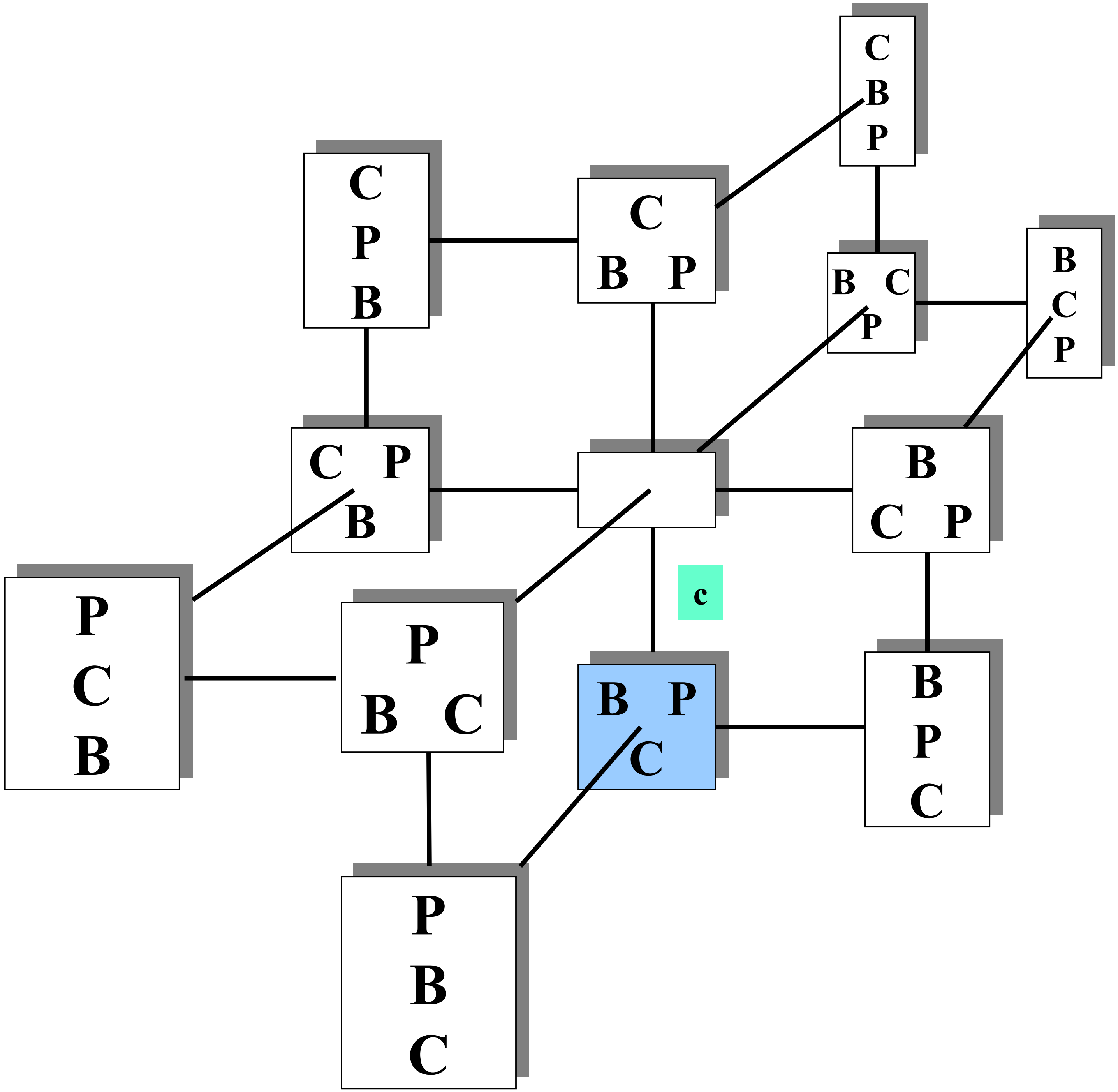
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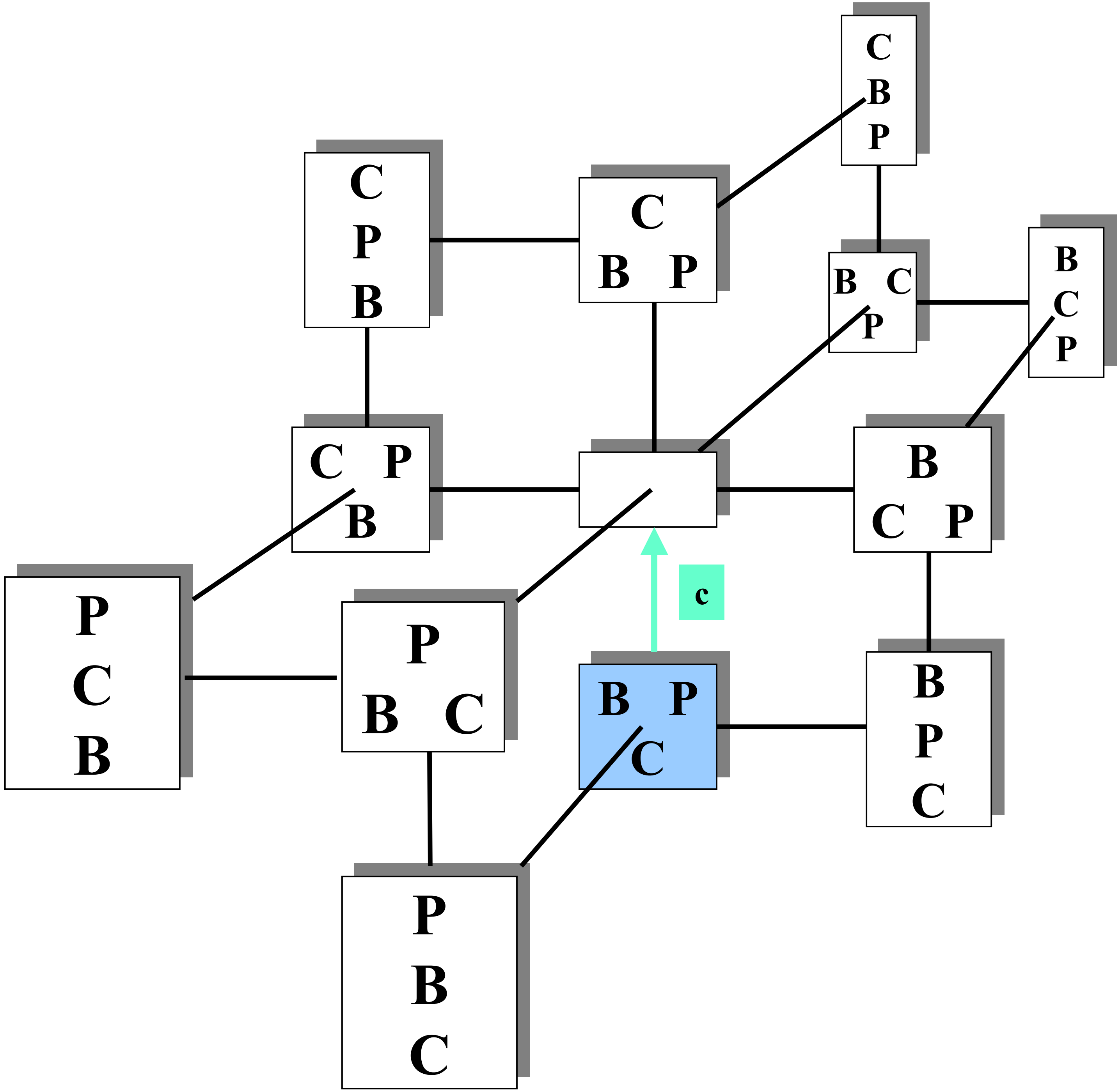
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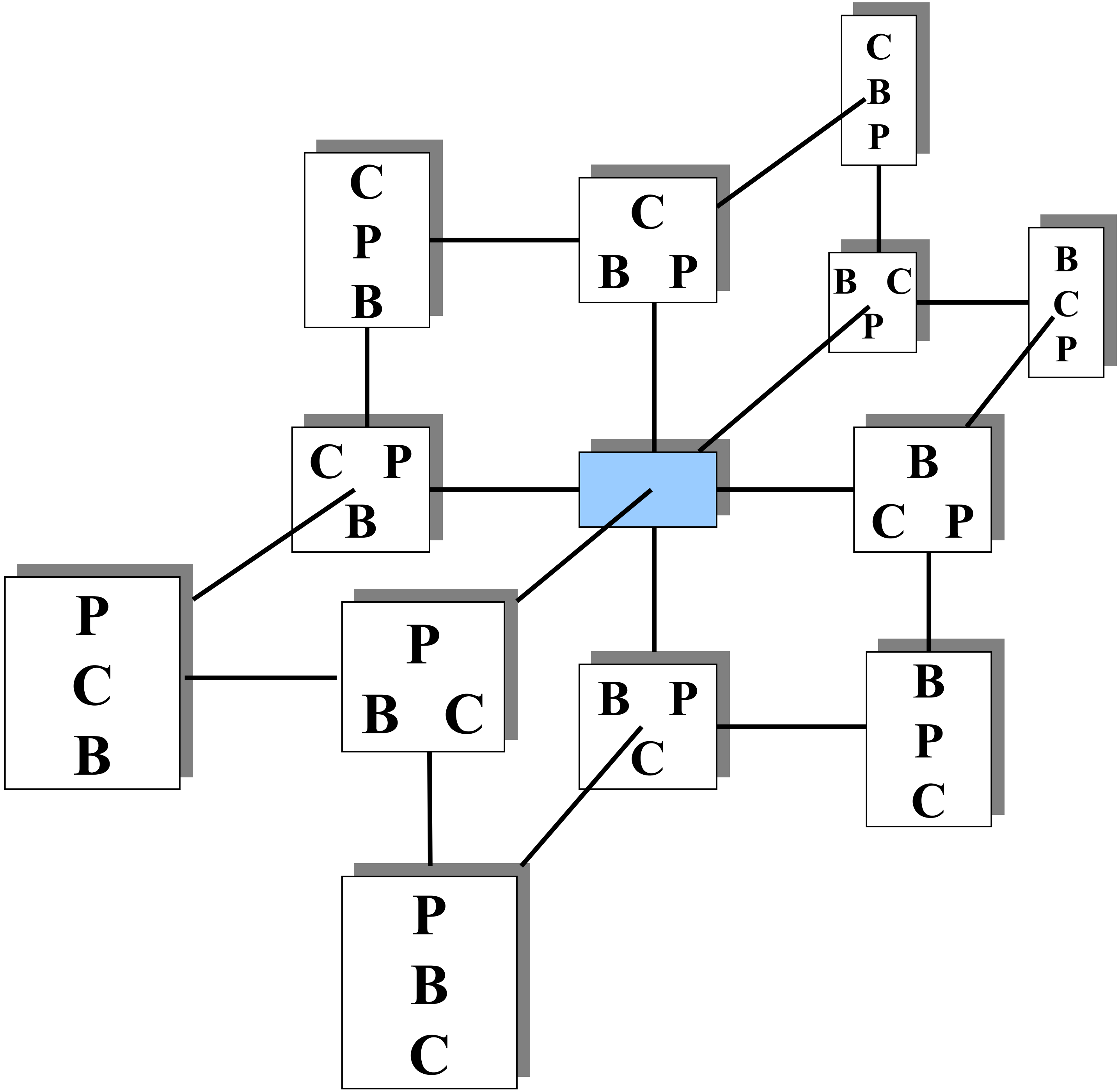
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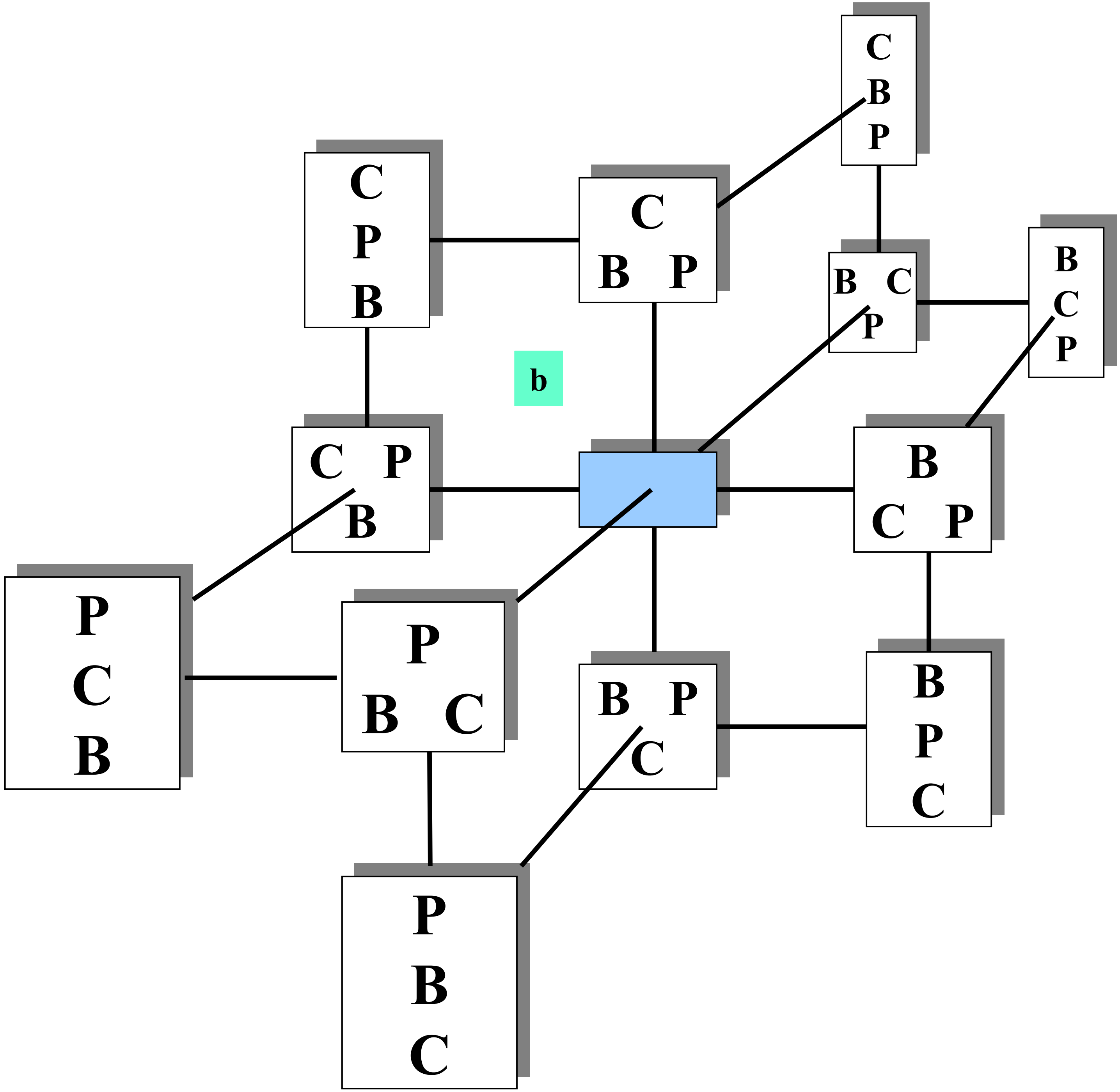
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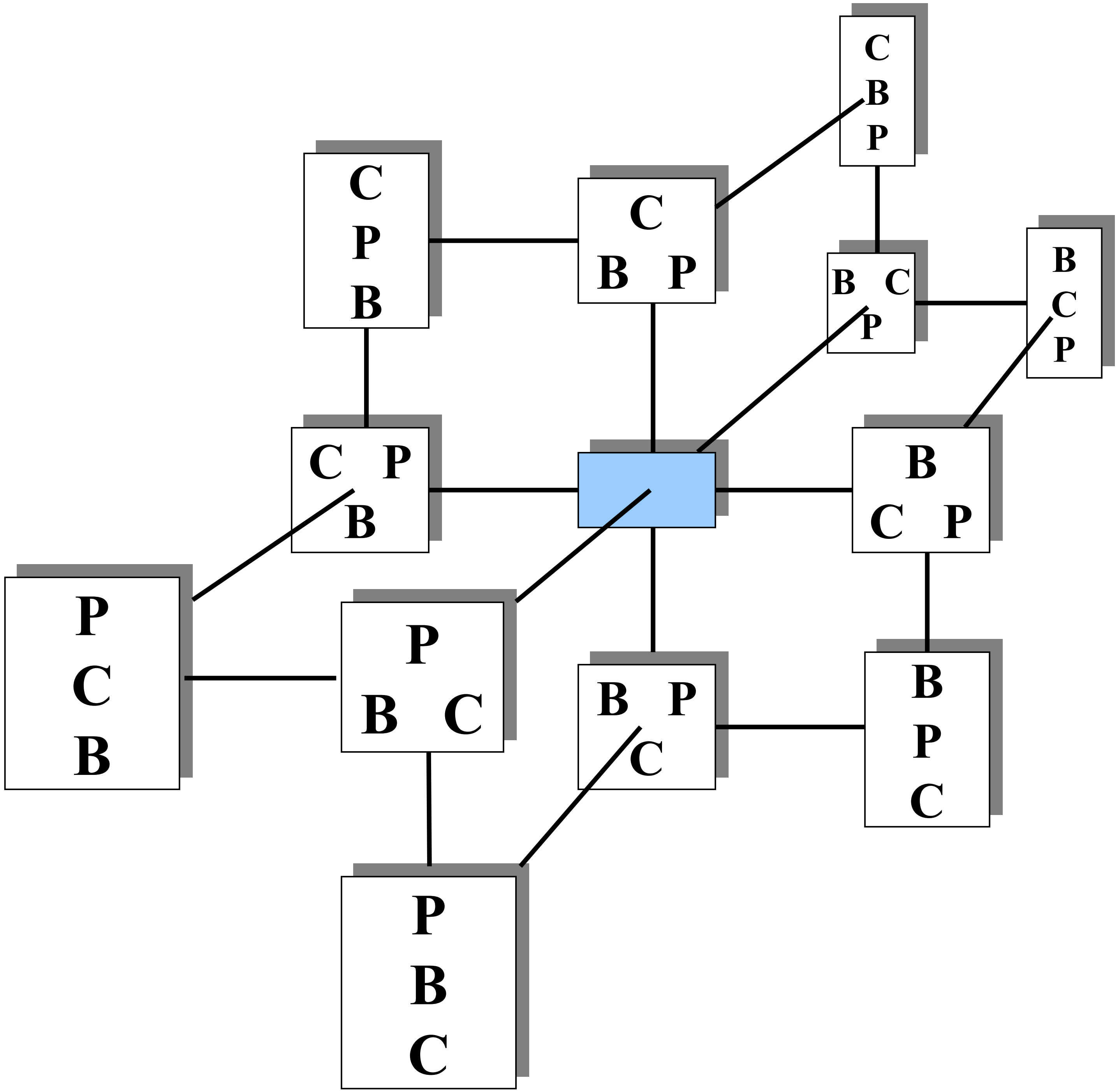
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Random Walk:

Theorem:

The asymptotic distribution exists and can be computed analytically

Some Interesting Parameters:

Positive Bias Ratio for Alternative i

$$\frac{\text{Probability of I}}{\text{Probability of i}}$$

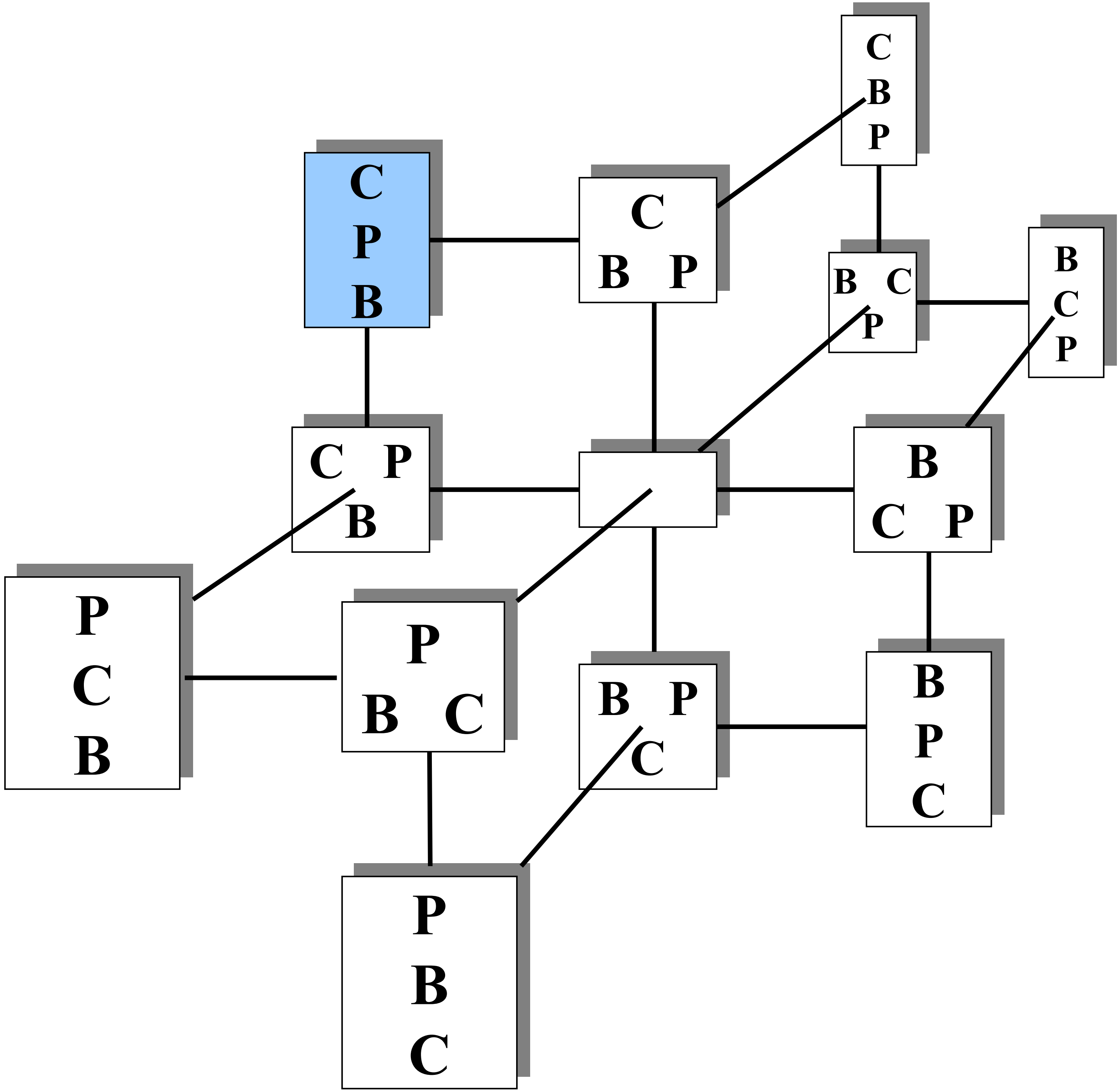
Negative Bias Ratio for Alternative i

$$\frac{\text{Probability of I}}{\text{Probability of i}}$$

Positive Bias Ratio for Clinton

$$\frac{\text{Probability of C}}{\text{Probability of c}}$$

Net tendency of information that moves Clinton to the top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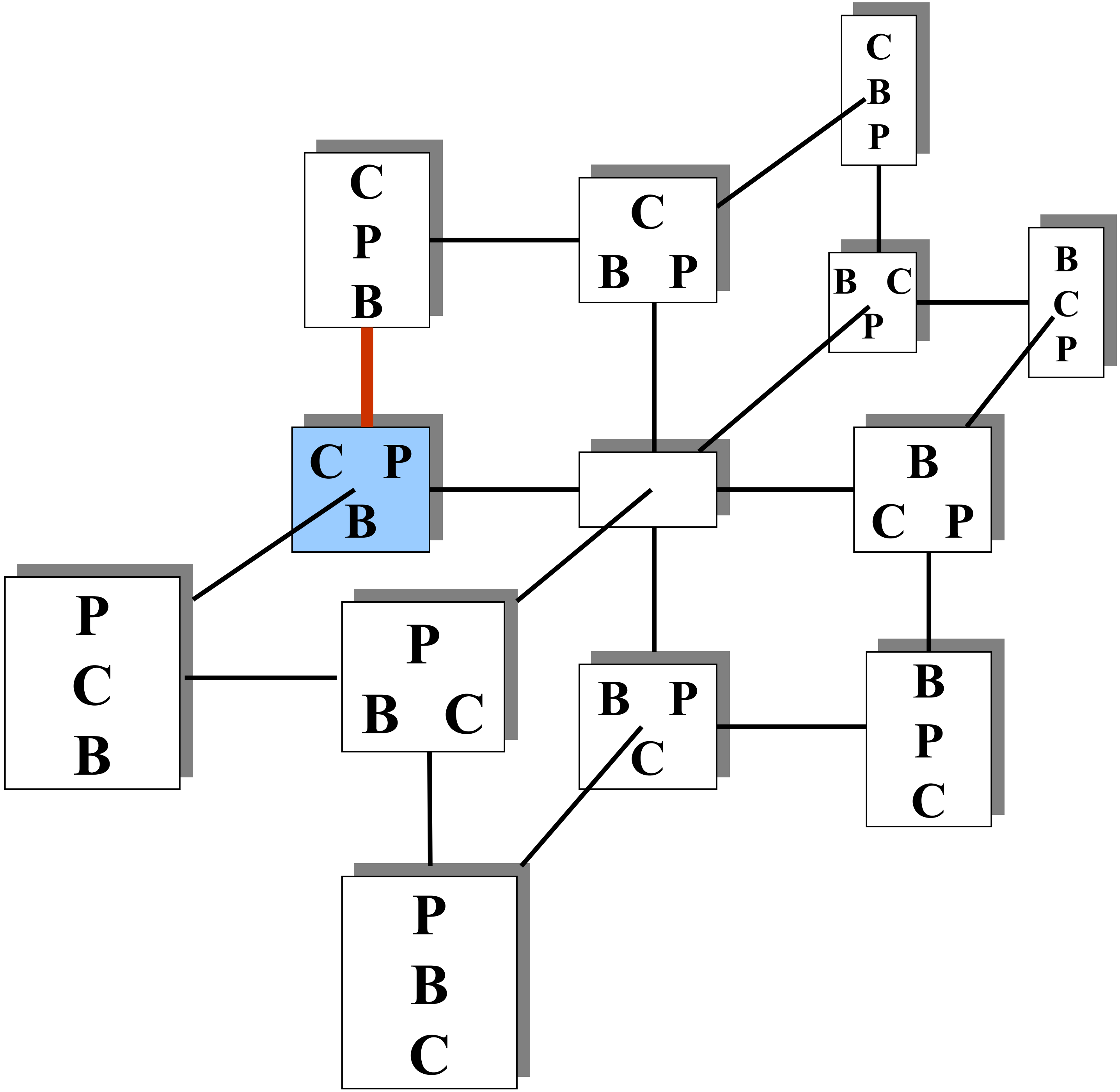
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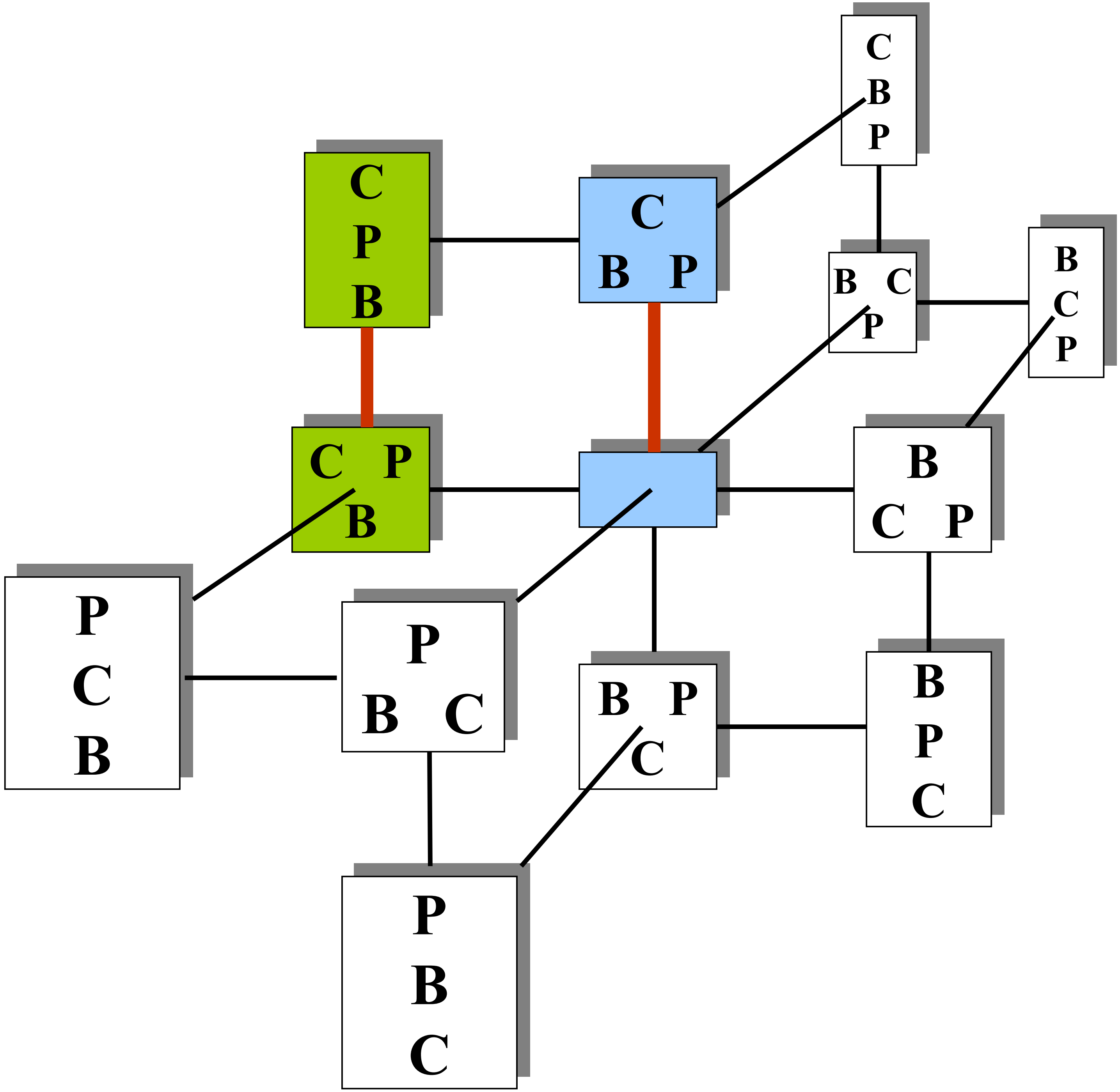
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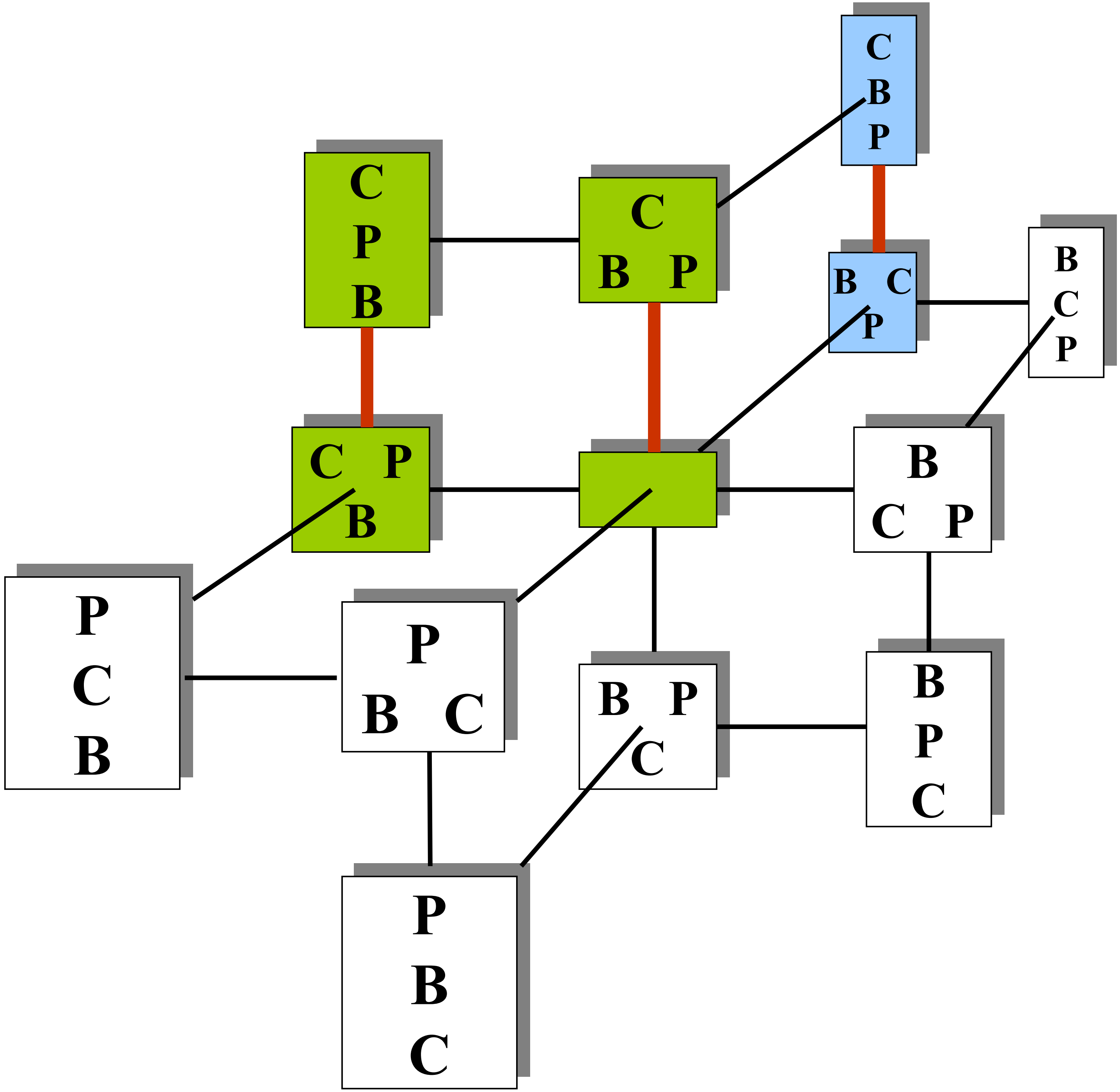

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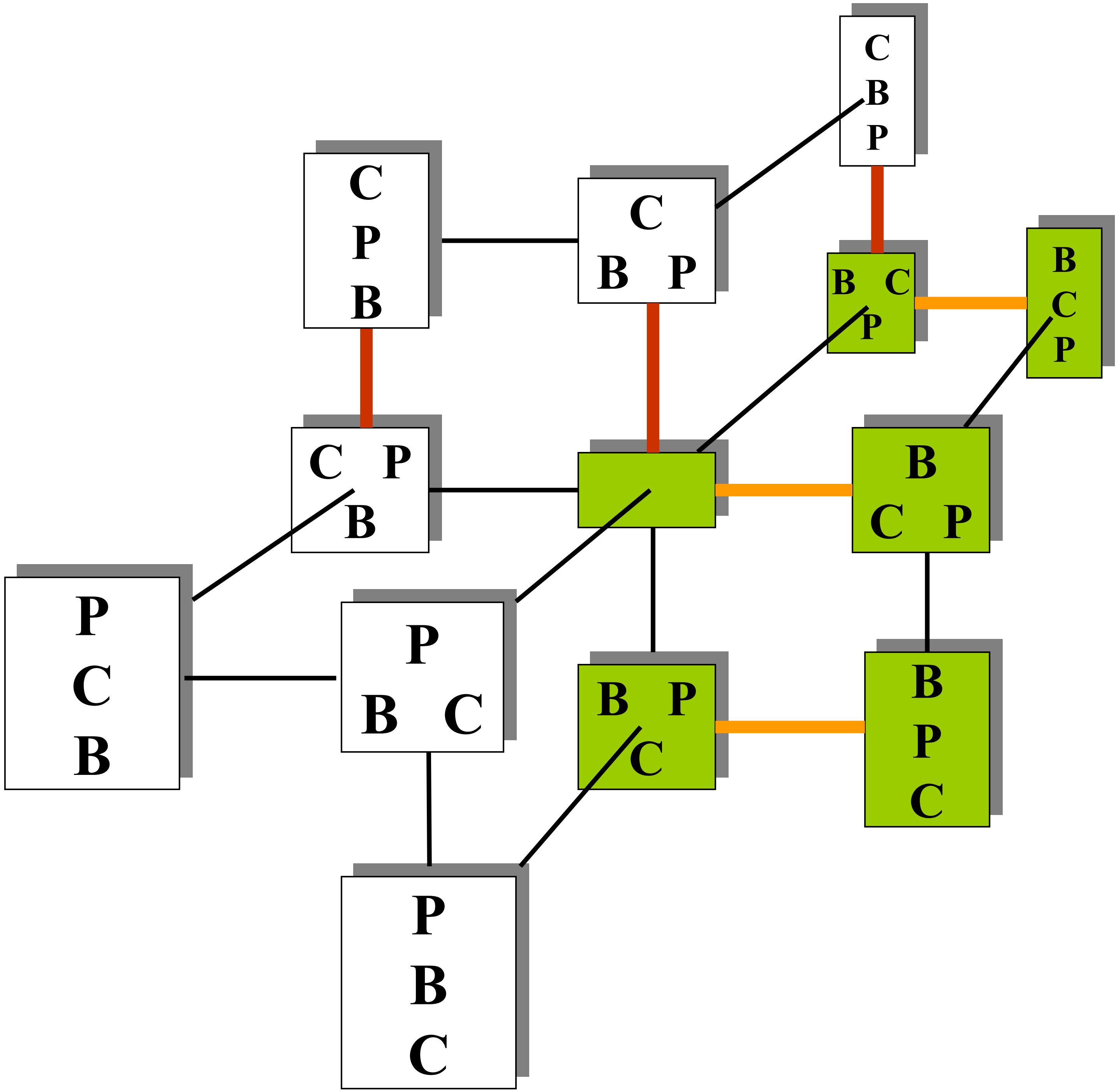
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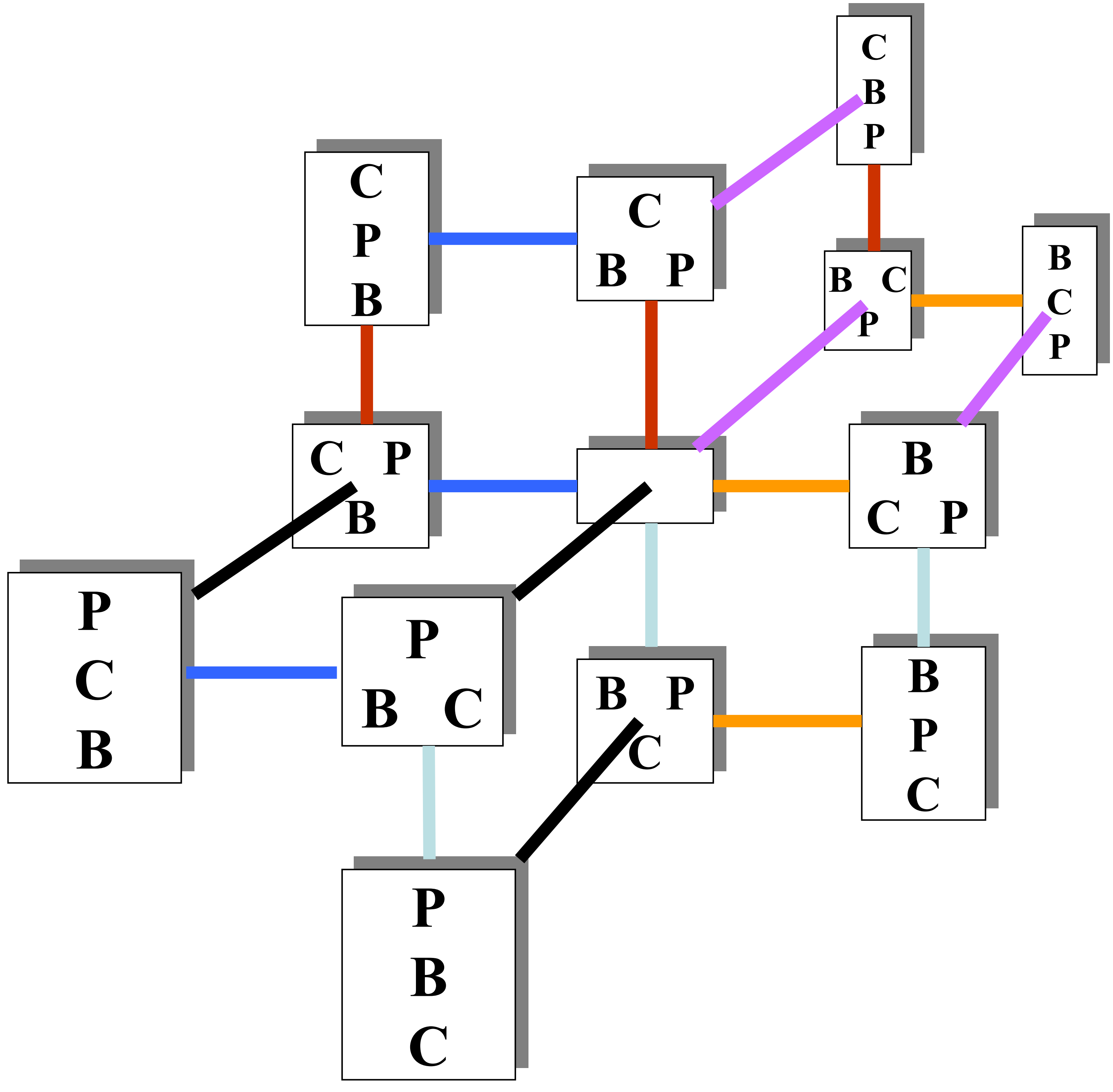

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Data:

ICPSR:

1992 NES Feeling Thermometer Ratings

- **before the election**
- **after the election**

Self-Ratings on Partisanship Scale

(Party ID, pre-election WO, post-election WO)

3x13x13

Goodness-of Fit of Asymptotic Model Vs. Single Time Data

	Fit	G^2	p-value (df)
Pre-Election	**Good**	**21.6**	**.25 (18)**
Post-Election	**Very poor**	**36.5**	**.006 (18)**

(MLE, N=2,024)

New process started between the 2 interviews.

Hypothesis Tests (92 Pre-election):

Asymptotic Submodels vs. Asymptotic Model	Reject/Retain Hypothesis	G^2	p-value (df)
Same Information Flow all Parties	**Reject**	**950**	**< .000006 (12)**

Hypothesis Tests (92 Pre-election):

Asymptotic Submodels vs. Asymptotic Model	Reject/Retain Hypothesis	G^2	p-value (df)
Same Information about <u>Perot</u> all Parties	Reject	12	.02 (5)
Same Information about <u>Perot</u> for Dem. & Rep.	Retain	5.6	.06 (2)

Full Stochastic Model & Submodels

		G^2	p-value (df)
Full Stochastic Model vs. Data	**Excellent Fit**	**268.2**	**.384 (262)**
Same Information Flow before and after Election	**Reject**	**47.9**	**.0001 (18)**

Overall Analysis

Hypothesis Tests & Parameter Estimates validated by literature about 92 campaign

Note:
We did not even glimpse at the mass media!

Conclusions

(Probabilistic) Binary Preference Relations
(Random) Utility Representations:

Powerful Framework

Towards General Theory of Decision Making

• Analysis of Social Choice in Practice
using an Inference Framework

Preference Aggregation Model Dependent
Where are the Majority Cycles??
Congruence among Social Choice Rules
Study Persuasion without Control of Stimuli