Fast-Converging Tatonnement Algorithms for One-Time and Ongoing Markets

Richard Cole

(joint work with Lisa Fleischer)

Bob Tarjan, Bell Labs Years (and NYU too)

1981-85

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A Potential Function based Analysis

$$\Phi = \sum_{i} \Phi_{i}$$

$$\Phi_i = \left(\frac{1-\lambda t_i}{a_1}\right) p_i \left[\begin{array}{c} \mathrm{span}\left\{x_i^+, x_i^-, \tilde{x}_i^+, \tilde{x}_i^-\right\} \\ + \frac{1}{a_2 E}\left(x_i^+ - x_i^- + \tilde{x}_i^+ - \tilde{x}_i^-\right) \\ + \left(\tilde{x}_i^+ - \tilde{x}_i^-\right) \end{array} \right] + a_3 p_i^* \,|\, \tilde{x}_i - s_i^* \,|$$

v_i^+ denotes the maximum value of v_i in time period $[0, t_i]$

v_i^- denotes the minimum value of v_i in time period $[0, t_i]$

$t_i \leq 1$ is time since last update to p_i

Informal Problem Definition

Input:

- n goods G_i, $1 \leq i \leq n$
- Buyers and sellers with initial endowments of money and goods

Goal:

- Find prices that balance supply and demand in all goods simultaneously

Basic assumption: Economies, more or less, are near equilibrium.

Papadimitriou (02): If so, (near)-equilibrium prices are surely P-Time computable.

Many P-time algorithms for finding exact and approximate equilibria for restricted markets

- L. Chen, Y. Ye, J. Zhang. A note on equilibrium pricing as convex optimization, WINE 07.
- N. Chen, X. Deng, X. Sun, A. Yao. Fisher equilibrium price with a class of concave utility functions, ESA 04.
- B. Codenotti, B. McCune, K. Varadarajan. Market equilibrium via the excess demand function, STOC 05.
- B. Codenotti, S. Pemmaraju, K. Varadarajan. On the polynomial time computation of equilibria for certain exchange economies, SODA 05.
- B. Codenotti and K. Varadarajan. Market equilibrium in exchange economies with some families of concave utility functions, DIMACS Workshop on Large Scale Games, 05.
- N.R. Devanur, V. V. Vazirani. The spending constraint model for market equilibrium: algorithm, existence and uniqueness results, STOC 04.
- N.R. Devanur, C.H. Papadimitriou, A. Saberi, V. V. Vazirani. Market equilibrium via a primal-dual-type algorithm, FOCS 02.
- R. Garg and S. Kapoor. Auction algorithms for market equilibrium, STOC 04.

More P-time algorithms

R.Garg, S.Kapoor, V.Vazirani. An auction-based market equilibrium algorithm for the separable gross substitutibility case, APPROX 04.

K. Jain. A polynomial time algorithm for computing the Arrow-Debreu market equilibrium for linear utilities, FOCS 04.

K. Jain, M. Mahdian, A. Saberi. Approximating market equilibria, APPROX 03.

K. Jain and K. Varadarajan. Equilibria for economies with production: constant-returns technologies and production planning constraints, SODA 06.

K. Jain and V.V. Vazirani. Eisenberg-Gale Markets: Algorithms and structural properties, STOC 07.

K. Jain, V.V. Vazirani. Y. Ye. Market-equilibria for homethetic,quasi-concave utilities and economies of scale in production, SODA 05.

Y. Ye. A path to the Arrow-Debreu competitive market equilibrium. *Math. Program*., 2008.

Papadimitriou (02): If so, (near)-equilibrium prices are surely P-Time computable.

Cole/Fleischer: And they are surely also readily computable by the markets themselves.

Papadimitriou (02): If so, (near)-equilibrium prices had better be P-Time computable.

Cole/Fleischer: And they had better be readily computable by the markets themselves.

Questions:

- What market-based price adjustment rules achieve this?
- What constraints on the markets ensure fast convergence using these rules?

Arrow-Debreu or Exchange Market

Goods $G_1, G_2, \ldots, G_n$
Prices $p_1, p_2, \ldots, p_n$
Agents $A_1, A_2, \ldots, A_m$
Utilities $u_1, u_2, \ldots, u_m$

u_j gives agent a_j's preferences

w_{ij}: initial allocation of G_i to a_j; $w_i = \sum_j w_{ij}$.
A_i seeks to maximize its utility at current prices.
$x_{ij}(\mathbf{p})$: demand of A_j for good G_i;

$x_i = \sum_j x_{ij}$, demand for G_i
Excess demand: $z_i = x_i - w_i$

Problem: Find prices $\mathbf{p}$ such that $x_i \leq w_i$ for all i.

Fisher Market

Agents are either *buyers* or *sellers*

- Sellers start with one good each and desire money alone
- Buyers start with money alone and desire a mix of goods (possibly including money)

One-Time Markets

- The above exchange and Fisher markets

Tatonnement

Prices adjust as follows:

- Excess supply: prices decrease
- Excess demand: prices increase

(1874, Leon Walras, Elements of Pure Economics)

Modeling Price Updates

- Virtual Price Setters
  - One per good

Self Adjusting Markets
Price Update Protocol Desiderata

- Limited information: The price setter for G_i knows only p_i, z_i, w_i and their history.
- Asynchrony
- Fast convergence
- Robustness
- Simple actions

Related to work on dynamic convergence to Nash equilibria:

H. Ackerman, H. Roglin, B. Vöcking. On the impact of combinatorial structure on congestion games, FOCS 06.

S. Chien and A. Sinclair. Convergence to approximate nash equilibria in congestion games, SODA 07.

S. Fischer, H. Räcke, B. Vöcking. Fast convergence to Wardrop equilibria by adaptive sampling methods, STOC 06.

M. Goemans, V. Mirrokni, A. Vetta. Sink equilibria and convergence, FOCS 05.

V.Mirrokni and A.Vetta. Convergence issues in competitive games, APPROX 04.

Difficulty

How to interpret tatonnement in market problem setting?

- Tatonnement occurs over time
- No notion of time in classic market problem

Original approach (Walras): auctioneer model

One perspective:

(Essentially) the same market repeats daily.

- What happens to excess demands?

Standard tatonnement amounts to:

- Ignore excess demands.
- We call the associated convergence rate, the *One-Time* analysis.

Ongoing Market

For each good G_i there is a capacity c_i warehouse.

Focus on Fisher setting.

- WLOG, one seller per good.
- Each day, buyers receive their demands at current prices.
- Excess demands are taken from/stored in the warehouses.

Ongoing Market, Cont.

For i-th warehouse have target content s_i^*,

$$0 < s_i^* < c_i.$$

Goals:

- Have warehouse contents converge to $\mathbf{s}^*$.
- Have prices converge to equilibrium values.

Notation: s_i denotes current contents of warehouse i.

Goals

- Give a price adjustment rule
- Identify constraints on the markets that enable fast convergence
- Analyze the convergence rate in these markets

Self Adjusting Markets
Price Update Protocol Desiderata

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- Asynchrony
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- Simple actions

Our Price Update Rules

For One-Time Markets:

$$p'_i \leftarrow p_i + \lambda_i \, p_i \min\{1, z_i / w_i\}$$

$z_i = x_i - w_i$, the excess demand

By contrast, Uzawa (1961) used the rule

$$p'_i \leftarrow p_i + \lambda_i \, z_i / w_i$$

Our Price Update Rules, Cont.

For Ongoing Markets

Define target demand: $\tilde{x}_i = w_i + \kappa_i (s_i - s_i^*)$

Update Rule: $p_i' = p_i + \lambda_i p_i \min\{1, (x_i - \tilde{x}_i) / w_i\}$

For simplicity, set: $\lambda_i = \lambda,\ \kappa_i = \kappa \quad \text{all } i$

Conditions enabling rapid convergence

- PPAD hard to find equilibrium in Exchange markets with Leontief utilities (Codenotti et al.)
- Samuelson's equation ($dp_i/dt = \lambda_i z_i/w_i$) is not always convergent
  - One condition assuring convergence: Weak Gross Substitutes (increasing one price only increases the demand for other goods).

Rapid convergence needs good response to price adjustment signals.

In the one time market this means:

$$|p_i - p_i^*| \text{ large} \quad \Leftrightarrow \quad |z_i| \text{ large}$$

In the Ongoing market this means:

$$|p_i - \tilde{p}_i| \text{ large} \quad \Leftrightarrow \quad |\tilde{z}_i| \text{ large}$$

$\tilde{p}_i$ are the price achieving demand $\tilde{x}_i$

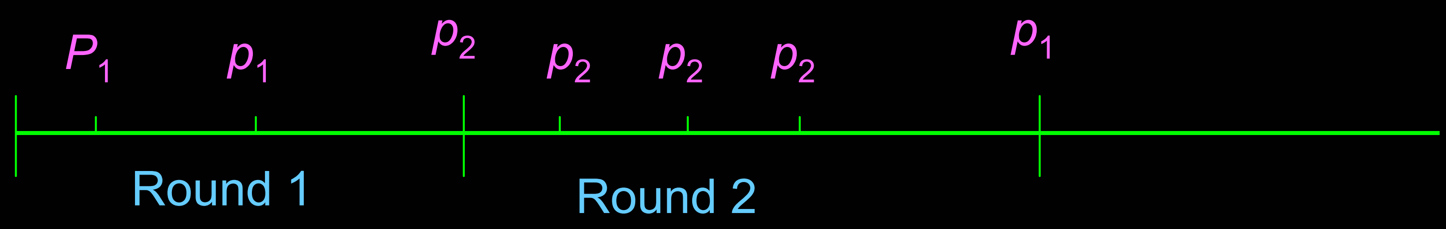
$\tilde{z}_i = x_i - \tilde{x}_i$

Entails parameters $E \geq 1$, $\beta \leq 1$.

Complexity Model

Rounds (from asynchronous distributed computing)

- A round is the minimal time interval in which every price updates at least once



How to measure convergence

Seek prices **p** such that

$$\max_i \frac{|p_i - p_i^*|}{p_i^*} \leq \delta$$

and in Ongoing markets in addition

$$r \max_i \frac{|s_i - s_i^*|}{w_i} \leq \delta$$

(*r* a scaling factor)

p*, **s*** denote equilibrium values

Our results

Theorem 1 In One-Time Fisher market with GS and parameters $\beta \le 1$ and $E \ge 1$, if $\lambda \le 1/(2E - 1)$, the worst price improves by one bit in $O(1/(\beta\lambda))$ rounds.

(Price update rule: $p'_i \leftarrow p_i + \lambda_i p_i \min\{1, z_i/w_i\}$)

For Cobb-Douglas utilities $\beta = 1$ and $E = 1$;
for CES utilities, $\beta = 1$ and $E = 1/(1 - \rho)$.

Our results, cont.

Theorem 2 In the Ongoing Fisher Market, with GS and parameters $\beta \le 1$ and $E \ge 1$, the worse of the worst price and worst warehouse stock improves by one bit in $O(1/(\beta\lambda) + 1/\kappa)$ rounds, if:

$$\lambda = O(1/E) \quad \text{and} \quad \kappa = O(\lambda)$$

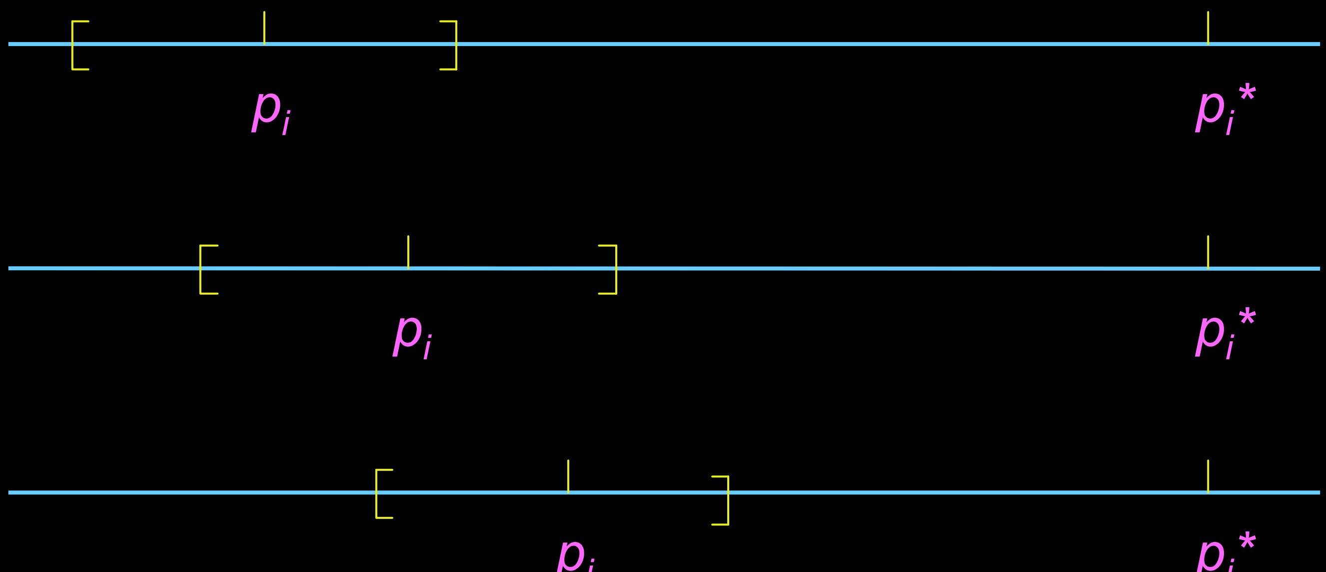
Price update rule:

$$p_i' = p_i + \lambda_i p_i \min\{1, (x_i - \tilde{x}_i)/w_i\}$$

- Motivation for price update rule: Price adjustment lies in zone where

  excess demand change = Θ(price change)

- Constraints on the market: Ensure zone large enough to achieve fast convergence



2 Phase Analysis

Phase 1: Ensures $x_i \leq 2s_i^*$, p_i / p_i^*, $p_i^* / p_i \leq 2$

Phase 2: Potential based argument showing misspending ($\Sigma_i [|z_i p_i| + c|s_i - s_i^*|p_i]$) decreases

c a suitable constant

The Potential Function

$$\Phi = \sum_i \Phi_i$$

$$\Phi_i = \left(\frac{1-\lambda t_i}{a_1}\right) p_i \left[\begin{array}{c} \mathrm{span}\left\{x_i^+, x_i^-, \tilde{x}_i^+, \tilde{x}_i^-\right\} \\ +\frac{1}{a_2 E}\left(x_i^+ - x_i^- + \tilde{x}_i^+ - \tilde{x}_i^-\right) \\ +\left(\tilde{x}_i^+ - \tilde{x}_i^-\right) \end{array}\right] + a_3 p_i^* \mid \tilde{x}_i - s_i^* \mid$$

v_i^+ denotes the maximum value of v_i in time period $[0, t_i]$

v_i^- denotes the minimum value of v_i in time period $[0, t_i]$

$t_i \leq 1$ is time since last update to p_i

At time $t_i = 0$,

$$\Phi_i = p_i[x_i - \tilde{x}_i] + a_3 p_i^* \,|\, \tilde{x}_i - s_i^* \,|$$

Recall: $$\tilde{x}_i = x_i + \kappa_i (s_i - s_i^*)$$

Reduction to One-Time Market

$$\Phi_i = \left(\frac{1-\lambda t_i}{a_1}\right) p_i \begin{bmatrix} \text{span}\left\{x_i^+, x_i^-, \tilde{x}_i^+, \tilde{x}_i^-\right\} \\ +\frac{1}{a_2 E}\left(x_i^+ - x_i^- + \tilde{x}_i^+ - \tilde{x}_i^-\right) \\ +\left(\tilde{x}_i^+ - \tilde{x}_i^-\right) \end{bmatrix} + \quad a_3 p_i^* \mid \tilde{x}_i - s_i^* \mid$$

$$\leq \left(\frac{1-\lambda t_i}{a_1}\right) p_i \begin{bmatrix} \text{span}\left\{x_i^+ + \tilde{x}_i - \tilde{x}_i^-, x_i^-, \tilde{x}_i\right\} \\ +\frac{1}{a_2 E}\left((x_i^+ + \tilde{x}_i - \tilde{x}_i^-) - x_i^-\right) \end{bmatrix} + \quad a_3 p_i^* \mid \tilde{x}_i - s_i^* \mid$$

If the price update is an increase

Further Issues

- What if goods and money are indivisible (Cole/Rastogi)?
- How might the parameters κ and λ be found by the market?
- Extend beyond markets obeying WGS.
- Extension to Arrow-Debreu setting involves parameter $0 \le \alpha \le 1$ ($\alpha = 1$ is the Fisher market, while $\alpha = 0$ corresponds to no money).
- How might one justify seller behavior in the Ongoing Market?