Bounds for the k-Dimension of Products of Special Posets

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The *k-dimension* of a poset P, written $\dim_k P$, is the least t such that P embeds in the product of t chains of size k; the usual order dimension imposes no bound on the chain size. For posets P and Q with unique maximum and minimum, Baker proved that $\dim(P \times Q) = \dim P + \dim Q$. Trotter proved that $\dim(S_n \times S_n) = 2n - 2$, where S_n is the "standard" n-dimensional poset with $2n$ elements. Trotter conjectured in 1982 that $\dim(P \times Q) \geq \dim P + \dim Q - 2$ when P and Q have connected diagrams. Reuter proved this for $(P, Q) = (S_m, S_n)$ and when $P = Q$ and $\dim P = 3$, but little other progress has been made.

To shed some light on the problem, we discuss upper bounds on $\dim_k$ for products of disconnected posets and lower bounds on $\dim_k$ for higher products of standard examples. Let mP denote the union of m disjoint copies of a poset P. We prove that $2\dim_k mP - \dim_k(mP \times mP)$ is unbounded if $k = 2$ or mP is an antichain. On the other hand, $\dim_k(S_m \times S_n) = m + n - \min\{2, k-2\}$. For higher-order products, $\dim_2(\prod_i S_{n_i}) = \sum_i n_i$. Our lower bounds for ordinary dimension are weaker: $\dim_k(\prod_i S_{n_i}) \geq \sum_i (n_i - 2)$.

These results are joint work with Michael Baym.

If time permits, the talk will conclude by mentioning several long-standing open problems in the combinatorics of posets, with a view toward stimulating their study.