Lower Bounds for Gap-Hamming-Distance and Consequences for Data Stream Algorithms

Amit Chakrabarti

(Joint work with Joshua Brody)

Dartmouth College

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Status of Certain Streaming Problems, Jan 2009

Problems:

- Distinct elements
- Frequency moments
- Empirical entropy

One-pass, randomized, ε-approximate:

- Space upper bound: $\widetilde{O}(\varepsilon^{-2})$
- Space lower bound: $\widetilde{\Omega}(\varepsilon^{-2})$

Do multiple passes help?

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Do multiple passes help? If not, why not?

The Gap-Hamming-Distance Problem

Input: Alice gets $x \in \{0,1\}^n$, Bob gets $y \in \{0,1\}^n$.

Output:

- $\text{GHD}(x,y) = 1$ if $\Delta(x,y) > \frac{n}{2} + \sqrt{n}$
- $\text{GHD}(x,y) = 0$ if $\Delta(x,y) < \frac{n}{2} - \sqrt{n}$

Problem: Design randomized, constant error protocol to solve this

Cost: Worst case number of bits communicated

$x =$	0	1	0	0	1	0	1	1	0	0	0	1
$y =$	0	0	0	0	0	0	1	1	1	0	0	1

$$n = 12; \quad \Delta(x,y) = 3 \in [6 - \sqrt{12}, 6 + \sqrt{12}]$$

The Reductions

E.g., Distinct Elements (Other problems: similar)

$x =$	0	1	0	0	1	0	1	1	0	0	0	1
σ :	(1,0)	(2,1)	(3,0)	(4,0)	(5,1)	(6,0)	(9,0)	(8,0)	(9,0)	(10,0)	(11,0)	(12,1)
$y =$	0	0	0	0	0	0	1	1	1	0	0	1
τ :	(1,0)	(2,0)	(3,0)	(4,0)	(5,0)	(6,0)	(9,0)	(8,0)	(9,1)	(10,0)	(11,0)	(12,1)

Alice: $x \longmapsto \sigma = \langle (1, x_1), (2, x_2), \ldots, (n, x_n) \rangle$

Bob: $y \longmapsto \tau = \langle (1, y_1), (2, y_2), \ldots, (n, y_n) \rangle$

Notice: $F_0(\sigma \circ \tau) = n + \Delta(x, y) = \begin{cases} < \frac{3n}{2} - \sqrt{n}, & \text{or} \\ > \frac{3n}{2} + \sqrt{n}. & \end{cases}$ Set $\varepsilon = \frac{1}{\sqrt{n}}$.

Communication to Streaming

p-pass streaming algorithm $\Longrightarrow$ $(2p-1)$-round communication protocol

messages = memory contents of streaming algorithm

And Thus

Previous results [Indyk-Woodruff'03], [Woodruff'04], [C.-Cormode-McGregor'07]:

- For one-round protocols, $\mathrm{R}^{\rightarrow}(\textsc{ghd}) = \Omega(n)$
- Implies the $\widetilde{\Omega}(\varepsilon^{-2})$ streaming lower bounds

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Key open questions:

- What is the unrestricted randomized complexity $\mathrm{R}(\textsc{ghd})$?
- Better algorithm for Distinct Elements (or F_k, or H) using two passes?

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Previous Results (Communication):

- One-round (one-way) lower bound: $\mathrm{R}^{\rightarrow}(\textsc{ghd}) = \Omega(n)$ [Woodruff'04]
- Simplification, clever reduction from $\textsc{index}$ [Jayram-Kumar-Sivakumar]

- Multi-round case: $\mathrm{R}(\textsc{ghd}) = \Omega(\sqrt{n})$ [Folklore]

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- Theorem 1: $\Omega(n)$ lower bound for any $O(1)$-round protocol

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What we show:

- Theorem 1: $\Omega(n)$ lower bound for any $O(1)$-round protocol

  Holds under uniform distribution
- Theorem 2: one-round, deterministic: $\mathrm{D}^{\rightarrow}(\textsc{ghd}) = n - \Theta(\sqrt{n}\log n)$
- Theorem 3: $\mathrm{R}^{\rightarrow}(\textsc{ghd}) = \Omega(n)$ (simpler proof, uniform distrib)

Technique: Round Elimination

Base Case Lemma: There is no "nice" 0-round GHD protocol.

Round Elimination Lemma: If there is a "nice" k-round GHD protocol, then there is a "nice" $(k-1)$-round GHD protocol.

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Round Elimination Lemma: If there is a "nice" k-round GHD protocol, then there is a "nice" $(k-1)$-round GHD$'$ protocol.

- The $(k-1)$-round protocol will be solving a "simpler" problem
- Parameters degrade with each round elimination step

Parametrized Gap-Hamming-Distance Problem

The problem:

$$\mathrm{GHD}_{c,n}(x,y) = \begin{cases} 1\,, & \text{if } \Delta(x,y) \geq n/2 + c\sqrt{n}\,, \\ 0\,, & \text{if } \Delta(x,y) \leq n/2 - c\sqrt{n}\,, \\ \star\,, & \text{otherwise.} \end{cases}$$

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$\mu_{c,n}$: uniform over (x,y) such that $|\Delta(x,y) - n/2| \geq c\sqrt{n}$

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Protocol assumptions (eventually, will lead to contradiction):

- Deterministic k-round protocol for $\mathrm{GHD}_{c,n}$
- Each message is $s \ll n$ bits
- Error probability $\le \varepsilon$, under distribution $\mu_{c,n}$

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- Protocol $\mathcal{Q}_1$:
  - Input: $x', y' \in \{0,1\}^A$ where $A \subseteq [n], \ |A| = n'$
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  - BAD_1: $\text{GHD}_{c',n'}(x',y') \neq \text{GHD}_{c,n}(x,y)$.
  - BAD_2: $\text{GHD}_{c,n}(x,y) \neq \mathcal{P}(x,y)$.

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VC-Dimension

Fixing Alice's first message:

- Call x good if $\Pr_y[\mathcal{P}(x,y) \neq \text{GHD}_{c,n}(x,y)] \leq 2\varepsilon$

  Then $\#\{\text{good } x\} \geq 2^{n-1}$ (Markov)

- Let $M = M_{\mathsf{m}} = \{\text{good } x : \text{Alice sends m on input } x\}$.

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Shattering:

- Say $S \subseteq \{0,1\}^n$ shatters $A \subseteq [n]$ if $\#\{x|_A : x \in S\} = 2^{|A|}$

- $\text{VCD}(S)$:= size of largest A shattered by S

Sauer's Lemma: If $\text{VCD}(S) < \alpha n$ then $|S| < 2^{nH(\alpha)}$.

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Extend $x' \to x$: pick $x \in M$ such that $x' = x|_A$

The First Bad Event

Recall BAD_1: $\textsc{ghd}_{c',n'}(x',y') \neq \textsc{ghd}_{c,n}(x,y)$.

Notation: $x = x' \circ x''$, $y = y' \circ y''$, $n = n' + n''$.

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Lemma: If x'', y'' nearly orthogonal and $c' \geq 2c$, then

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Corollary: $\Pr[BAD_1] < 1/8$.

The Second Bad Event

Recall BAD_2: $\textsc{ghd}_{c,n}(x,y) \neq \mathcal{P}(x,y)$.

Bounding $\Pr[BAD_2]$ is subtle:

- x is good, so $\Pr[\mathcal{P} \text{ errs} \mid x] \leq 2\varepsilon$
  - But this requires $(x,y) \sim \mu_{c,n}$
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- Actual distrib (fixed x, random y):
  - $(x, y) \sim (\mu_{c',n'} \mid x) \otimes \text{Unif}_{n''}$
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Lemma: $\Pr[BAD_2] = O(\varepsilon)$.

Round Elimination, First Attempt (Recap)

Putting it together:

- $\mathcal{P}$ is k-round ε-error protocol for $\text{GHD}_{c,n}$
- $\mathcal{Q}_1$ is $(k-1)$-round ε'-error protocol for $\text{GHD}_{c',n'}$ with
  - $c' = 2c, \; n' = n/3$
  - $\varepsilon' = 1/8 + O(\varepsilon)$

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Second attempt: protocol $\mathcal{Q}$:

- Repeat $\mathcal{Q}_1$ $2^{O(k)}$ times in parallel, take majority
- Blows up communication by $2^{O(k)}$
- Error is now $\varepsilon' = O(\varepsilon)$
  - Analysis even more subtle: not just a Chernoff bound

Eventual Round Elimination Lemma

Lemma: If there is a k-round, ε-error protocol for $\text{GHD}_{c,n}$ in which each player sends $s \ll n$ bits, then there is a $(k-1)$-round, $O(\varepsilon)$-error protocol for $\text{GHD}_{2c,n/3}$ in which each player sends $2^{O(k)}s$ bits.

Recall Base Case Lemma: There is no zero-round protocol with error $< 1/2$.

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Consequence: Main Theorem

Theorem: There is no $o(n)$-bit, $\frac{1}{3}$-error, $O(1)$-round randomized protocol for $\text{GHD}_{c,n}$. In other words, $\text{R}^{O(1)}(\text{GHD}) = \Omega(n)$.

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More Specific: $\text{R}^{k}(\text{GHD}) = n/2^{O(k^2)}$.

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- Rectangle-based methods (discrepancy/corruption)
- Approximate polynomial degree
- Pattern matrix, Factorization norms [Sherstov'08], [Linial-Shraibman'07]
- Information complexity [C.-Shi-Wirth-Yao'01], [BarYossef-J.-K.-S.'02]

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  Hmm! Can't see a concrete obstacle

Why Did This Take So Long?

Multi-pass lower bounds for Distinct Elements and F_k has been an important open question since at least 2003. Why did it remain open for so long?

Underlying communication problem thorny! Resists the "usual" attacks:

- Rectangle-based methods (discrepancy/corruption)

  Matrix has large near-monochromatic rectangles

- Approximate polynomial degree

  Underlying predicate has approx degree $\widetilde{O}(\sqrt{n})$

- Pattern matrix, Factorization norms [Sherstov'08], [Linial-Shraibman'07]

  Quantum communication upper bound $O(\sqrt{n}\log n)$

- Information complexity [C.-Shi-Wirth-Yao'01], [BarYossef-J.-K.-S.'02]

  Hmm! Can't see a concrete obstacle

  I'm biased (I helped invent it, so it's my pet technique)

Open Problems

1. The key problem here: Settle $\mathrm{R}(\textsc{ghd})$.
2. More generally: Understand communication complexity of "gap problems" better.
3. This should help with other streaming problems, e.g., longest increasing subsequence.