The Effect of Spatial Correlations on MIMO Capacity: A (not so) Large N Analytical Approach:

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Outline

- Aim:
  - Calculate statistics of MIMO Capacities with Correlated Channels and Interference
  - Use partial knowledge of channel (at transmitter) to optimize throughput
- Method (large antenna N analysis)
- Results/Examples
  - Gaussian character of capacity
- Applications

Mutual Information with Correlations & Interference

$$I = \log\left[\frac{\det\left(\nu + GQG^{+} + HH^{+}\right)}{\det\left(\nu + HH^{+}\right)}\right]$$

- G : n_R x n_T Channel matrix
- H : n_I x n_T Interferer matrix (known to receiver)
- ν : n_R x n_R Noise (due to interference with channel unknown to receiver)
- Q : Transmitted Signal covariance

- G (and thus I) are random: Treat them statistically

- Assumption: Temporal Average = Spatial Average over channel realizations

Statistical Treatment of *G*

- Small parameter λ/ℓ = Wavelength/Mean Free Path <= 0.01-0.1
- Leading correction in systematic expansion in λ/ℓ :
  - E[G G$^+$] since E[G]=0 (diffuson approximation)
  - G, H: Gaussian random (not generally iid)

$$E\left[G_{ia}G_{jb}^{*}\right]=T^{s}{}_{ij}\,\frac{\rho_s}{n_T}\,R^{s}{}_{ab}$$

$$E\left[H_{ia}H_{jb}^{*}\right]=T^{I}{}_{ij}\,\frac{\rho_I}{n_I}\,R^{I}{}_{ab}$$

$$\rho_s = S/N$$

$$\rho_I = I/N$$

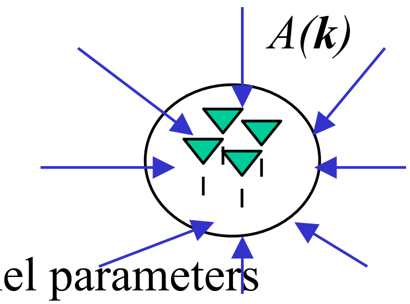
In general not separable:
e.g. MUD or multi-keyholes

$$E\left[G_{ia}G_{jb}^{*}\right]=\sum_k T^{k}{}_{ij}\,\frac{\rho_k}{n_T}\,R^{k}{}_{ab}$$

Correlation Matrix $\boldsymbol{R}$ (and analogously $\boldsymbol{T}$)

- Response $\chi_\alpha(\boldsymbol{k})$ of Antenna α to Incoming Wave $\boldsymbol{k}$ of Amplitude $A(\boldsymbol{k})$

$$r_\alpha = \int d\hat{k}\ A(k)\,\chi_\alpha(k)$$



- $$R_{\alpha\beta} = E\left[r^*_\alpha r_\beta\right] = \int d\hat{k}\ \ \chi_\alpha{}^*(k)\chi_\beta(k)w(k)$$

- *w(k)* wave weight: depends on angle-spread & other channel parameters

$$w(\varphi) = \exp\left[-\frac{\varphi^2}{2\delta^2}\right]$$

- Based on antenna/array and channel properties can *determine* R_{ab}
- correlated antennas: rank R = 1;
  - (one big antenna) beamforming
- uncorrelated antennas: eigs of R equal
  - no interference (each antenna "sees" different field)

Statistics of Mutual Information

- Aim: Calculate Statistical Properties of *I*

- Method: Large (but finite) Number Antenna Approximation

- Calculate Moments of ***I*** (e.g. **E[I]**, **Var[I]**, **Sk[I]**):

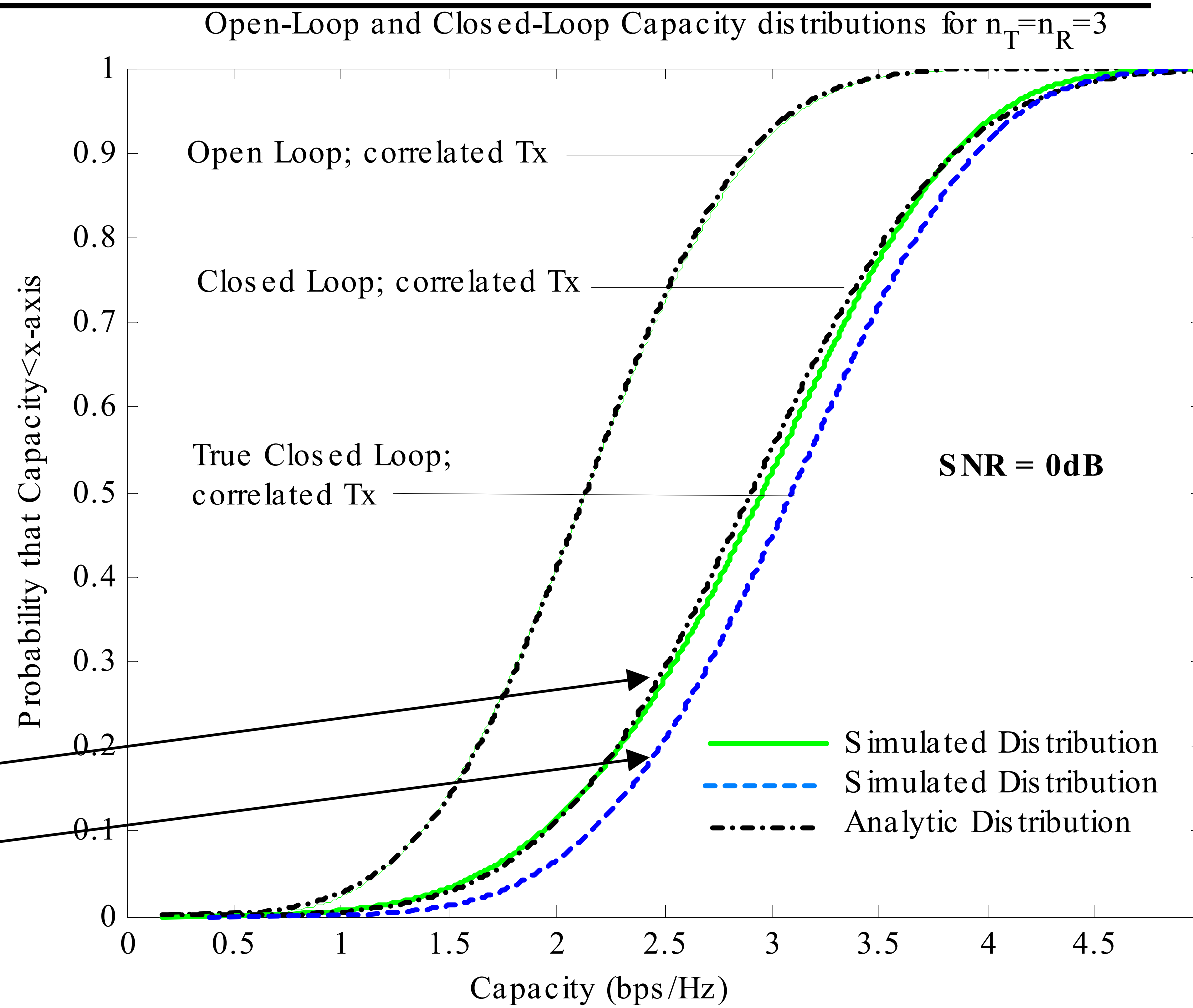
- Quantities of Interest:
  - **E[I]**
  - Outage Capacity (Solve for $\mathbf{Prob(I>I_0)=p_0}$)
- Optimize Transmission given partial channel information
  - $\mathbf{C = max_Q\ E[I \mid Q]}$ (versus $\mathbf{C' = E[\ max_Q\ (I \mid Q)\ ]}$)
  - Optimization based on quasi-static channel statistics ***T, R***:
    - **Statistical Waterpouring**
  - C is *Realistic* closed loop capacity

Gaussianity of the PDF(I)

Approximation:

$$PDF(I) \approx N\left(E[I], Var[I]\right)$$

- Nominally valid for large antenna numbers
- Surprisingly accurate even for $n_T=n_R=3$ and correlated channels
    - Angle spread 5°
    - Linear array with $d_{min}=\lambda$
- $C=\max_Q E[\ I \mid Q]$
- $C=E[\max_Q I|Q]$
- Statistical Waterpouring gives good results

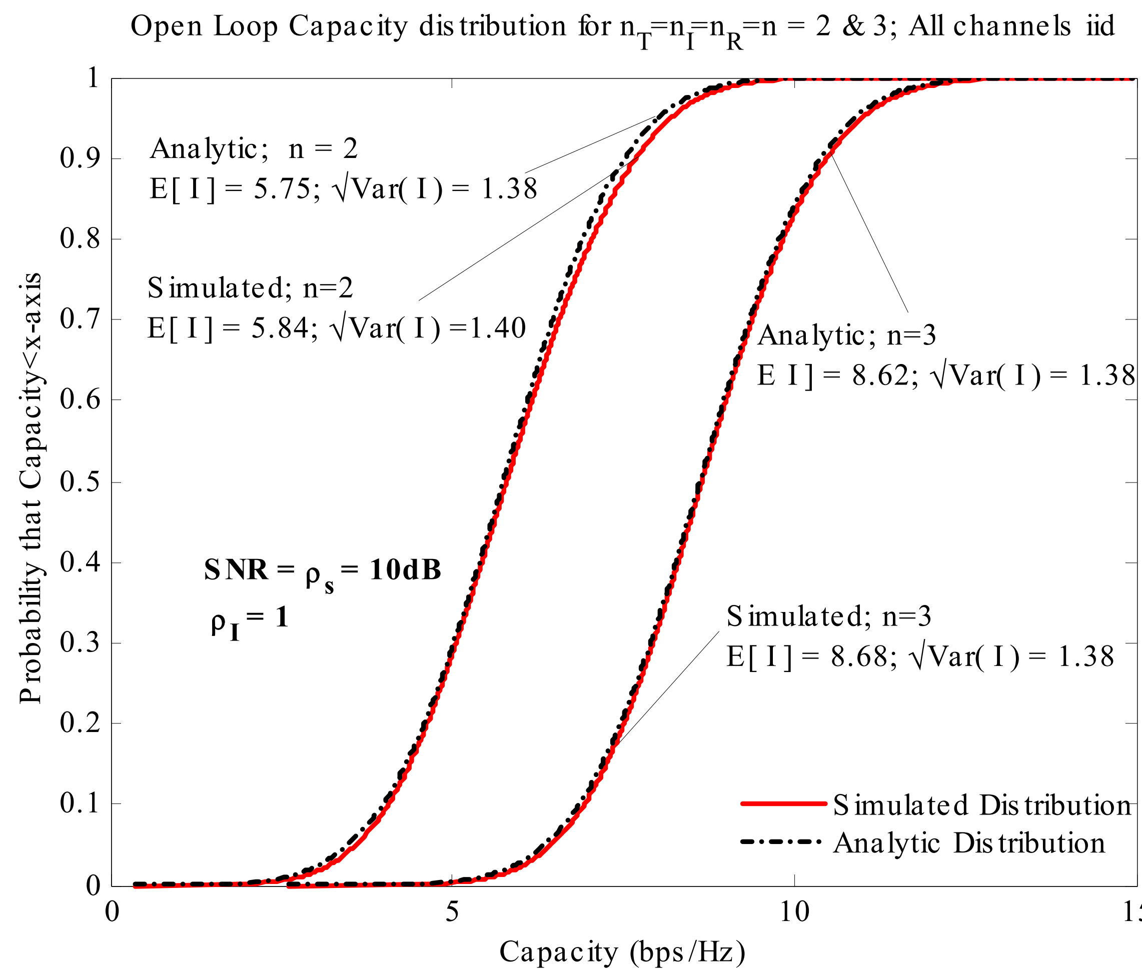


Gaussianity of the PDF(I)

Approximation:

$$PDF(I) \approx N(E[I], Var[I])$$

- Nominally valid for large antenna numbers
- Accurate even with small $n_T = n_R = n_I = 2, 3$



Why is N(E[I],Var[I]) so accurate?

Main Reason:

- E[I], Var[I]:
  - For finite n_T, n_R, n_I they are finite and n – dependent : the leading terms capture (most) n – dependence
- Higher moments (Skewness etc): O(1/n)
  - Also correction to E[I] = O(1/n), Var[I]=O(1/n^2)
- Additional small parameter (for not too large SIR) ?

Method:

Calculate generating function:

$$g(\mu) = E\left[e^{-\mu\ I}\right] \xrightarrow[\mu \to 0^+]{} \exp\left[-\mu\ E[I] + \frac{\mu^2}{2} Var[I] - \frac{\mu^3}{6} Sk[I] \cdots\right]$$

$$g(\mu) = E\left[\frac{\det\left(\nu + GQG^+ + HH^+\right)^{-\mu}}{\det\left(\nu + HH^+\right)^{-\mu}}\right]$$

Replica Trick:

- Generate μ replicas of determinants (Sengupta & Mitra)
- Then analytically continue μ to zero (Parisi)

Method:

$$g(\mu) = E\left[\det\left(\nu + GQG^{+} + HH^{+}\right)^{-\mu} \det\left(\nu + HH^{+}\right)^{+\mu}\right]$$

- To average over G use identity:

$$\det\left(\nu + GQG^{+} + HH^{+}\right)^{-1} = \int dX \quad e^{-X^{+}(\nu + GQG^{+} + HH^{+})X}$$

- To average over H tricky – Need to combine positive and negative powers of dets:

$$\det\left(\nu + HH^{+}\right)^{+1} = \int dA \quad e^{A^{+}(\nu + HH^{+})A}$$

- Thus A have to be Grassman variables:

$$\int dA = 0; \qquad \int AdA = 1; \qquad AB = -BA; \qquad A^2 = 0$$

Method:

- Represent determinant by integrals
- Integrate out G, H (correlated Gaussian)
- Introduce auxiliary μ x μ matrix variables $\{t, r\}$ to represent $g(\mu)$:

$$g(\mu) = \int \{dt, dr\} \exp[-S(t,r)]$$

- Analytically continue μ to zero and find saddle point of S for large n

$$t_{\alpha\beta} \rightarrow t\delta_{\alpha\beta} + \delta t_{\alpha\beta}; \qquad \alpha, \beta = 1, \cdots \mu$$

- Saddle point S_0 gives ergodic capacity
- Corrections $(\delta t_{\alpha\beta})^{2k+2}$ give systematically higher moments, e.g. Var, Sk
- Need to solve few algebraic equations to find saddle point

Equations to Solve. Example: ($n_I = 0$)

$$E\left[G_{ia}G_{jb}^{*}\right] = T_{ij}\frac{\rho_s}{n_T}R_{ab}$$ ← Given

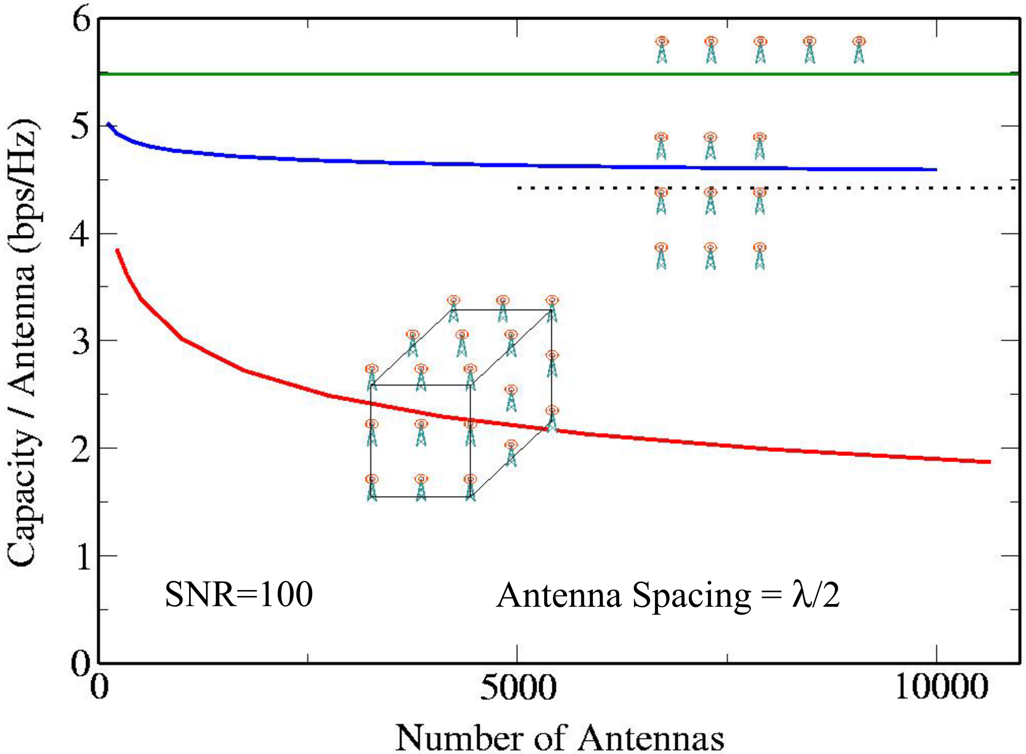
$$E[I] = Tr\log\left[1 + tT\sqrt{\rho_s}\right] + Tr\log\left[1 + rR\sqrt{\rho_s}\right] - n_T tr$$ ← Answer

$$t = \frac{1}{n_T}Tr\left[\frac{R\sqrt{\rho_s}}{1 + rR\sqrt{\rho_s}}\right]$$

$$r = \frac{1}{n_T}Tr\left[\frac{T\sqrt{\rho_s}}{1 + tT\sqrt{\rho_s}}\right]$$

← Solve for t, r

Results: *Ergodic Capacity E[I]: Open Loop (Q=1)*



Large N approach valid
For $n_T>2$
n_T dependence of Capacity due to antenna correlations = Real

3D : Capacity ~ $N^{2/3}$
Only Antennas on the Surface Resolve Incoming Directions

Results: *Ergodic Capacity E[I]*

Open Loop vs. Closed Loop:

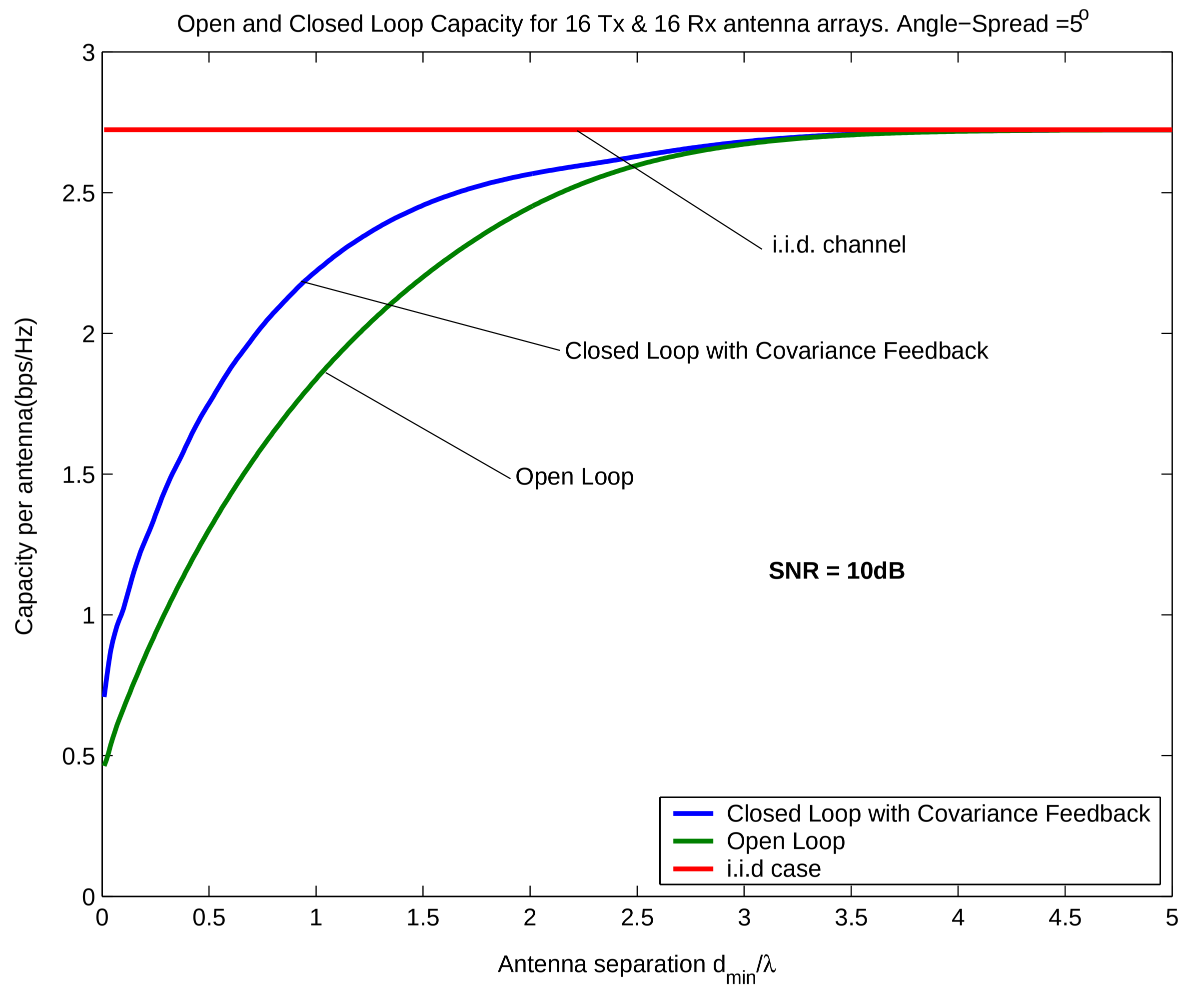
Calculate $\mathbf{C = max_Q\ E[\ I\ |\ Q]}$

Determine optimal Q analytically

No Interferers

Result: SNR dependent

High SNR=10: Low gain



Results: *Ergodic Capacity E[I]*

Open Loop vs. Closed Loop:

Calculate $\mathbf{C = max_Q\ E[\ I \mid Q]}$

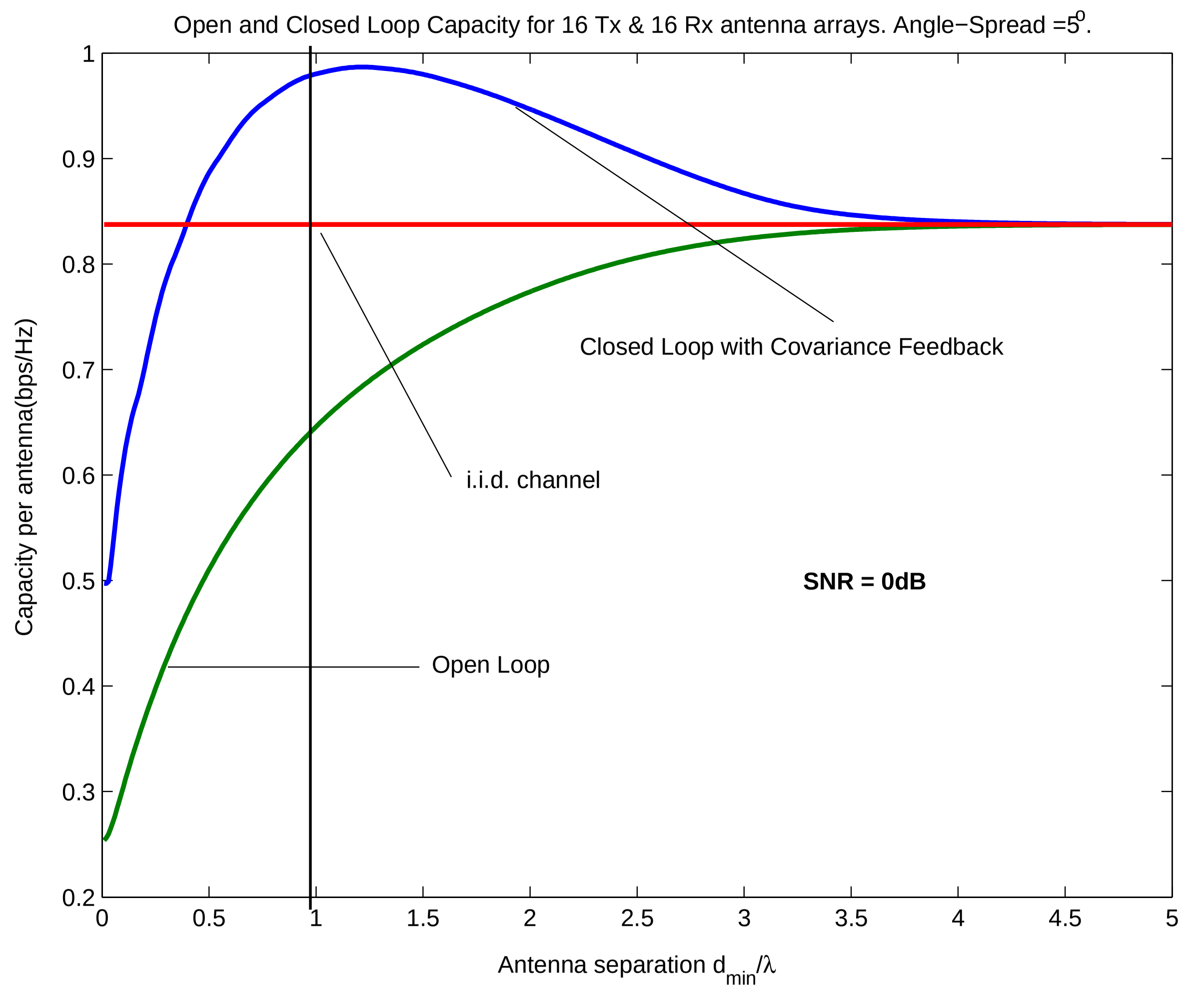
No Interferers

Result: SNR dependent

Low SNR=1: High gain

Beamforming beneficial

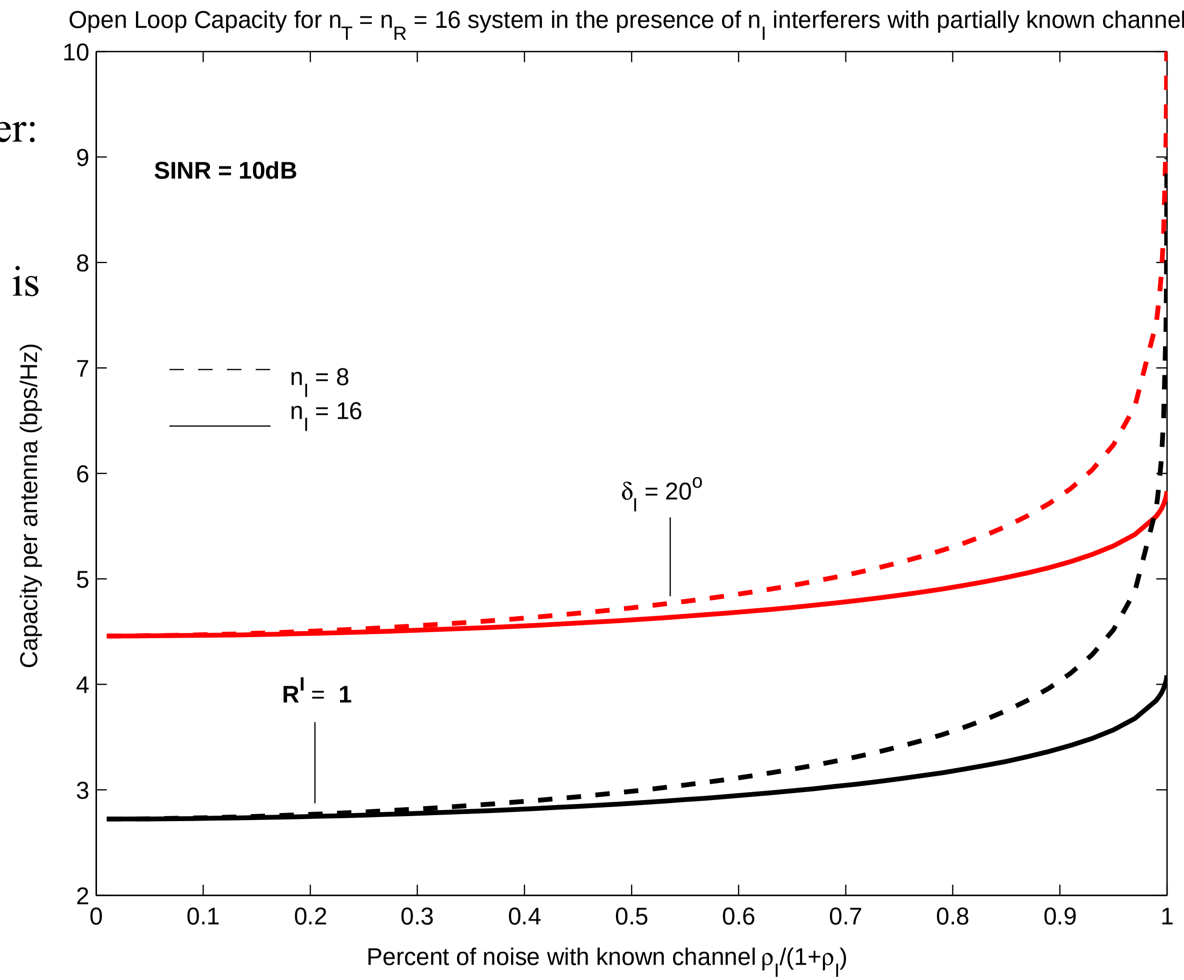
at low SNR



Results: *Ergodic Capacity E[I]*

- Gain of knowing the channel of the interferer at the receiver:
- Knowing **H** vs. ν (E(**HH***))
- Gain is substantial if channel is known accurately ($INR=\rho_I \to \infty$)
- Result is Angle-Spread (δ) dependant
- $\nu \to 0$ and $n_I < n_T$ results to: $C \to \infty$
- If $n_I = n_T$ ($=n_R$) and $\nu \to 0$:

$$C = 2n_T \log_2 \left[1 + \sqrt{SNR}\right]$$



Results: *Higher Moments: Var[I], Sk[I], …*

Variance:

- Calculate using $\left(\delta t_{\alpha\beta}\right)^2; \quad \alpha, \beta = 1, \cdots \mu$
- O(1) in n_T, n_R etc
  - (O(1/N) for $n_T \ll n_R$ (or vice-versa))
- Diverges (logarithmically) at large SNR (small parameter?)
- Large N result gives very accurate prediction

Skewness (and correction to E[I]):

- From $\left(\delta t_{\alpha\beta}\right)^3$ corrections
- O(1/N) (for $n_T \sim n_R$ etc)

Higher moments:

- $Cum_k = O(N^{2-k})$ $\longrightarrow$ $I \approx N(E[I], Var[I])$

Summary – Applications

- Powerful method to calculate MIMO capacities
  - with even few antennas
  - with correlated channels/noise and interference
  - applicable to other cases (MUD etc)
- Straightforward method to find optimal transmission scheme based on partial channel information.
- Gaussian approximation of capacity
  - accurate even for few antennas.
  - simplifies System Level Analysis.
    - analytic results for system level capacity
    - scheduling Hochwald *et al*
  - allows Outage Capacity calculation.