Dynamic Pricing for Non-Perishable Products with Demand Learning

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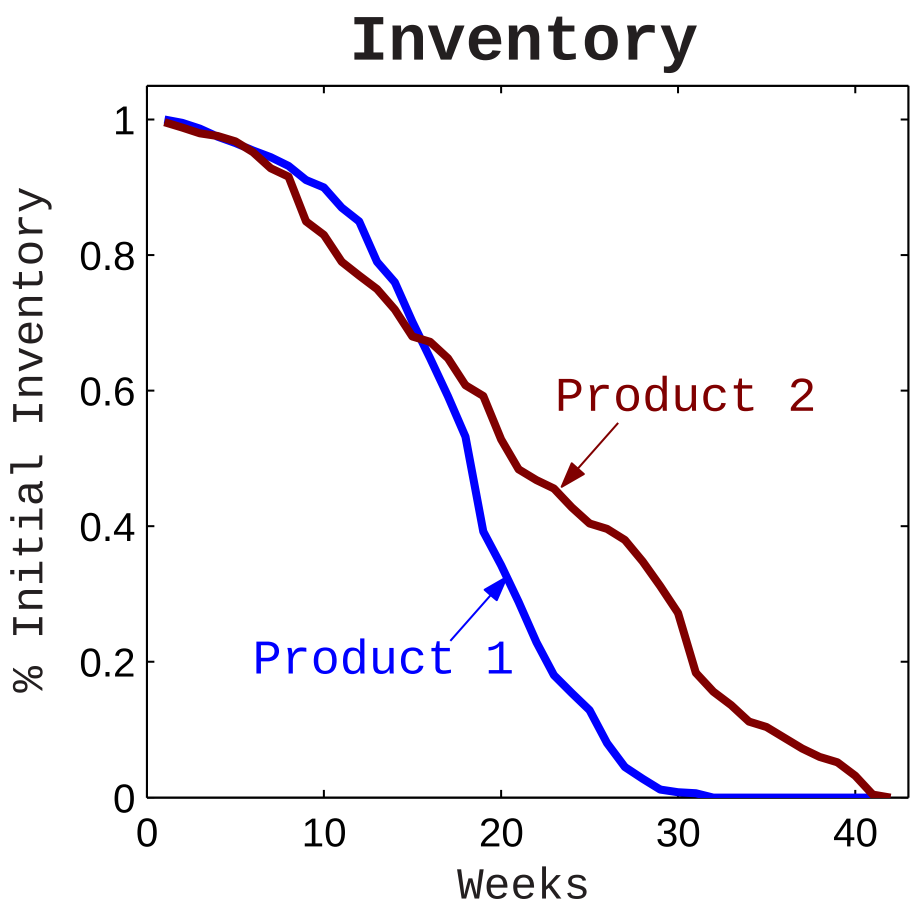
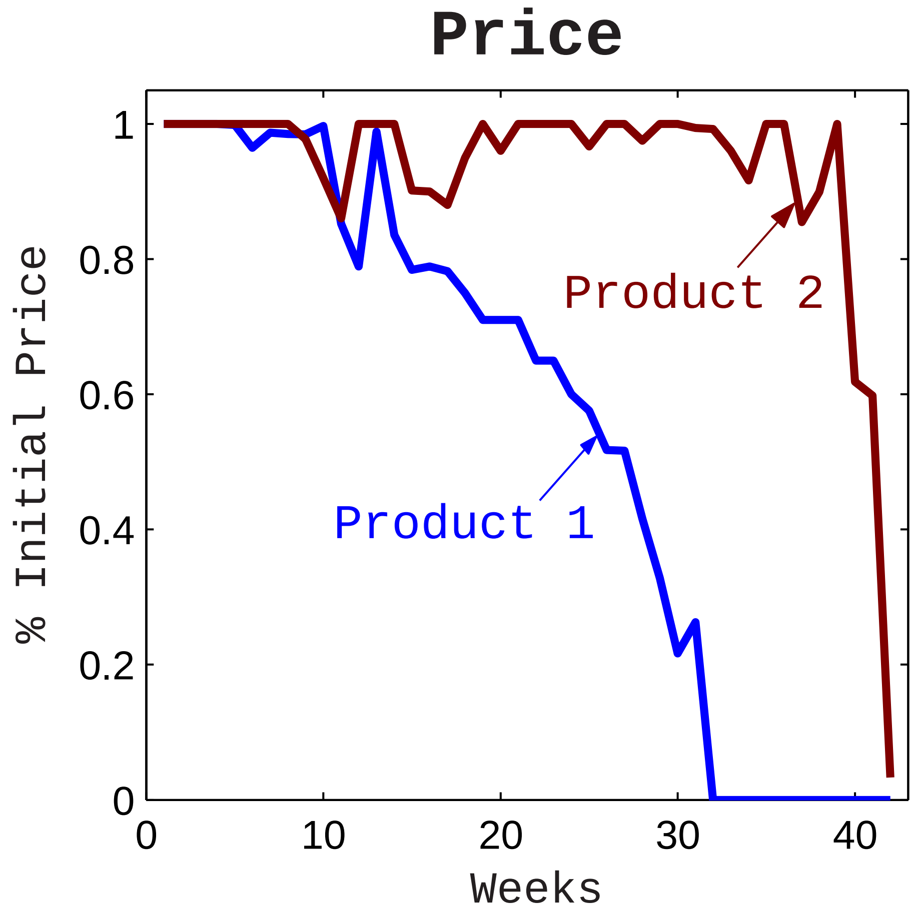
DIMACS Workshop on Yield Management and Dynamic Pricing

Rutgers University

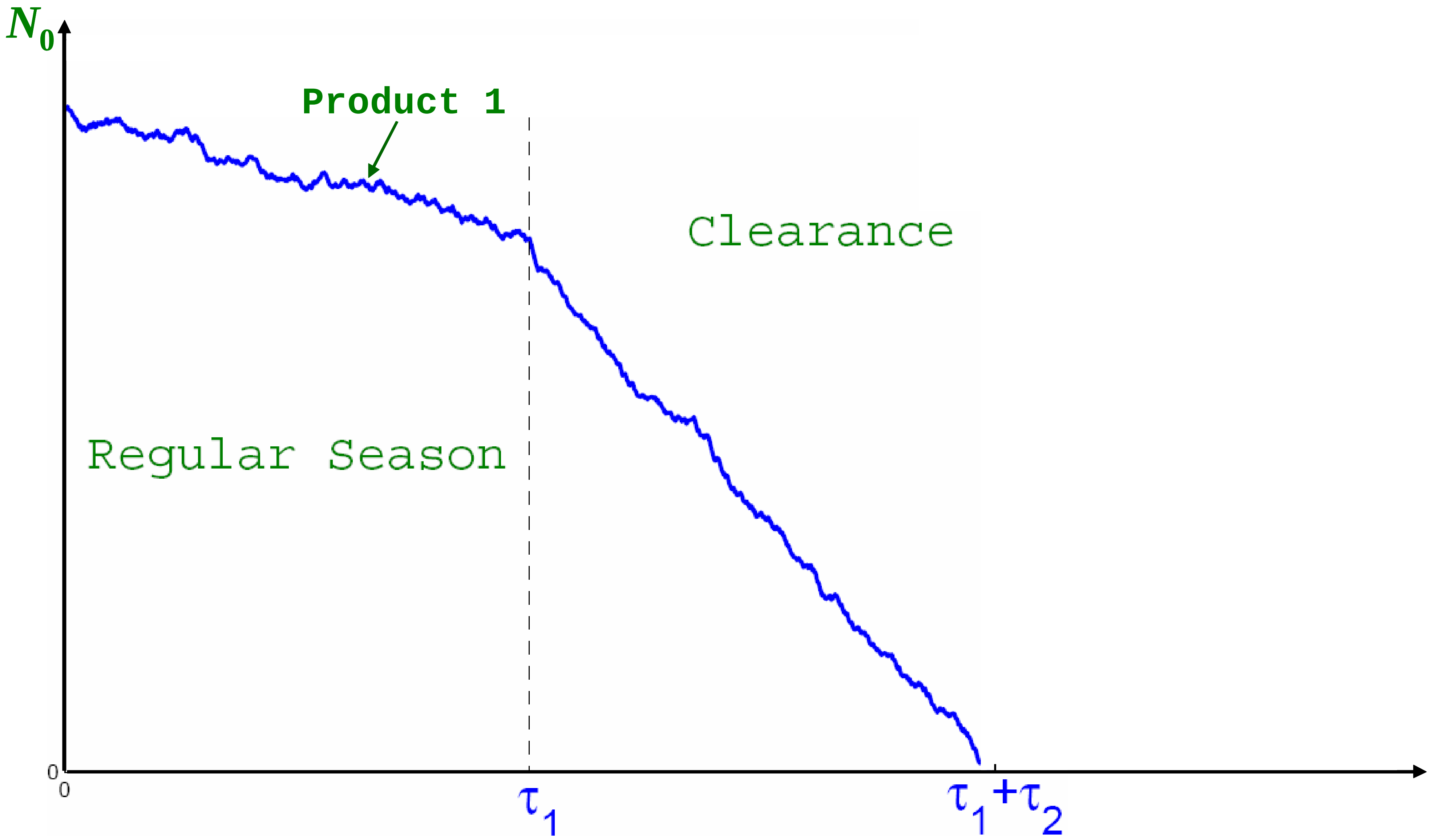
August, 2005

Motivation

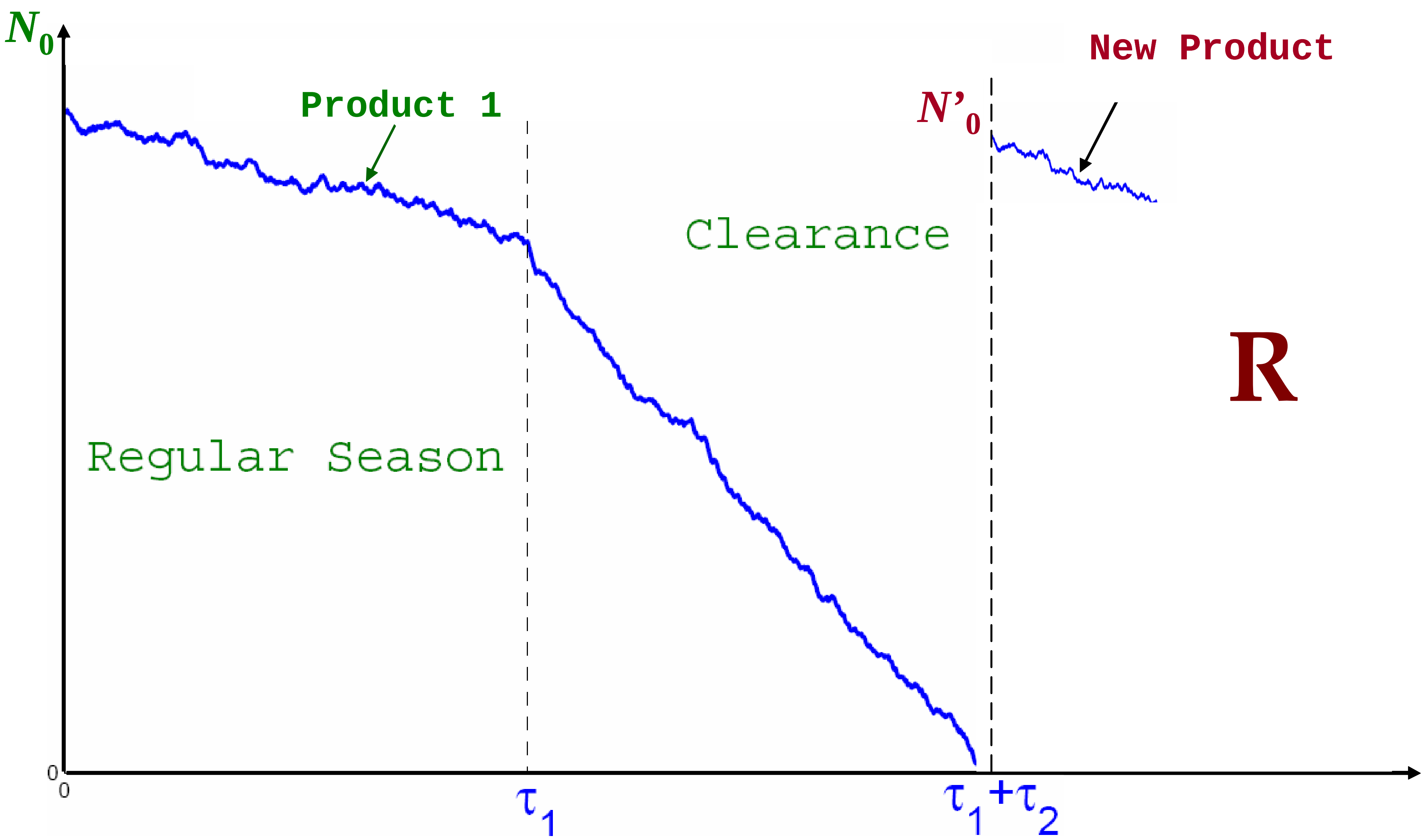

Inventory
% Initial Inventory
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
Weeks
Product 2
Product 1


Price
% Initial Price
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
Weeks
Product 2
Product 1

Motivation



Motivation



N0
New Product
Product 1
N'0
Clearance
R
Regular Season
0
0
τ1
τ1+τ2

Motivation

- For many retail operations "capacity" is measured by store/shelf space.

- A key performance measure in the industry is

  Average Sales per Square Foot per Unit Time.

- Trade-off between short-term benefits and the opportunity cost of assets.

  Margin vs. Rotation.

- As opposed to the airline or hospitality industries, selling horizons are flexible.

- In general, most profitable/unprofitable products are new items for which there is little demand information.

Outline

✓ Model Formulation.

✓ Perfect Demand Information.

✓ Incomplete Demand Information.

- Inventory Clearance
- Optimal Stopping ("outlet option")

✓ Conclusion.

Model Formulation

I) Stochastic Setting:

- A probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- A standard Poisson process $D(t)$ under $\mathbb{P}$ and its filtration $\mathcal{F}_t = \sigma(D(s) \, : \, 0 \leq s \leq t)$.

- A collection $\{\mathbb{P}_\alpha \, : \, \alpha > 0\}$ such that $D(t)$ is a Poisson process with intensity α under $\mathbb{P}_\alpha$.

- For a process f_t, we define $I_f(t) := \int_0^t f_s \, \mathrm{d}s$.

II) Demand Process:

- Pricing strategy, a nonnegative (adapted) process p_t.

- A price-sensitive unscaled demand intensity

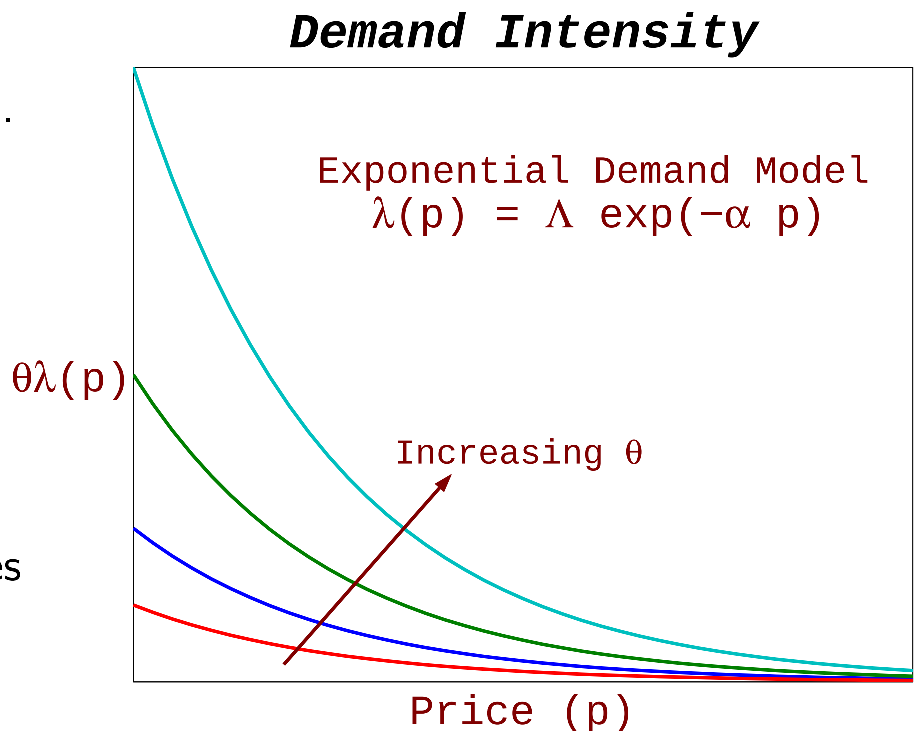
$$\lambda_t := \lambda(p_t) \iff p_t = p(\lambda_t).$$

- A (possibly unknown) demand scale factor $\theta > 0$.

- Cumulative demand process $D(I_\lambda(t))$ under $\mathbb{P}_\theta$.

- Select $\lambda \in \mathcal{A}$ the set of admissible (adapted) policies

$$\lambda_t : \mathbb{R}_+ \to [0, \Lambda].$$



Model Formulation

III) Revenues:

- Unscaled revenue rate $c(\lambda) := \lambda\, p(\lambda)$, $\lambda^* := \text{argmax}_{\lambda \in [0,\Lambda]}\{c(\lambda)\}$, $c^* := c(\lambda^*)$.
- Terminal value (opportunity cost): R Discount factor: r
- Normalization: $c^* = r\, R$.

IV) Selling Horizon:

- Inventory position: $N_t = N_0 - D(I_\lambda(t))$.
- $\tau_0 = \inf\{t \geq 0 \ : \ N_t = 0\}$, $\mathcal{T} := \{\mathcal{F}_t - \text{stopping times } \tau \text{ such that } \tau \leq \tau_0\}$

V) Retailer's Problem:

$$\max_{\lambda \in \mathcal{A},\ \tau \in \mathcal{T}} \quad \mathbb{E}_\theta\left[\int_0^\tau e^{-r\,t}\, p(\lambda_t)\, \mathrm{d}D(I_\lambda(t)) + e^{-r\,\tau}\, R\right]$$

$$\text{subject to} \quad N_t = N_0 - D(I_\lambda(t)).$$

Full Information

Suppose $\theta > 0$ is known at $t = 0$ and an inventory clearance strategy is used, *i.e.*, $\tau = \tau_0$.

Define the value function

$$W(n;\theta) = \max_{\lambda \in \mathcal{A}} \quad \mathbb{E}_\theta \left[\int_0^{\tau_0} e^{-r\,t}\, p(\lambda_t)\, \mathrm{d}D(I_\lambda(t)) + e^{-r\,\tau}\, R \right]$$

$$\text{subject to} \quad N_t = n - D(I_\lambda(t)) \quad \text{and} \quad \tau_0 = \inf\{t \geq 0 \ : \ N_t = 0\}.$$

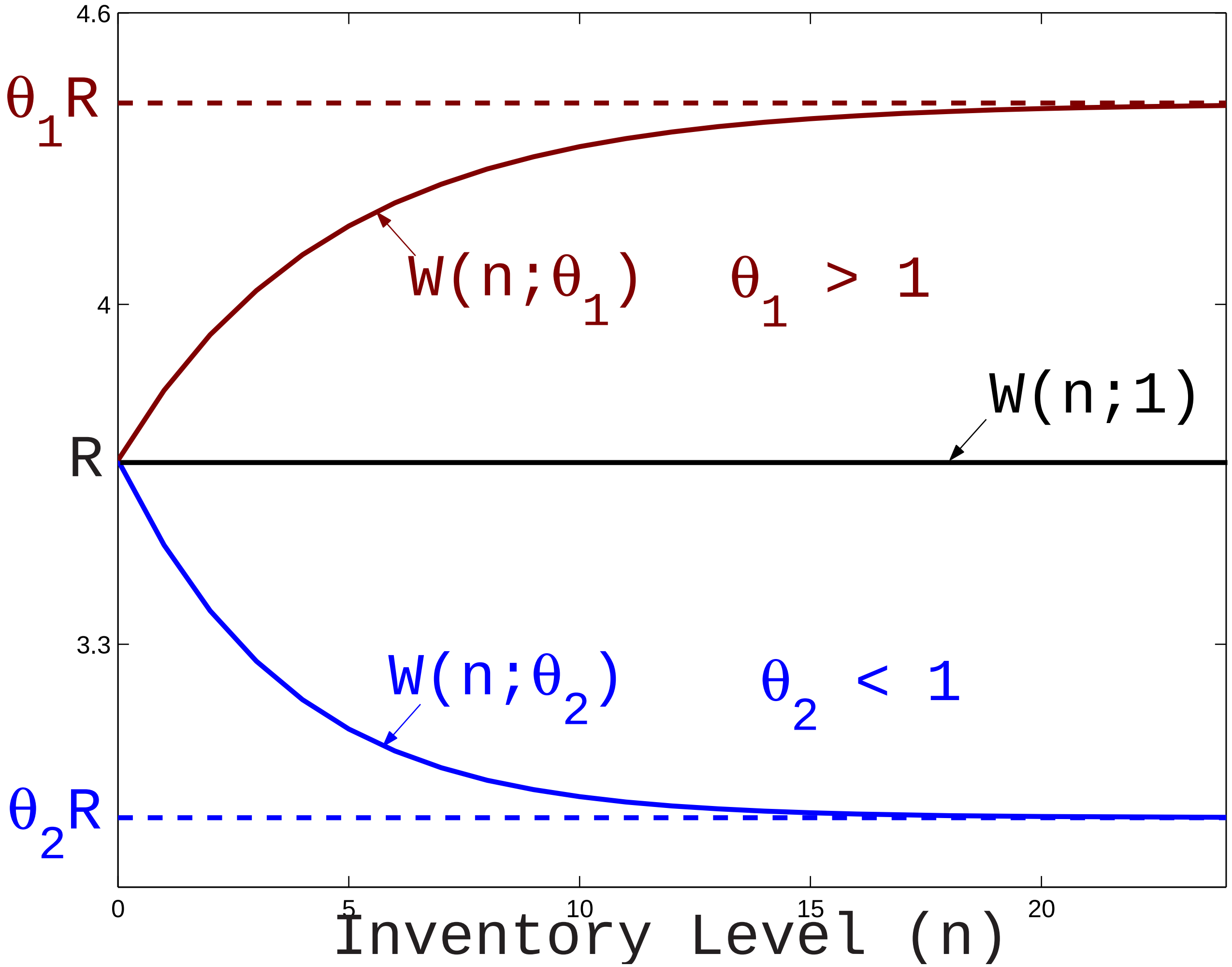
The solution satisfies the recursion $\dfrac{r\,W(n;\theta)}{\theta} = \Psi(W(n-1;\theta) - W(n;\theta))$ and $W(0;\theta) = R$,

$$\text{where} \quad \Psi(z) \triangleq \max_{0 \leq \lambda \leq \Lambda} \{\lambda\, z + c(\lambda)\}.$$

Proposition. For every $\theta > 0$ and $R \geq 0$ there is a unique solution $\{W(n) \ : \ n \in \mathbb{N}\}$.

- If $\theta \geq 1$ then the value function W is increasing and concave as a function of n.
- If $\theta \leq 1$ then the value function W is decreasing and convex as a function of n.
- $\lim_{n \to \infty} W(n) = \theta R$.

Full Information



Value function for two values of θ and an exponential demand rate $\lambda(p) = \Lambda\ \exp(-\alpha\, p)$.
The data used is $\Lambda = 10$, $\alpha = 1$, $r = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.8$, $R = \Lambda\ \exp(-1)/(\alpha\, r) \approx 3.68$.

Full Information

Corollary. Suppose $c(\lambda)$ is strictly concave.

The optimal sales intensity satisfies:

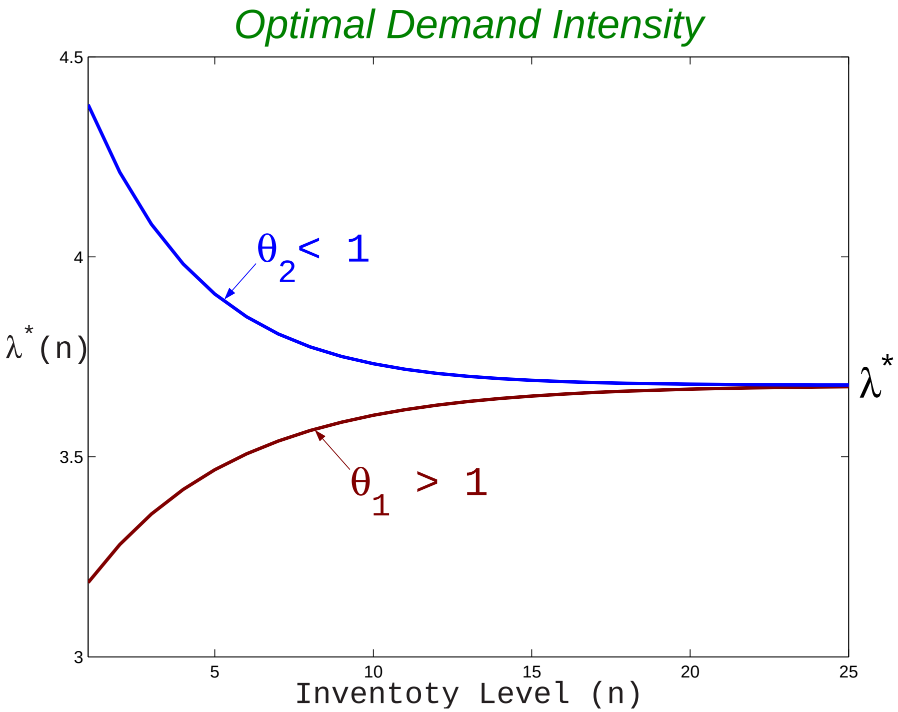
$$\lambda^*(n;\theta) = \underset{0\,\le\,\lambda\,\le\,\Lambda}{\text{argmax}} \; \{\lambda\,(W(n-1;\theta)-W(n;\theta))+c(\lambda)\}.$$

- If $\theta \geq 1$ then $\lambda^*(n;\theta) \uparrow n$.

- If $\theta \leq 1$ then $\lambda^*(n;\theta) \downarrow n$.

- $\lambda^*(n;\theta) \downarrow \theta$.

- $\lim_{n\to\infty} \lambda^*(n,\theta) = \lambda^*$.



Exponential Demand $\lambda(p) = \Lambda\ \exp(-\alpha\, p)$.
$\Lambda = 10$, $\alpha = r = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.8$, $R = 3.68$.

What about inventory turns (rotation)?

Proposition. Let $s(n,\theta) \triangleq \theta\,\lambda^*(n,\theta)$ be the optimal sales rate for a given θ and n.

$$\text{If} \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda\, p'(\lambda)) \leq 0, \qquad \text{then} \quad s(n,\theta) \uparrow \theta.$$

Full Information

SUMMARY:

- A tractable dynamic pricing formulation for the inventory clearance model.

- $W(n;\theta)$ satisfies a simple recursion based on the Fenchel-Legendre transform of $c(\lambda)$.

- With full information products are divided in two groups:
  - High Demand Products with $\theta \geq 1$: $W(n,\theta)$ and $\lambda^*(n)$ increase with n.
  - Low Demand Products with $\theta \leq 1$: $W(n,\theta)$ and $\lambda^*(n)$ decrease with n.

- High Demand products are sold at a higher price and have a higher selling rate.

- If the retailer can stop selling the product at any time at no cost then:
  - If $\theta < 1$ stop immediately ($\tau = 0$).
  - If $\theta > 1$ never stop ($\tau = \tau_0$).

- In practice, a retailer rarely knows the value of θ at $t = 0$!

Incomplete Information: Inventory Clearance

Setting:

- The retailer does not know θ at $t = 0$ but knows $\theta \in \{\theta_L, \theta_H\}$ with $\theta_L \leq 1 \leq \theta_H$.

- The retailer has a prior belief $q \in (0, 1)$ that $\theta = \theta_L$.

- We introduce the probability measure $\mathbb{P}_q = q\,\mathbb{P}_{\theta_L} + (1-q)\,\mathbb{P}_{\theta_H}$.

- We assume an inventory clearance model, *i.e.*, $\tau = \tau_0$.

Retailer's Beliefs:

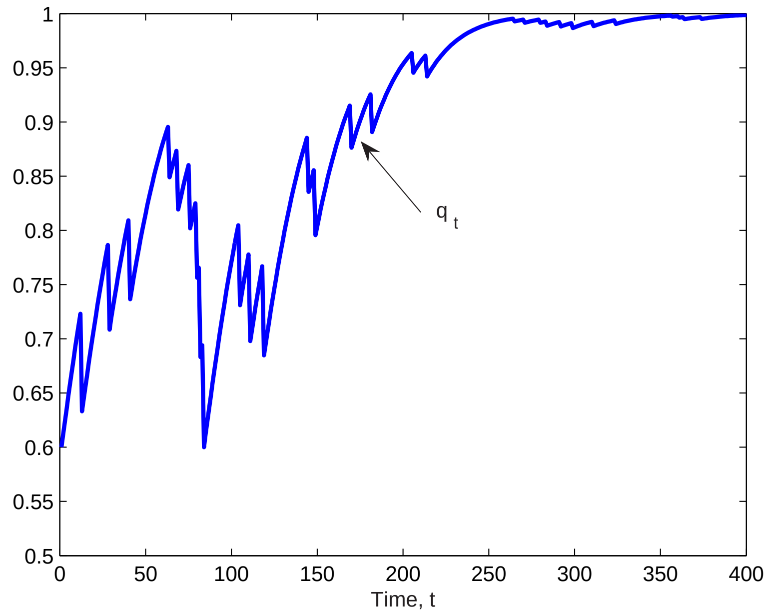
Define the belief process $\quad q_t := \mathbb{P}_q[\theta \mid \mathcal{F}_t]$.

Proposition. q_t is a $\mathbb{P}_q$-martingale that satisfies the SDE

$$dq_t = -\eta(q_{t-})\,[dD_t - \lambda_t\,\bar{\theta}(q_{t-})dt],$$

$$\text{where} \quad \bar{\theta}(q) := \theta_L\,q + \theta_H\,(1-q)$$

$$\text{and} \quad \eta(q) := \frac{q\,(1-q)\,(\theta_H - \theta_L)}{\theta_L q + \theta_H\,(1-q)}.$$



Incomplete Information: Inventory Clearance

Retailer's Optimization:

$$V(N_0, q) = \sup_{\lambda \in \mathcal{A}} \mathbb{E}_q \left[\int_0^{\tau_0} e^{-r\,t}\, p(\lambda_t)\, \mathrm{d}D(I_\lambda(s)) + e^{-r\,\tau_0}\, R \right]$$

subject to

$$N_t = N_0 - \int_0^t \mathrm{d}D(I_\lambda(s)),$$

$$\mathrm{d}q_t = -\eta(q_{t-})\,[\mathrm{d}D_t - \lambda_t\,\bar{\theta}(q_{t-})\mathrm{d}t], \quad q_0 = q,$$

$$\tau_0 = \inf\{t \geq 0 \;:\; N_t = 0\}.$$

HJB Equation:

$$rV(n,q) = \max_{0 \leq \lambda \leq \Lambda} \left[\lambda\,\bar{\theta}(q)[V(n-1, q-\eta(q)) - V(n,q) + \eta(q)V_q(n,q)] + \bar{\theta}(q)\,c(\lambda) \right],$$

with boundary condition $V(0,q) = R$, $V(n,0) = W(n;\theta_H)$, and $V(n,1) = W(n;\theta_L)$.

Recursive Solution:

$$V(0,q) = R, \qquad V(n,q) + \Phi\left(\frac{r\,V(n,q)}{\bar{\theta}(q)}\right) - \eta(q)\,V_q(n,q) = V(n-1, q-\eta(q)).$$

Incomplete Information: Inventory Clearance

Proposition.

-) The value function $V(n,q)$ is

a) monotonically decreasing and convex in q,

b) bounded by

$$W(n;\theta_L) \le V(n,q) \le W(n;\theta_H), \quad \text{and}$$

c) uniformly convergent as $n \uparrow \infty$,

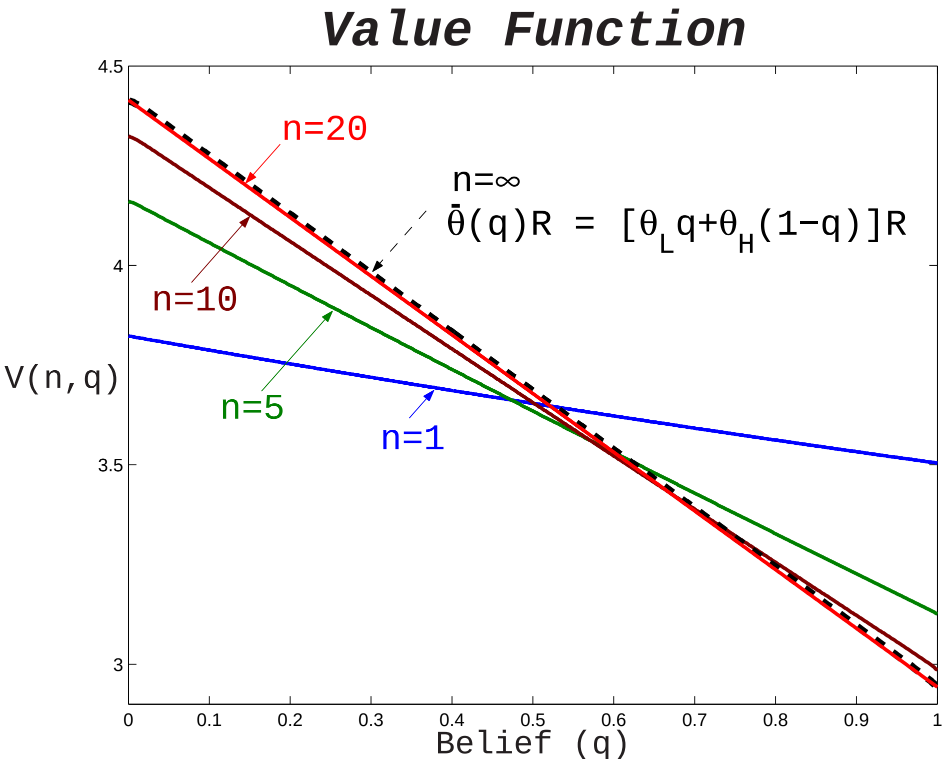
$$V(n,q) \overset{n\to\infty}{\longrightarrow} R\,\bar{\theta}(q), \quad \text{uniformly in } q.$$

-) The optimal demand intensity satisfies

$$\lim_{n\to\infty} \lambda^*(n,q) = \lambda^*.$$

Conjecture:

The optimal sales rate $\bar{\theta}(q)\,\lambda^*(n,q) \downarrow q$.



Incomplete Information: Inventory Clearance

ASYMPTOTIC APPROXIMATION: Since

$$\lim_{n\to\infty} V(n,q) = R\,\bar{\theta}(q) = \lim_{n\to\infty} \{q\,W(n,\theta_L) + (1-q)\,W(n,\theta_H)\},$$

we propose the following approximation for $V(n,q)$

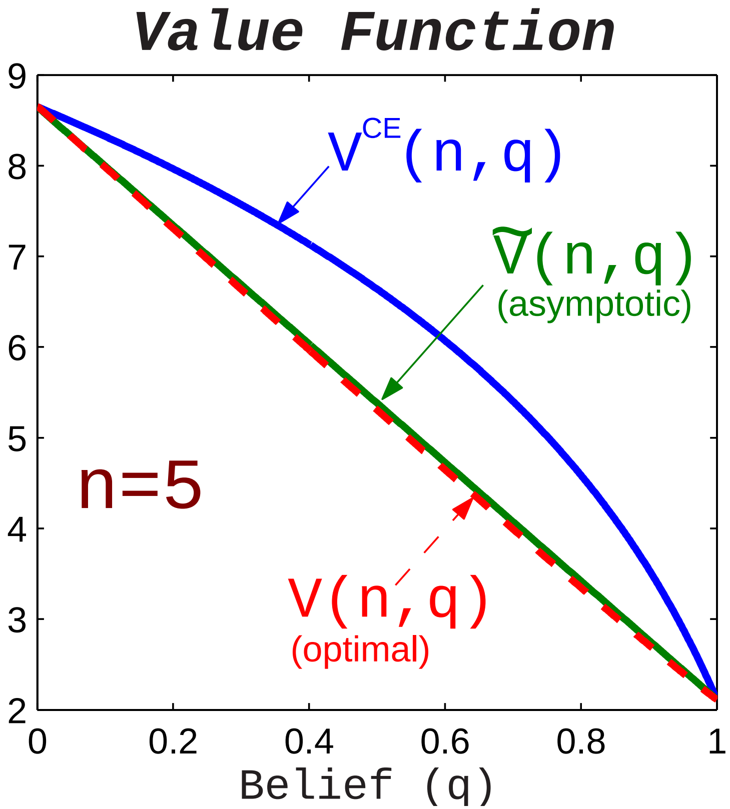
$$\widetilde{V}(n,q) := q\,W(n,\theta_L) + (1-q)\,W(n,\theta_H).$$

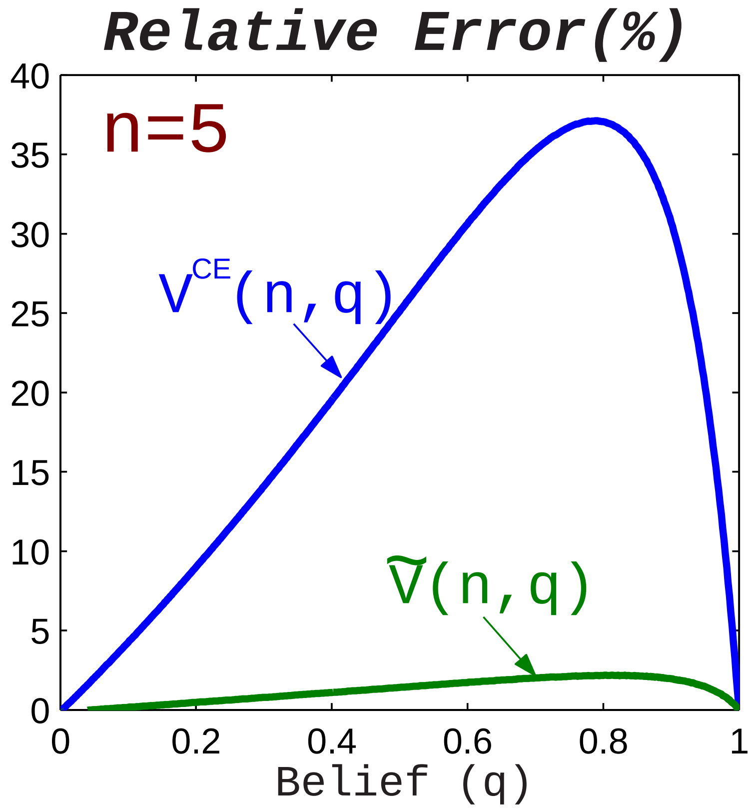
SOME PROPERTIES OF $\widetilde{V}(n,q)$:

- Linear approximation easy to compute.
- Asymptotically optimal as $n \to \infty$.
- Asymptotically optimal as $q \to 0^+$ or $q \to 1^-$.
- $\widetilde{V}(n,q) = \mathbb{E}_q[W(n,\theta)] \neq W(n,\mathbb{E}_q[\theta]) =: V^{\mathsf{CE}}(n,q)$ = Certainty Equivalent.

Incomplete Information: Inventory Clearance

$$\text{Relative Error (\%)} := \frac{V^{\text{approx}}(n,q) - V(n,q)}{V(n,q)} \times 100\%.$$





Exponential Demand $\lambda(p) = \Lambda \exp(-\alpha\, p)$: Inventory $= 5$, $\Lambda = 10$, $\alpha = r = 1$, $\theta_H = 5.0$, $\theta_L = 0.5$.

Incomplete Information: Inventory Clearance

For any approximation $V^{\text{approx}}(n, q)$, define the corresponding demand intensity using the HJB

$$\lambda^{\text{approx}}(n, q) := \arg\max_{0 \le \lambda \le \Lambda} [\lambda\, \bar{\theta}(q)[V^{\text{approx}}(n-1, q-\eta(q)) - V^{\text{approx}}(n, q)] + \lambda\, \kappa(q) V_q^{\text{approx}}(n, q) + \bar{\theta}(q)\, c(\lambda)].$$

$$\text{Relative Price Error (\%)} := \frac{p(\lambda^{\text{approx}}) - p(\lambda^*)}{p(\lambda^*)} \times 100\%.$$

Asymptotic Approximation (%)

	Inventory (n)				
q	1	5	10	25	100
0.0	0.0	0.0	0.0	0.0	0.0
0.2	2.7	-0.2	-0.3	-0.6	-0.5
0.4	6.9	**0.8**	**-0.6**	**-0.9**	-0.7
0.6	12.5	**2.4**	**-0.2**	**-0.7**	-1.0
0.8	19.4	**3.3**	**0.1**	**-0.4**	-0.6
1.0	0.0	0.0	0.0	0.0	0.0

Certainty Equivalent (%)

	Inventory (n)				
q	1	5	10	25	100
0.0	0.0	0.0	0.0	0.0	0.0
0.2	5.3	2.6	2.7	2.4	-0.4
0.4	14.4	**11.6**	**12.0**	**10.1**	-0.5
0.6	29.9	**28.2**	**28.0**	**17.6**	-1.0
0.8	54.6	**46.2**	**37.4**	**11.1**	-0.7
1.0	0.0	0.0	0.0	0.0	0.0

Relative price error for the exponential demand model $\lambda(p) = \Lambda \exp(-\alpha\, p)$, with $\Lambda = 20$ and $\alpha = 1$.

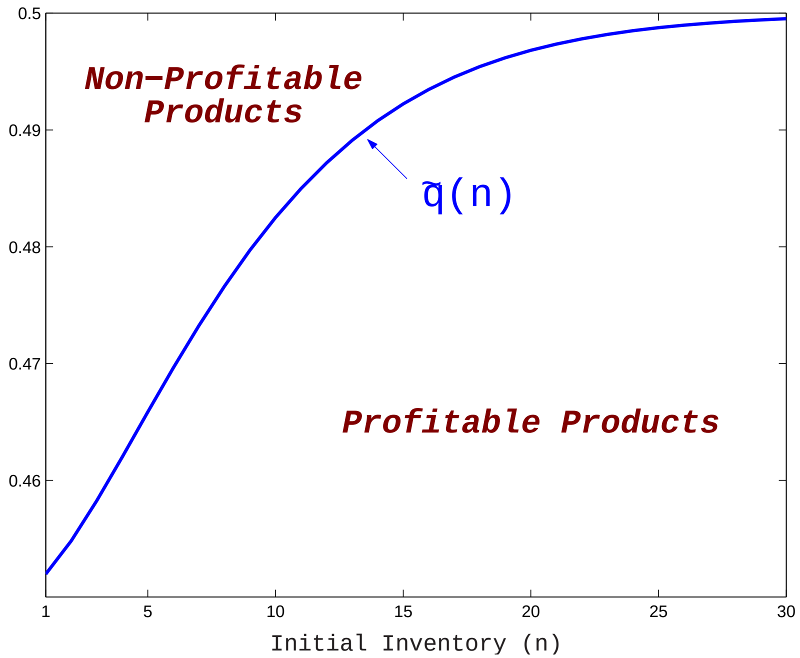
Incomplete Information: Inventory Clearance

When should the retailer engage in selling a given product?

When $V(n, q) \geq R$.

Using the asymptotic approximation $\widetilde{V}(n, q)$, this is equivalent to

$$q \leq \widetilde{q}(n) := \frac{W(n; \theta_H) - R}{[W(n; \theta_H) - R] + [R - W(n; \theta_L)]}.$$



Exponential demand rate $\lambda(p) = \Lambda \exp(-\alpha p)$.
Data: $\Lambda = 10$, $\alpha = 1$, $r = 1$, $\theta_H = 1.2$, $\theta_L = 0.8$.

$$\widetilde{q}(n) \to \widetilde{q}_\infty := \frac{\theta_H - 1}{\theta_H - \theta_L}, \text{ as } n \to \infty.$$

Incomplete Information: Inventory Clearance

SUMMARY:

- Uncertainty in market size (θ) is captured by a new state variable q_t (a jump process).

- $V(n,q)$ can be computed using a recursive sequence of ODEs.

- Asymptotic approximation $\widetilde{V}(n,q) := \mathbb{E}_q[W(n,\theta)]$ performs quite well.
  - Linear approximation easy to compute.
  - Value function: $V(n,q) \approx \widetilde{V}(n,q)$.
  - Pricing strategy: $p^*(n,q) \approx \widetilde{p}(n,q)$.

- Products are divided in two groups as a function of (n,q):
  - Profitable Products with $q < \widetilde{q}(n)$ and
  - Non-profitable Products with $q > \widetilde{q}(n)$.

- The threshold $\widetilde{q}(n)$ increases with n, that is, the retailer is willing to take more risk for larger orders.

Incomplete Information: Optimal Stopping

Setting:

- Retailer does not know θ at $t = 0$ but knows $\theta \in \{\theta_L, \theta_H\}$ with $\theta_L \leq 1 \leq \theta_H$.
- Retailer has the option of removing the product at any time, "Outlet Option".

Retailer's Optimization:

$$\begin{aligned} U(N_0, q) &= \max_{\lambda \in \mathcal{A},\ \tau \in \mathcal{T}} \mathbb{E}_q \left[\int_0^\tau e^{-r\,t}\, p(\lambda_t)\, \mathrm{d}D(I_\lambda(t)) + e^{-r\,\tau}\, R \right] \\ \text{subject to} \quad & N_t = N_0 - D(I_\lambda(t)), \\ & \mathrm{d}q_t = -\eta(q_{t-})\, [\mathrm{d}D(I_\lambda(t)) - \lambda_t\, \bar{\theta}(q_{t-})\mathrm{d}t], \quad q_0 = q. \end{aligned}$$

Optimality Conditions:

$$\begin{cases} U(n,q) + \Phi(\frac{r\,U(n,q)}{\bar{\theta}(q)}) - \eta(q)\, U_q(n,q) = U(n-1, q-\eta(q)) & \text{if } U \geq R \\ U(n,q) + \Phi(\frac{r\,U(n,q)}{\bar{\theta}(q)}) - \eta(q)\, U_q(n,q) \leq U(n-1, q-\eta(q)) & \text{if } U = R. \end{cases}$$

Incomplete Information: Optimal Stopping

Proposition.

a) There is a unique continuously differentiable solution $U(n,\cdot)$ defined on $[0,1]$ so that $U(n,q) > R$ on $[0,q_n^*)$ and $U(n,q) = R$ on $[q_n^*, 1]$, where q_n^* is the unique solution of

$$R + \Phi\left(\frac{r\,R}{\bar{\theta}(q)}\right) = U(n-1, q-\eta(q)).$$

b) q_n^* is increasing in n and satisfies

$$\frac{\theta_H - 1}{\theta_H - \theta_L} \leq q_n^* \overset{n\to\infty}{\longrightarrow} q_\infty^* \leq \text{Root}\left\{\Phi\left(\frac{r\,R}{\bar{\theta}(q)}\right) = \frac{\eta(q)}{q}\,(\theta_H - 1)\,R\right\} < 1.$$

c) The value function $U(n,q)$

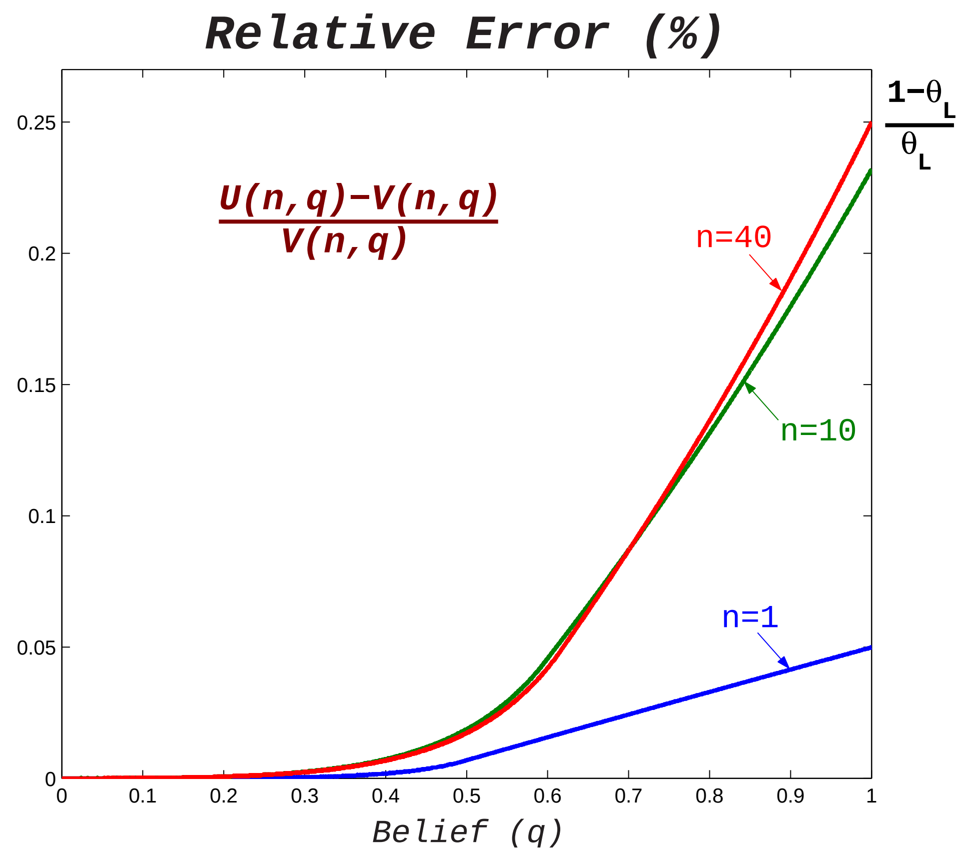
- Is decreasing and convex in q on $[0,1]$
- Increases in n for all $q \in [0,1]$ and satisfies

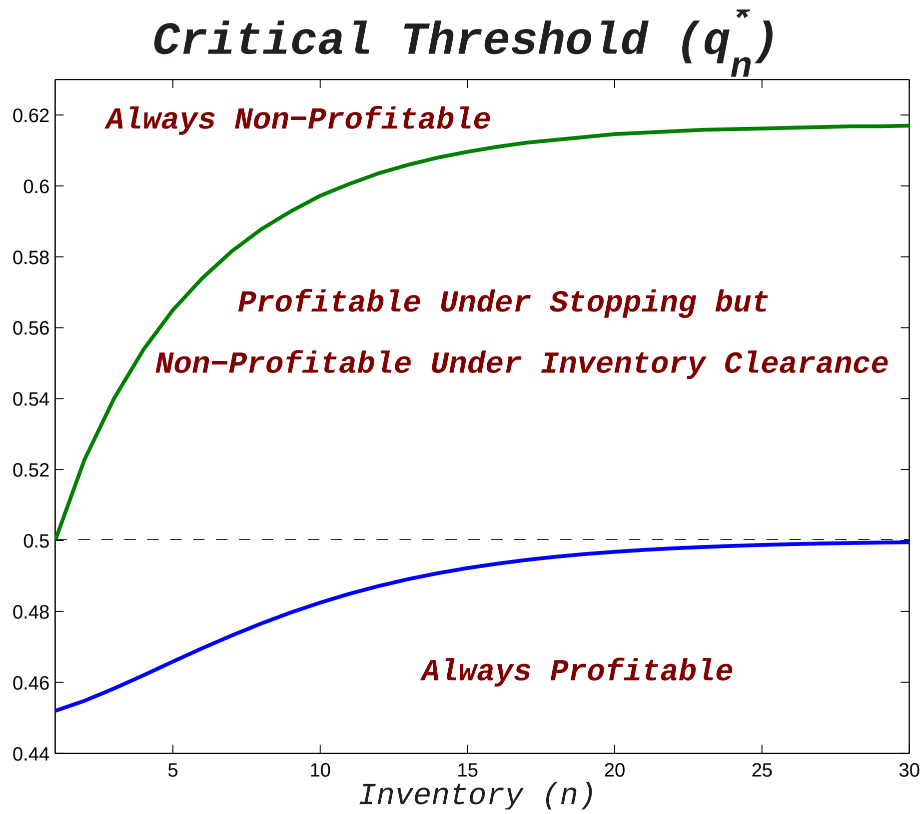
$$\max\{R, V(n,q)\} \leq U(n,q) \leq \max\{R, m(q)\} \quad \text{for all } q \in [0,1],$$

where
$$m(q) := W(n,\theta_H) - \frac{(W(n,\theta_H) - R)}{q_n^*}\,q.$$

- Converges uniformly (in q) to a continuously differentiable function, $U_\infty(q)$.

Incomplete Information: Optimal Stopping





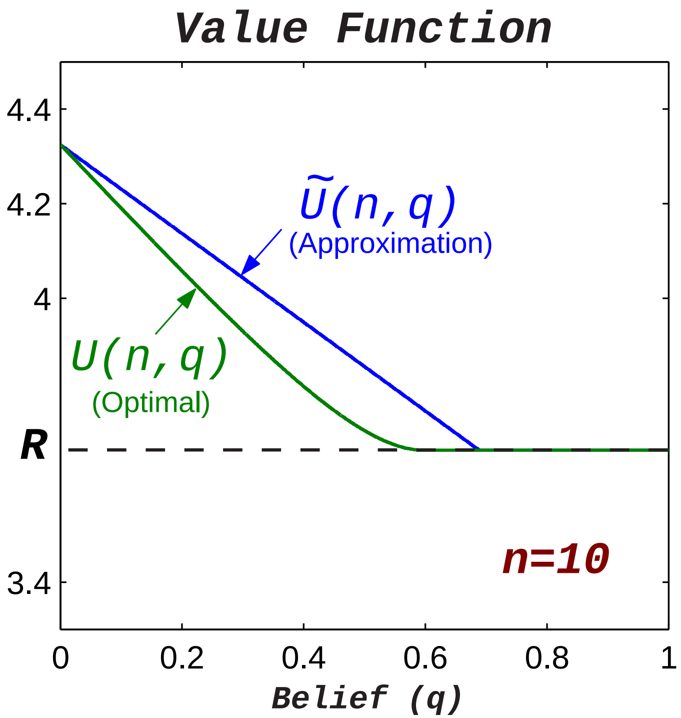
Exponential demand rate $\lambda(p) = \Lambda \, \exp(-\alpha \, p)$. Data: $\Lambda = 10$, $\alpha = 1$, $r = 1$, $\theta_H = 1.2$, $\theta_L = 0.8$.

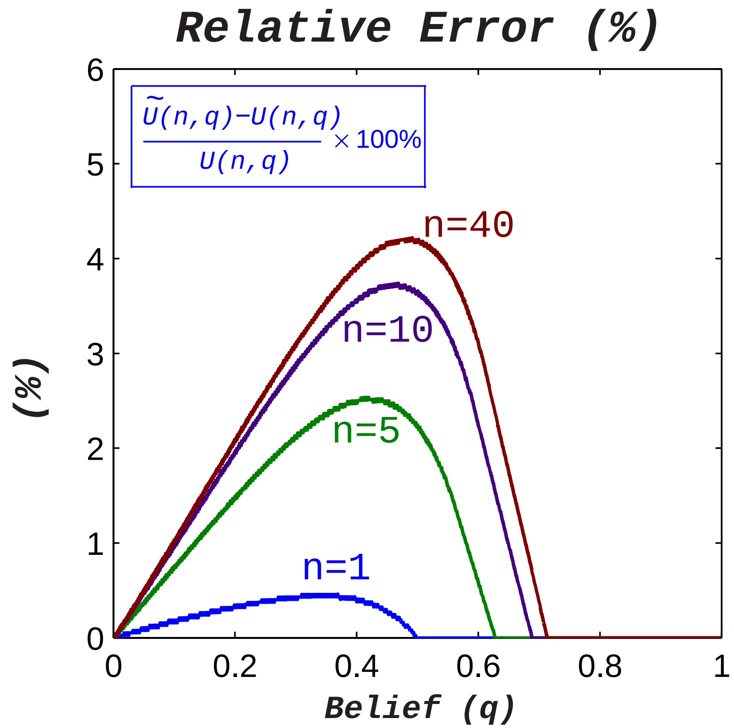
Incomplete Information: Optimal Stopping

APPROXIMATION:

$$\widetilde{U}(n,q) := \max\{R, W(n,\theta_H) - \frac{(W(n,\theta_H) - R)}{\tilde{q}_n}\, q\}$$

where $\tilde{q}_n$ is the unique solution of $\qquad R + \Phi\left(\frac{r\,R}{\bar{\theta}(q)}\right) = \widetilde{U}(n-1, q-\eta(q)).$





Exponential demand rate $\lambda(p) = \Lambda\, \exp(-\alpha\, p)$. Data: $\Lambda = 10$, $\alpha = 1$, $r = 1$, $\theta_H = 1.2$, $\theta_L = 0.8$.

Incomplete Information: Optimal Stopping

<u>SUMMARY:</u>

- $U(n, q)$ can be computed using a recursive sequence of ODEs with free-boundary conditions.
- For every n there is a critical belief q_n^* above which it is optimal to stop.
- Again, the sequence q_n^* is increasing with n, that is, the retailer is willing to take more risk for larger orders.
- The sequence q_n^* is bounded by

$$\frac{\theta_H - 1}{\theta_H - \theta_L} \leq q_n^* \leq \hat{q} := \text{Root}\left\{ \Phi\left(\frac{r\,R}{\bar{\theta}(q)}\right) = \frac{\eta(q)}{q}\,(\theta_H - 1)\,R \right\}$$

- The "outlet option" increases significantly the expected profits and the range of products (n, q) that are *profitable*.

$$0 \leq U(n, q) - V(n, q) \leq (1 - \theta_L)^+\,R.$$

- A simple piece-wise linear approximation works well.

$$\widetilde{U}(n, q) := \max\{R, W(n, \theta_H) - \frac{(W(n, \theta_H) - R)}{\tilde{q}_n}\,q\}$$

Concluding Remarks

- A simple dynamic pricing model for a retailer selling non-perishable products.
- Captures two common sources of uncertainty:
  - Market size measured by $\theta \in \{\theta_H, \theta_L\}$.
  - Stochastic arrival process of price sensitive customers.
- Analysis gets simpler using the Fenchel-Legendre transform of $c(\lambda)$ and its properties.
- We propose a simple approximation (linear and piecewise linear) for the value function and corresponding pricing policy.
- Some properties of the optimal solution are:
  - Value functions $V(n, q)$ and $U(n, q)$ are decreasing and convex in q.
  - The retailer is willing to take more risk ($\uparrow q$) for higher orders ($\uparrow n$).
  - The optimal demand intensity $\lambda^*(n, q) \uparrow q$ and the optimal sales rate $\bar{\theta}(q)\, \lambda^*(n, q) \downarrow q$.
- Extension: $R(n) = R + \nu\, n - K\, \mathbb{1}(n > 0)$.