STANDARD 8 — NUMERICAL OPERATIONS

K-12 Overview

> All students will understand, select, and apply various methods of performing numerical operations.

Descriptive Statement

Numerical operations are an essential part of the mathematics curriculum. Students must be able to select and apply various computational methods, including mental math, estimation, paper-and-pencil techniques, and the use of calculators. Students must understand how to add, subtract, multiply, and divide whole numbers, fractions, and other kinds of numbers. With calculators that perform these operations quickly and accurately, however, the instructional emphasis now should be on understanding the meanings and uses of the operations, and on estimation and mental skills, rather than solely on developing paper-and-pencil skills.

Meaning and Importance

The wide availability of computing and calculating technology has given us the opportunity to significantly reconceive the role of computation and numerical operations in our school mathematics programs. Up until this point in our history, the mathematics program has called for the expenditure of tremendous amounts of time in helping children to develop proficiency with paper-and-pencil computational procedures. Most people defined *proficiency* as a combination of speed and accuracy with the standard algorithms. Now, however, adults who need to perform calculations quickly and accurately have electronic tools that are more accurate and more efficient than any human being. It is time to re-examine the reasons to teach paper-and-pencil computational algorithms to children and to revise the curriculum in light of that re-examination. Mental mathematics, however, should continue to be stressed; students should be able to carry out simple computations without resort to either paper-and-pencil or calculators. Fourth-graders must know the basic facts of the multiplication table, and seventh-graders must be able to evaluate in their heads simple fractions, such as *What's two-thirds of 5 tablespoons?*

K-12 Development and Emphases

At the same time that technology has made the traditional focus on paper-and-pencil skills less important, it has also presented us with a situation where some numerical operations, skills, and concepts are much more important than they have ever been. **Estimation** skills, for example, are critically important if one is to be a competent user of calculating technology. People must know the range in which the answer to a given problem should lie before doing any calculation, they must be able to assess the reasonableness of the results of a string of computations, and they should be able to be satisfied with the results of an estimation when an exact answer is unnecessary. They should also be able to work quickly and easily with changes in order of magnitude, using powers of ten and their multiples. **Mental mathematics** skills also play a more important role in a highly technological world. Simple two-digit computations or operations that involve powers of ten should be performed mentally by a mathematically literate adult. Students should have enough confidence in their ability with such computations to do them mentally rather than using either a calculator or paper and pencil. Most importantly, a student's **knowledge of the meanings and uses of the various arithmetic operations** is essential. Even with the best of computing devices, it is still the human who must decide which operations need to be performed and in what order to answer the question at hand. The construction of solutions to life's everyday problems, and to society's larger ones, will require students to be thoroughly familiar with when and how the mathematical operations are used.

The major shift in this area of the curriculum, then, is one away from drill and practice of paper-and-pencil procedures and toward real-world applications of operations, wise choices of appropriate computational strategies, and integration of the numerical operations with other components of the mathematics curriculum. *So what is the role of paper-and-pencil computation in a mathematics program for the year 2000? Should children be able to perform any calculations by hand? Are those procedures worth any time in the school day?* Of course they should and of course they are.

Most simple **paper-and-pencil procedures** should still be taught and one-digit **basic facts** should still be committed to memory. We want students to be proficient with two- and three-digit addition and subtraction and with multiplication and division involving two-digit factors or divisors, but there should be changes both in the way we teach those processes and in where we go from there. The focus on the learning of those procedures should be on understanding the procedures themselves and on the development of accuracy. There is no longer any need to concentrate on the development of speed. To serve the needs of understanding and accuracy, non-traditional paper-and-pencil algorithms, or algorithms devised by the children themselves, may well be better choices than the standard algorithms. The extensive use of drill in multi-digit operations, necessary in the past to enable people to perform calculations rapidly and automatically, is no longer necessary and should play a much smaller role in today's curriculum.

For procedures involving larger numbers, or numbers with a greater number of digits, the intent ought to be to bring students to the point where they understand a paper-and-pencil procedure well enough to be able to extend it to as many places as needed, but certainly not to develop an old-fashioned kind of proficiency with such problems. In almost every instance where the student is confronted with such numbers in school, **technology** should be available to aid in the computation, and students should understand how to use it effectively. Calculators are the tools that real people in the real world use when they have to deal with similar situations and they should not be withheld from students in an effort to further an unreasonable and antiquated educational goal.

IN SUMMARY, numerical operations continue to be a critical piece of the school mathematics curriculum and, indeed, a very important part of mathematics. But, there is perhaps a greater need for us to rethink our approach here than to do so for any other component. An enlightened mathematics program for today's children will empower them to use all of today's tools rather than require them to meet yesterday's expectations.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 8 — Numerical Operations — Grades K-2

Overview

The wide availability of computing and calculating technology has given us the opportunity to significantly reconceive the role of computation and numerical operations in our elementary mathematics programs, but, in kindergarten through second grade, the effects will not be as evident as they will be in all of the other grade ranges. This is because the numerical operations content taught in these grades is so basic, so fundamental, and so critical to further progress in mathematics that much of it will remain the same. The approach to teaching that content, however, must still be changed to help achieve the goals expressed in the *New Jersey Mathematics Standards*.

Learning the **meanings of addition and subtraction,** gaining facility with **basic facts**, and mastering some **computational procedures for multi-digit addition and subtraction** are still the topics on which most of the instructional time in this area will be spent. There will be an increased conceptual and developmental focus to these aspects of the curriculum, though, away from a traditional drill-and-practice approach, as described in the K-12 Overview; nevertheless, students will be expected to be able to respond quickly and easily when asked to recall basic facts.

By the time they enter school, most young children can use counters to act out a mathematical story problem involving addition or subtraction and find a solution which makes sense. Their experiences in school need to build upon that ability and deepen the children's understanding of the **meanings of the operations**. School experiences also need to strengthen the children's sense that modeling such situations as a way to understand them is the right thing to do. It is important that they be exposed to a variety of different situations involving addition and subtraction. Researchers have separated problems into categories based on the kind of relationships involved (Van de Walle, 1990, pp. 75-6); students should be familiar with problems in all of the following categories:

Join problems

- Mary has 8 cookies. Joe gives her 2 more. *How many cookies does Mary have in all?*
- Mary has some cookies. Joe gives her 2 more. Now she has 8. *How many cookies did Mary have to begin with?* (Missing addend)
- Mary has 8 cookies. Joe gives her some more. Now Mary has 10. *How many cookies did Joe give Mary?* (Missing addend)

Separate problems

- Mary has 8 cookies. She eats 2. *How many are left?* (Take away)
- Mary has some cookies. She eats 2. She has 6 left. *How many cookies did Mary have to begin with?*
- Mary has 8 cookies. She eats some. She has 6 left. *How many cookies did Mary eat?* (Missing addend)

Part-part-whole problems

- Mary has 2 nickels and 3 pennies. *How many coins does she have?*
- Mary has 8 coins. Three are pennies, the rest nickels. *How many nickels does Mary have?*

Compare problems

- Mary has 6 books. Joe has 4. *How many more books does Mary have than Joe?*

- Mary has 2 more books than Joe. Mary has 6 books. *How many books does Joe have?*
- Joe has 2 fewer books than Mary. He has 4 books. *How many books does Mary have?*

Basic facts in addition and subtraction continue to be very important. Students should be able to quickly and easily recall one-digit sums and differences. The most effective way to accomplish this has been shown to be the focused and explicit use of basic fact strategies—conceptual techniques that make use of the child's understanding of number parts and relationships to help recover the appropriate sum or difference. By the end of second grade, students should not only be able to use *counting on*, *counting back, make ten,* and *doubles* and *near doubles* strategies, but also explain why these strategies work by modeling them with counters. Building on their facility with learning doubles like $7 + 7 = 14$, children recast $7 + 8$ as $7 + 7 + 1$, which they then recognize as 15 (*near doubles*). *Make ten* involves realizing that in adding $8 + 5$, you need two to make ten, and recasting the sum as $8 + 2 + 3$ which is $10 + 3$ or 13. *Counting on* involves starting with the large number and counting on the smaller number so that adding $9 + 3$, for example, would involve counting on *10, 11,* and then *12*. *Counting back* is used for subtraction, so that finding $12 - 4$, the child might count *11*, *10*, *9*, and then *8*.

Students must still be able to perform **multi-digit addition and subtraction** with paper and pencil, but the widespread availability of calculators has made the particular procedure used to perform the calculations less important. It need no longer be the single fastest, most efficient algorithm chosen without respect to the degree to which children understand it. Rather, the teaching of multi-digit computation should take on more of a problem solving approach, a more conceptual, developmental approach. Students should first use the models of multi-digit number that they are most comfortable with (base ten blocks, popsicle sticks, bean sticks) to explore the new class of problems. Students who have never formally done two-digit addition might be asked to use their materials to help figure out how many second graders there are in all in the two second grade classes in the school. Other similar real-world problems should follow, some involving regrouping and others not. After initial exploration, students share with each other all of the strategies they've developed, the best ways they've found for working with the tens and ones in the problems, and their own approaches (and names!) for regrouping. Most students can, with direction, take the results of those discussions and create their own paper-and-pencil procedures for addition and subtraction. The discussions can, of course, include the traditional approaches, but these ought not to be seen as *the only right way* to do these operations.

Kindergarten through second grade teachers are also responsible for setting up an atmosphere where **estimation** and **mental math** are seen as reasonable ways to do mathematics. Of course students at these grade levels do almost exclusively mental math until they reach multi-digit operations, but estimation should also comprise a good part of the activity. Students regularly involved in real-world problem solving should begin to develop a sense of when estimation is appropriate and when an exact answer is necessary.

Technology should also be an important part of the environment in primary classrooms. Calculators provide a valuable teaching tool when used to do student-programmed skip counting, to offer estimation and mental math practice with *target games*, and to explore operations and number types that the students have not formally encountered yet. They should also be used routinely to perform computation in problem solving situations that the students may not be able to perform otherwise. This use prevents the need to artificially contrive the numbers in real-world problems so that their answers are numbers with which the students are already comfortable.

The topics that should comprise the numerical operations focus of the kindergarten through second grade mathematics program are:

addition and subtraction basic facts
multi-digit addition and subtraction

Standard 8 — Numerical Operations — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Develop meaning for the four basic arithmetic operations by modeling and discussing a variety of problems.**
   - Students use unifix cube towers of two colors to show all the ways to make *"7"* (for example: *3+4, 2+5, 0+7*, and so on). This activity focuses more on developing a sense of *"sevenness"* than on addition concepts, but a good sense of each individual number makes the standard operations much easier to understand.
   - Kindergartners and first graders use workmats depicting various settings in which activity takes place to make up and act out story problems. On a mat showing a vacant playground, for instance, students place counters to show 3 kids on the swings and 2 more in the sandbox. *How many kids are there in all? How many more are on the swings than in the sandbox? What are all of the possibilities for how many are boys and how many are girls?*
   - Students work through the *Sharing a Snack* lesson that is described in the Introduction to this *Framework*. It challenges students to find a way to share a large number of cookies fairly among the members of the class, promoting discussion of early division, fraction, and probability ideas.
   - Students learn about addition as they read *Too Many Eggs* by M. Christina Butler. They place eggs in different bowls as they read and then make up addition number sentences to find out how many eggs were used in all.
   - Kindergarteners count animals and learn about addition as they read *Adding Animals* by Colin Hawkins. This book uses addends from one through four and shows the number sentences that go along with the story.
   - Students are introduced to the take-away meaning for subtraction by reading *Take Away Monsters* by Colin Hawkins. Students see the partial number sentence (e.g., $5 - 1 =$), count to find the answer, and then pull the tab to see the result.
   - Students explore subtraction involving missing addend situations as they read *The Great Take-Away* by Louise Mathews. This book tells the story of one lazy hog who decides to make easy money by robbing the other pigs in town. The answers to five subtraction mysteries are revealed when the thief is captured.
   - Students make booklets containing original word problems that illustrate different addition or subtraction situations. These may be included in a portfolio or evaluated independently.

2. **Develop proficiency with and memorize basic number facts using a variety of fact strategies**

(such as “counting on” and “doubles”).

- Students play *one more than* dominoes by changing the regular rules so that a domino can be placed next to another only if it has dots showing *one more than* the other. Dominoes of any number can be played next to others that show *6* (or *9* in a set of double nines). *One less than* dominoes is also popular.
- Students work through the *Elevens Alive* lesson that is described in the Introduction to this *Framework*. It asks them to consider the parts of eleven and the natural, random, occurrence of different pairs of addends when tossing eleven two-colored counters.
- Second graders regularly use the *doubles* and *near doubles*, the *make ten*, and the *counting on* and *counting back* strategies for addition and subtraction. Practice sets of problems are structured so that use of all of these strategies is encouraged and the students are regularly asked to explain the procedures they are using.
- Students play games like *addition war* to practice their basic facts. Each of two children has half of a deck of playing cards with the face cards removed. They each turn up a card and the person who wins the trick is the first to say the sum (or difference) of the two numbers showing. Calculators may be used to check answers, if necessary.
- Students use the calculator to count *one more than* by pressing + *1*= = =. The display will increase by one every time the student presses the = key. Any number can replace the *1* key.
- Students use two dice to play board games (*Chutes and Ladders* or home-made games). These situations encourage rapid recall of addition facts in a natural way. In order to extend practice to larger numbers, students may use 10-sided dice.
- Students use computer games such as *Math Blaster Plus* or *Math Rabbit* to practice basic facts.

3. Construct, use, and explain procedures for performing whole number calculations in the various methods of computation.

- Second graders use popsicle sticks bundled as tens and ones to try to find a solution to the first two-digit addition problem they have formally seen: *Our class has 27 children and Mrs. Johnson's class has 26. How many cupcakes will we need for our joint party?* Solution strategies are shared and discussed with diversity and originality praised. Other problems, some requiring regrouping and others not, are similarly solved using the student-developed strategies.
- Students use calculators to help with the computation involved in a first-grade class project: to see how many books are read by the students in the class in one month. Every Monday morning, student reports contribute to a weekly total which is then added to the monthly total.
- Students look forward to the hundredth day of school, on which there will be a big celebration. On each day preceding it, the students use a variety of procedures to determine how many days are left before day 100.

- As part of their assessment, students explain how to find the answer to an addition or subtraction problem (such as *18 + 17*) using pictures and words.
- Students find the answer to an addition or subtraction problem in as many different ways as they can. For example, they might solve *28 + 35* in the following ways:

  8 + 5 = 13 and 20 + 30 = 50, so 13 + 50 = 63

  28 + 30 = 58. Two more is 60, and 3 more is 63

  25 + 35 = 60 and 3 more is 63.
- Students use estimation to find out whether a package of 40 balloons is enough for everyone in the class of 26 to have two balloons. They discuss the strategies they use to solve this problem and decide if they should buy more packages.

4. Use models to explore operations with fractions and decimals.

- Kindergartners explore part/whole relations with pattern blocks by seeing which shapes can be created using other blocks. You might ask: *Can you make a shape that is the same as the yellow hexagon with 2 blocks of some other color? with 3 blocks of some other color? with 6 blocks of some other color?* and so on.
- Students use paper folding to begin to identify and name common fractions. You might ask: *If you fold this rectangular piece of paper in half and then again and then again, how many equal parts are there when you open it up?* Similarly folded papers, each representing a different unit fraction, allow for early comparison activities.
- Second graders use fraction circles to model situations involving fractions of a pizza. For example: *A pizza is divided into six pieces. Mary eats two pieces. What fraction of the pizza did Mary eat? What fraction is left?*
- Students use manipulatives such as pattern blocks or Cuisenaire rods to model fractions. For example: *If the red rod is one whole, then what number is represented by the yellow rod?*

5. Use a variety of mental computation and estimation techniques.

- Students regularly practice a variety of oral counting skills, both forward and backward, by various steps. For instance, you might instruct your students to: *Count by ones — start at 1, at 6, at 12, from 16 to 23; Count by tens — start at 10, at 30, at 110, at 43, at 67, from 54 to 84*, and so on.
- Students estimate sums and differences both before doing either paper-and-pencil computation or calculator computation and after so doing to confirm the reasonableness of their answers.
- Students are given a set of index cards on each of which is printed a two-digit addition pair (*23+45, 54+76, 12+87,* and so on). As quickly as they can they sort the set into three piles: *more than 100, less than 100,* and *equal to 100.*
- Students play "*Target 50*" with their calculator. One student enters a two-digit number and the other must add a number that will get as close as possible to *50*.

6. **Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.**

   - The daily *calendar routine* provides the students with many opportunities for computation. Questions like these arise almost every day: *There are 27 children in our class. Twenty-four are here today. How many are absent? Fourteen are buying lunch; how many brought their lunch?* or *It's now 9:12. How long until we go to gym at 10:30?* The students are encouraged to choose a computation method with which they feel comfortable; they are frequently asked why they chose their method and whether it was important to get an exact answer. Different solutions are acknowledged and praised.
   - Students regularly have *human vs. calculator races*. Given a list of addition and subtraction basic facts, one student uses mental math strategies and another uses a calculator. They quickly come to realize that the human has the advantage.
   - Students regularly answer multiple choice questions like these with their best guesses of the most reasonable answer: *A regular school bus can hold: 20 people, 60 people, 120 people? The classroom is: 5 feet high, 7 feet high, 10 feet high?*
   - As part of an assessment, students tell how they would solve a particular problem and why. They might circle a picture of a calculator, a head (for mental math), or paper-and-pencil for each problem.

7. **Understand and use relationships among operations and properties of operations.**

   - Students explore three-addend problems like *4 + 5 + 6 =*. First they check to see if adding the numbers in different orders produces different results and, later, they look for pairs of compatible addends (like *4* and *6*) to make the addition easier.
   - Students make up humorous stories about adding and subtracting zero. *I had 27 cookies. My mean brother took away zero. How many did I have left?*
   - Second graders, exploring multiplication arrays, make a *4* x *5* array of counters on a piece of construction paper and label it: *4 rows, 5 in each row = 20*. Then they rotate the array *90°* and label the new array, *5 rows, 4 in each row = 20*. Discussions follow which lead to intuitive understandings of commutativity.

References

Butler, M. Christina. *Too Many Eggs.* Boston: David R. Godine Publisher, 1988.

Hawkins, Colin. *Adding Animals.* New York: G. P. Putnam’s Sons, 1984.

Hawkins, Colin. *Take Away Monsters.* New York: G. P. Putnam’s Sons, l984.

Mathews, Louise. *The Great Take-Away.* New York: Dodd, Mead, & Co., 1980.

Van de Walle, J. A. *Elementary School Mathematics: Teaching Developmentally.* New York: Longman, 1990.

Software

Math Blaster Plus. Davidson.

Math Rabbit. The Learning Company.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 8 — Numerical Operations — Grades 3-4

Overview

The widespread availability of computing and calculating technology has given us the opportunity to reconceive the role of computation and numerical operations in our third and fourth grade mathematics programs. Traditionally, tremendous amounts of time were spent at these levels helping children to develop proficiency and accuracy with paper-and-pencil procedures. Now, adults needing to perform calculations quickly and accurately have electronic tools that are both more accurate and more efficient than those procedures. At the same time, though, the new technology has presented us with a situation where some numerical operations, skills, and concepts are much more important than they used to be. As described in the K-12 Overview, **estimation, mental computation,** and **understanding the meanings of the standard arithmetic operations** all play a more significant role than ever in the everyday life of a mathematically literate adult.

The major shift in the curriculum that will take place in this realm, therefore, is one away from drill and practice of paper-and-pencil procedures with symbols and toward real-world applications of operations, wise **choices of appropriate computational strategies**, and integration of the numerical operations with other components of the mathematics curriculum.

Third and fourth graders are primarily concerned with cementing their understanding of addition and subtraction and developing new **meanings for multiplication and division.** They should be in an environment where they can do so by modeling and otherwise representing a variety of real-world situations in which these operations are appropriately used. It is important that the variety of situations to which they are exposed include all the different scenarios in which multiplication and division are used. There are several slightly different taxonomies of these types of problems, but minimally students at this level should be exposed *to repeated addition and subtraction, array, area,* and *expansion* problems. Students need to recognize and model each of these problem types for both multiplication and division.

Basic facts in multiplication and division continue to be very important. Students should be able to quickly and easily recall quotients and products of one-digit numbers. The most effective approach to enabling them to acquire this ability has been shown to be the focused and explicit use of basic fact strategies—conceptual techniques that make use of the child's understanding of the operations and number relationships to help recover the appropriate product or quotient. *Doubles* and *near doubles* are useful strategies, as are discussions and understandings regarding the regularity in the *nines* multiplication facts, the roles of *one* and *zero* in these operations, and the roles of *commutativity and distributivity.*

Students must still be able to perform **two-digit multiplication and division** with paper and pencil, but the widespread availability of calculators has made the particular procedure used to perform the calculations less important. It need no longer be the single fastest, most efficient algorithm chosen without respect to the degree to which children understand it. Rather, the teaching of two-digit computation should take on more of a problem solving approach, a more conceptual, developmental approach. Students should first use the models of multi-digit numbers that they are most comfortable with (base ten blocks, money) to explore this new class of problems. Students who have never formally done two-digit multiplication might be asked to use their materials to help figure out how many pencils are packed in the case just received in the school office. There are 24 boxes with a dozen pencils in each box. *Are there enough for every student in the* *school to have one?* Other, similar, real-world problems would follow, some involving regrouping and others not.

After initial exploration, students share with each other all of the strategies they've developed, the best ways they've found for working with the tens and ones in the problem, and their own approaches to dealing with the place value issues involved. Most students can, with direction, take the results of those discussions and create their own paper-and-pencil procedures for multiplication and division. The discussions can, of course, include the traditional approaches, but these ought not to be seen as *the only right way* to perform these operations.

Estimation and **mental math** become critically important in these grade levels as students are inclined to use calculators for more and more of their work. In order to use that technology effectively, third and fourth graders must be able to use estimation to know the range in which the answer to a given problem should lie before doing any calculation. They also must be able to assess the reasonableness of the results of a computation and be satisfied with the results of an estimation when an exact answer is unnecessary. Mental mathematics skills, too, play a more important role in third and fourth grade. Simple two-digit addition and subtraction problems and those involving powers of ten should be performed mentally. Students should have enough confidence in their ability with these types of computations to do them mentally instead of relying on either a calculator or paper and pencil.

Technology should be an important part of the environment in third and fourth grade classrooms. Calculators provide a valuable teaching tool when used to do student-programmed repeated addition or subtraction, to offer estimation and mental math practice with *target games*, and to explore operations and number types that the students have not yet formally encountered. Students should also use calculators routinely to find answers to problems that they might not be able to find otherwise. This use prevents the need to artificially contrive real-world problems so that their answers are numbers with which the students are already comfortable.

The topics that should comprise the numerical operations focus of the third and fourth grade mathematics program are:

multiplication and division basic facts
multi-digit whole number addition and subtraction
two-digit whole number multiplication and division
decimal addition and subtraction
explorations with fraction operations

Standard 8—Numerical Operation—Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. Develop meaning for the four basic arithmetic operations by modeling and discussing a variety of problems.

- Students broaden their initial understanding of multiplication as repeated addition by dealing with situations involving arrays, expansions, and combinations. Questions of these types are not easily explained through repeated addition: *How many stamps are on this 7 by 8 sheet? How big would this painting be if it was 3 times as big? How many outfits can you make with 2 pairs of pants and 3 shirts?*
- Students use counters to model both repeated subtraction (*There are 12 cookies. How many bags of 3?*) and sharing (*There are 12 cookies and 3 friends. How many cookies each?*) meanings for division and write about the difference in their journals.
- Students work through the *Sharing Cookies* lesson that is described in the First Four Standards of this *Framework*. They investigate division by using 8 cookies to be shared equally among 5 people, and discuss the problem of simplifying the number sentence which describes the amount of each person's share.
- From the beginning of their work with division, children are asked to make sense out of remainders in problem situations. The answers to these three problems are different even though the division is the same: *How many cars will we need to transport 19 people if each car holds 5? How many more packages of 5 ping-pong balls can be made if there are 19 balls left in the bin? How much does each of 5 children have to contribute to the cost of a $19 gift?*
- Students explore division by reading *The Doorbell Rang* by Pat Hutchins. In this story, Victoria and Sam must share 12 cookies with increasing numbers of friends. Students can use counters to show how many cookies each person gets.
- Students learn about multiplication as an array by reading *One Hundred Hungry Ants* by Elinor Pinczes, *Lucy and Tom's 1, 2, 3* by Shirley Hughes or *Number Families* by Jane Srivastava.
- Students make books showing things that come in *3's, 4's, 5's, 6's, or 12's*.

2. Develop proficiency with and memorize basic number facts using a variety of fact strategies (such as "counting on" and "doubles").

- Students use *streets and alleys* as both a mental model of multiplication and a useful way to recover facts when needed. It simply involves drawing a series of horizontal lines (streets) to represent one factor and a series of vertical lines (alleys) crossing them to represent the other. The number of intersections of the *streets* and *alleys* is the product!

- Students use a *double maker* on a calculator for practice with doubles. They enter x *2 =* on the calculator. Any number pressed then, followed by the equal sign, will show the number's double. Students work together to try to say the double for each number before the calculator shows it.

- Students regularly use *doubles, near doubles,* and *use a related fact* strategies for multiplication; they are using the *near doubles* strategy when they calculate a sum like $15 + 17$ by recognizing that it is *2* more than double *15*. More generally, they are using the *use a related fact* strategy when they use any fact they happen to remember, like $8 + 4 = 12$, to make a related calculation like $8 + 5 = 12 + 1 = 13$. They also recover facts by knowledge of the role of zero and one in multiplication, of commutativity, and of the regular patterned behavior of multiples of nines. Practice sets of problems are structured so that use of all these strategies is encouraged and the students are regularly asked to explain the procedures they are using.

- Pairs of students play *Circles and Stars* (Burns, 1991). Each student rolls a die and draws as many circles as the number shown, then rolls again and puts that number of stars in every circle, and then writes a multiplication number sentence and records how many stars there are all together. Each student takes seven turns, and adds the total. The winner is the student with the most stars.

- Students use color tiles to show how a given number of candies can be arranged in a rectangular box.

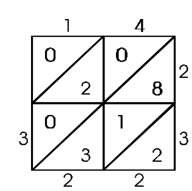
- Students play *multiplication war,* using a deck of cards with kings and queens removed. All of the cards are dealt out. Each player turns up two cards and multiplies their values (Jacks count as *0*; aces count as *1*). The "general" draws a target number from a hat. The player closest to the target wins a point. The first player to get *10* points wins the game.

- Students use computer programs such as *Math Workshop* to practice multiplication facts.

3. Construct, use, and explain procedures for performing whole number calculations in the various methods of computation.

- Students work through the *Product and Process* lesson that is described in the Introduction to this *Framework*. It challenges students to use calculators and four of the five digits *1, 3, 5, 7,* and *9* to discover the multiplication problem that gives the largest product.

- Students explore lattice multiplication and try to figure out how it works. For example, the figure at the right shows $14 \cdot 23 = 322$.



- Students use the skills they've developed with *arrow puzzles* (See Standard 6—Number Sense—Grades 3-4—Indicator 3) to practice mental addition and subtraction of 2- and 3-digit numbers. To add *23* to *65,* for instance, they start at *65* on their "mental hundred number chart," go down twice and to the right three times.

- Students use base ten blocks to help them decide how many blocks there would be in each group if they divided 123 blocks among 3 people. The students describe how they used the blocks to help them solve the problem and compare their solutions and solution strategies.

4. Use models to explore operations with fractions and decimals.

- Students use *fraction circle* pieces (each unit fraction a different color) to begin to explore addition of fractions. Questions like: *Which of these sums are greater than 1?* and *How do you know?* are frequent.
- Students use the base ten models that they are most familiar with for whole numbers and relabel the components with decimal values. Base ten blocks represent 1 whole, 1 tenth, 1 hundredth, and 1 thousandth. Coins, which had represented a whole number of cents, now represent hundredths of dollars.
- Students operate a school store with school supplies available for sale. Other students, using play money, decide on purchases, pay for them, receive and check on the amount of change.
- In groups, students each roll a number cube and use dimes to represent the decimal rolled. For example, a student rolling a *4* would take 4 dimes to represent 4 tenths of a dollar. When a student gets 10 dimes, he turns them in for a dollar. The first student to get $5 wins the game.
- Students use money to represent fractions. For example, a quarter and a quarter equals half a dollar.
- Students demonstrate equivalent fractions using pattern blocks. For example, if a yellow hexagon is one whole, then three green triangles (*3/6*) is the same size as one red trapezoid (*1/2*). Pattern blocks may also be used to represent addition and subtraction of fractions.

5. Use a variety of mental computation and estimation techniques.

- Students frequently do warm-up drills that enhance their mental math skills. Problems like: *3,000* x *7 =* , *200* x *6 =* , and *5,000* x *5 + 5 =* are put on the board as individual children write the answers without doing any paper-and-pencil computation.
- Students make appropriate choices from among *front-end*, *rounding*, and *compatible numbers* strategies in their estimation work depending on the real-world situation and the numbers involved. *Front end* strategies involve using the first digits of the largest numbers to get an estimate, which of course is too low, and then adjusting up. *Compatible numbers* involves finding some numbers which can be combined mentally, so that, for example, *762 + 2,444 + 248* is about (*750 + 250) + 2,500*, or *3,500*.
- Students use money and shopping situations to practice estimation and mental math skills. *Is $20.00 enough to buy items priced at $12.97, $4.95, and 3.95? About how much would 4 cans of beans cost if each costs $0.79?*
- Students explore estimation involving division as they read *The Greatest Guessing Game: A Book about Dividing* by Robert Froman. A little girl and her three friends solve a variety of problems, estimating first and discussing what to do with remainders.

6. Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.

- Students play *addition max out*. Each student has a 2 x 3 array of blanks (in standard 3-digit addition form) into each of which will be written a digit. One student rolls a die and everyone must write the number showing into one of their blanks. Once the number is written in, it can not be changed. Another roll — another number written, and so on. The object is to be the player with the largest sum when all six digits have been written. If a player has the largest possible sum that can be made from the six digits rolled, there is a bonus for *maxing out*.

- Students discuss this problem from the NCTM Standards (p. 45): *Three fourth grade teachers decided to take their classes on a picnic. Mr. Clark spent $26.94 for refreshments. He used his calculator to see how much the other two teachers should pay him so that all three could share the cost equally. He figured they each owed him $13.47. Is his answer reasonable?* As a follow-up individual assessment, they write about how they might find an answer.

7. Understand and use relationships among operations and properties of operations.

- Students take *7x8* block rectangular grids printed on pieces of paper. They each cut along any one of the *7* block-long segments to produce two new rectangles, for example, a *7x6* and a *7x2* rectangle. They then discuss all of the different rectangle pairs they produced and how they are all related to the original one.

- Students write a letter to a second grader explaining why *2+5* equals *5+2* to demonstrate their understanding of commutativity.

- Students explore modular, or clock, addition as an operation that behaves differently from the addition they know how to do. For example: *6 hours after 10 o'clock in the morning is 4 o'clock in the afternoon, so 10 + 6 = 4 on a 12-hour clock. How is clock addition different from regular addition? How is it the same? How would modular subtraction and multiplication work?*

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On-Line Resources

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Standard 8 — Numerical Operations — Grades 5-6

Overview

As indicated in the K-12 Overview, the widespread availability of computing and calculating technology has given us the opportunity to significantly reconceive the role of computation and numerical operations in our fifth and sixth grade mathematics programs. Some skills are less important while others, such as **estimation, mental computation,** and **understanding the meanings of the standard arithmetic operations,** all play a more significant role than ever in the everyday life of a mathematically literate adult.

The major shift in the curriculum that will take place in grades 5 and 6, therefore, is one away from drill and practice of paper-and-pencil symbolic procedures and toward real-world applications of operations, wise **choices of appropriate computational strategies**, and integration of the numerical operations with other components of the mathematics curriculum. At these grade levels, students are consolidating their understanding of whole number operations (especially multiplication and division) and beginning to develop computational skills with fractions and decimals. A sample unit on fractions for the sixth-grade level can be found in Chapter 17 of this *Framework.*

Much research in the past decade has focused on students' understandings of operations with large whole numbers and work with fractions and decimals. Each of these areas requires students to restructure their simple conceptions of number that were adequate for understanding whole number addition and subtraction.

Multiplication requires students to think about different meanings for the two factors. The first factor in a multiplication problem is a "multiplier." It tells how many groups one has of a size specified by the second factor. Thus, students need different understandings of the roles of the two numbers in the operation of multiplication than their earlier understandings of addition, in which both addends meant the same thing.

A similar restructuring is necessary for dealing appropriately with operations involving fractions and decimals. This restructuring revolves around the role of the "unit" in these numbers. In earlier grades, students thought about *5* or *498* as numbers that represented that many *things*. The understood unit, one, is the number which was used to count a group of objects. With fractions and decimals, though, the unit, still one and still understood, is a harder concept to deal with because its essential use is to help define the fraction or decimal rather than as a counter. When we speak of *5 poker chips* or *35 students*, our message is reasonably clear to elementary students. But when we speak of *2/3 of the class* or *0.45 of the price of the sweater*, the meaning is significantly less clear and we must be much more explicit about the role being played by the unit.

The topics that should comprise the numerical operations focus of the fifth and sixth grade mathematics program are:

multi-digit whole number multiplication and division
decimal multiplication and division
fraction operations
integer operations

Standard 8 — Numerical Operations — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

6*. Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.

- Fifth and sixth grade students have calculators available to them at all times, but frequently engage in competitions to see whether it is faster to do a given set of computations with the calculators or with the mental math techniques they've learned.
- Fifth graders make rectangular arrays with base-ten blocks to try to figure out how to predict how many square foot tiles they will need to tile a *17' by 23'* kitchen floor.
- Students are challenged to answer this question and then discuss the appropriate use of estimation when an exact answer is almost certain to be wrong: *The Florida's Best Orange Grove has 15 rows of 21 orange trees. Last year's yield was an average of 208.3 oranges per tree. How many oranges might they expect to grow this year? What factors might affect that number?*
- Students play *multiplication max out*. Each student has a *2* x *2* array of blanks (in the standard form of a 2-digit multiplication problem) into each of which a digit will be written. One student rolls a die and everyone must write the number showing into one of the blanks. Once a number is written, it cannot be moved. Another roll—another number written, and so on. The object is to be the player with the largest product when all four digits have been written. If a player has the largest possible product that can be made from the four digits rolled, there is a bonus for *maxing out*.

8. Extend their understanding and use of arithmetic operations to fractions, decimals, integers, and rational numbers.

- Students work in groups to explore fraction multiplication and division. They use *fraction circles* and *fraction strips* to solve problems like: *How can you divide four cakes among five people evenly?* They solve the problems and then write in their math journals about the methods they used and the reasons they believe their answers to be good ones.
- Students complete their study of fraction addition and subtraction by reading about Egyptian fractions. The Egyptians wrote every fraction as a unit fraction or the sum of a series of unit fractions with different denominators, for example, *7/8 = 1/2 + 1/4 + 1/8*. They try to find Egyptian fractions for *2/3 (1/2 + 1/6); 2/5 (1/3 + 1/15);* and *4/5 (1/2 + 1/5 + 1/10)*.
- Students demonstrate their understanding of division of fractions on a test by drawing a picture to show that *"1½ ÷ ½"* means: *How many halves are there in 1½?*

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*Activities are included here for Indicator 6, which is also listed for grade 4, since the Standards specify that students demonstrate continued progress in this indicator.

- Students use two-color chips to explore addition of integers. They each take ten chips and toss them ten times. Each time, the students record the number of yellow chips as a positive number (points earned) and the number of red chips as a negative number (points lost). For each toss, the student writes a number sentence, such as $6 + -4 = 2$ for *6* points earned and *4* lost. Students may also keep a running total of points overall.
- Students read and discuss *If You Made a Million* by David Schwartz, relating money to decimals.
- Students read and discuss "Beasts of Burden" in *The Man Who Counted: A Collection of Mathematical Adventures* by Malba Tahan. In this story, three brothers must divide their Father's 35 camels so that one gets 1/2 of the camels, another 1/3, and the last 1/9. The narrator and a wise mathematician help them solve the problem by adding their camel to the 35, making 36. One brother then gets 18, another 12, and the third receives 4 making a total of 34. The narrator and mathematician take back their camel as well as the one left over.
- Students demonstrate their understanding of addition and subtraction of unlike fractions on a test by finding the errors made by a fictitious student and explaining to that student what he/she did wrong.
- Students read Shel Silverstein's poem *A Giraffe and a Half* and make up stories about mixed fractions.
- Students discuss whether $2/3 \times 5/4$ is more or less than $2/3$. They explain their reasoning.
- Students listen to John Ciardi's poem "Little Bits" and discuss the concept that a whole can be described by an infinite number of equivalent names, such as $1/2 + 1/4 + 1/8 + 1/8$ or $3/3$.
- Students solve *missing link* problems, like the one below, in which they must find number(s) and/or symbols that will make a true sentence:

$$\_\_\_\_\_\_ + 3.1 - \_\_\_\_\_\_ - 5.4 = 8.7$$

9. Extend their understanding of basic arithmetic operations on whole numbers to include powers and roots.

- Students explore the exponent key, the x^2 key, and the square root key on their calculators. The groups are challenged to define the function of each key, to tell how each works, and to create a keypress sequence using these keys, the result of which they predict before they key it in.
- Students work through *The Powers of the Knight* lesson that is described in the Introduction to this *Framework*. It introduces a classic problem of geometric growth which engages them as they encounter notions of exponential notation.
- Students work through the *Pizza Possibilities* lesson that is described in the First Four Standards of this *Framework*. In it, students discover that the number of pizzas possible doubles every time another choice of topping is added. They work through the *Two-Toned Towers* lesson that is also described in the First Four Standards and note the similarities in the problems and in their solutions.
- Students join the midpoints of the sides of a 2 x 2 square on a geoboard to form a smaller

square. They determine the area of the smaller square and explore the lengths of its four sides.

10. Develop, apply, and explain procedures for computation and estimation with whole numbers, fractions, decimals, integers, and rational numbers.

- Students working in groups develop a method to estimate the products of two-digit whole numbers and decimals by using the kinds of base-ten block arrays described in Indicator 6 above. Usually just focusing on the “flats” results in a reasonable estimate.
- Students follow up a good deal of experience with concrete models of fraction operations using materials such as *fraction bars* or *fraction squares* by developing and defending their own paper-and-pencil procedures for completing those operations.
- Students develop rules for integer operations by using *postman stories*, as described in Robert Davis' *Discovery in Mathematics*. The teacher plays the role of a postman who delivers mail to the students. Sometimes the mail delivered contains money (positive integers) and sometimes bills (negative integers). Sometimes they are delivered to the students (addition) and sometimes picked up from them (subtraction).
- Students model subtraction with two-color chips by adding pairs of red and yellow chips. (First, they must agree that an equal number of red and yellow chips has a value of *0*.) For example, to show $4 - (-2)$, they lay out 4 red chips, add 2 pairs of red and yellow chips (whose value is *0*), and then take away 2 yellow chips. They note that $4 - (-2)$ and $4 + 2$ give the same answer and try to explain why this is so.

11. Develop, apply, and explain methods for solving problems involving proportions and percents.

- Students develop an estimate of π by carefully measuring the diameter and circumference of a variety of circular objects (cans, bicycle tires, clocks, wooden blocks). They list the measures in a table and discuss observations and possible relationships. After the estimate is made, π is used to solve a variety of real-world circle problems.
- Students use holiday circulars advertising big sales on games and toys to comparison shop for specific items between different stores. *Is the new Nintendo game, Action Galore, cheaper at Sears where it is 20% off their regular price of $49.95 or at Macy’s where it's specially priced at $41.97?*
- One morning, as the students arrive at school, they see a giant handprint left on the blackboard overnight. They measure it and find it to be almost exactly one meter long. *How big was the person who left the print? Could she have fit in the room to make the print, or did she have to reach in through the window? How could you decide how much she weighs?*
- Students read *Jim and the Giant Beanstalk* by Raymond Briggs. Jim helps the aging giant by measuring his head and getting giant eyeglasses, false teeth and a wig. The students use the measures given in the book to find the size of the giant’s hand and then his height.
- Students develop a sampling strategy and use proportions to determine the population of *Bean City* (*NCTM Addenda Booklet, Grade 6*), whose inhabitants consist of three types of beans.
- Students discuss different ways of finding “*easy*” percents, such as *50%* of *30* or *15%* of

25. They then generate percent exercises that can be solved mentally and share them with their classmates.

12. Understand and apply the standard algebraic order of operations.

- Students bring in calculators from home to examine their differences. Among other activities, they each key in " *6 + 2* x *4 =* " and then compare their calculator displays. Some of the displays show *32* and others show *14*. *Why? Which is right? Are the other calculators broken?*
- Students play *rolling numbers*. They use four white dice and one red one to generate four working numbers and one target number. They must combine all of the working numbers using any operations they know to formulate an expression that equals the target number. For example, for *2, 3, 4, 5* with target number *1*, the following expression works: *(2+5)/(3+4)=1*. Questions about order of operations and about appropriate use of parentheses frequently arise.

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Standard 8 — Numerical Operations — Grades 7-8

Overview

Traditionally, tremendous amounts of time were spent at these grade levels helping students to finish their development of complex paper-and-pencil procedures for the four basic operations with whole numbers, fractions, and decimals. While some competency with paper-and-pencil computation is necessary, **estimation**, **mental computation**, and **understanding the meanings of the standard arithmetic operations** all play a more significant role than ever in the everyday life of a mathematically literate adult.

As indicated in the K-12 Overview, then, the major shift in the curriculum that will take place at these grade levels is one away from drill and practice of paper-and-pencil symbolic procedures and toward **real-world applications of operations**, wise **choices of appropriate computational strategies**, and **integration of the numerical operations with other components of the mathematics curriculum**.

Seventh- and eighth-graders are relatively comfortable with the unit shift discussed in this standard's Grades 5-6 Overview. Operations on fractions and decimals, as well as whole numbers, should be relatively well developed by this point, allowing the focus to shift to a more holistic look at operations. "Numerical operations" becomes less a specific object of study and more a process, a set of tools for problem setting. It is critical that teachers spend less time focused on numerical operations, per se, so that the other areas of the *Standards*-based curriculum receive adequate attention.

One important set of related topics that needs to receive some significant attention here, however, is **ratio, proportion,** and **percent**. Seventh and eighth graders are cognitively ready for a serious study of these topics and to begin to incorporate proportional reasoning into their set of problem solving tools. Work in this area should start out informally, progressing to the student formulation of procedures that make proportions and percents the powerful tools they are.

Two other topics that receive greater attention here, even though they have been informally introduced earlier, are **integer operations** and **powers and roots**. Both of these types of operations further expand the students' knowledge of the types of numbers that are used and the ways in which they are used.

Estimation, mental math, and **technology use** begin to mature in seventh and eighth grades as students use these strategies in much the same way that they will as adults. If earlier instruction in these skills has been successful, students will be able to make appropriate choices about which computational strategies to use in given situations and will feel confident in using any of these in addition to paper-and-pencil procedures. For example, students should evaluate simple problems involving fractions, such as *what's two-thirds of 5 tablespoons?* using mental math. Students need to continue to develop alternatives to paper-and-pencil as they learn more about operations on other types of numbers, but the work here is primarily on the continuing use of all of the strategies in rich real-world problem solving settings.

The topics that should comprise the numerical operations focus of the seventh and eighth grade mathematics program are:

rational number operations	powers and roots
integer operations	proportion and percent

Standard 8 — Numerical Operations — Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

6*. **Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results**.

- Students choose a stock from the New York Stock Exchange and estimate and then compute the net gain or loss each week for a $1,000 investment in the company.
- Students use spreadsheets to "program" a set of regular, repeated, calculations. They might, for example, create a prototype on-line order blank for a school supply company that lists each of the ten items available, the individual price, a cell for each item in which to place the quantity ordered, the total computed price for each item, and the total price for the order.
- Students regularly have *human vs. calculator races*. Given a list of specially selected computation exercises (e.g., *53* x *20, 40* x *10, 95 + 17 +5*), one student uses mental math strategies and another uses a calculator. They quickly come to realize that the human has the advantage in many situations.

8. **Extend their understanding and use of arithmetic operations to fractions, decimals, integers, and rational numbers.**

- Given a decimal or a fractional value for a piece of a tangram puzzle, the students determine a value for each of the other pieces and a value for the whole puzzle.
- Students use *fraction squares* to show why the multiplication of two fractions less than one results in a product that is less than either.
- Students demonstrate their understanding of operations on rational numbers by formulating their own reasonable word problems to accompany given number sentences such as *3/4 divided by 1/2 = 1½.*

9. **Extend their understanding of basic arithmetic operations on whole numbers to include powers and roots.**

- Students play *powers max out*. Each student has a set of 5 blanks, into each of which will be written a digit. They are in the form $VW^X + YZ$. One student rolls a die and everyone must write the *number* showing into one of their blanks. Once written, a number can not be moved. Another roll — another number written, and so on. The object is to be the player

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*Activities are included here for Indicator 6, which is also listed for grade 4, since the Standards specify that students demonstrate continued progress in this indicator.

with the largest-valued expression when all five digits have been written. If a player has the largest possible value that can be made from the five digits rolled, there is a bonus for *maxing out*.

- Students develop their own "rules" for operations on numbers raised to powers by rewriting the expressions without exponents. For example, $7^2 \times 7^4 = (7 \times 7) \times (7 \times 7 \times 7 \times 7) = 7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^6$. *You just add the exponents!*

- Students read *The King's Chessboard, The Rajah's Rice: A Mathematical Folktale from India*, or *A Grain of Rice*. All of these stories involve a situation in which a quantity is doubled each day. Students use the story to discuss powers of *2* and to look for patterns in the sums of the powers of *2*.

- Students use the relationship between the area of a square and the length of one of its sides to begin their study of roots. Starting with squares on a geoboard with areas of *1*, *4*, *9*, and *16*, they then are asked to find squares whose areas are *2*, *5*, and *13*.

- Students work through the *Rod Dogs* lesson that is described in the First Four Standards of this *Framework*. They investigate how the surface area and volume of an object changes as it is enlarged by various scale factors.

10. Develop, apply, and explain procedures for computation and estimation with whole numbers, fractions, decimals, integers, and rational numbers.

- Students use a videotape of a youngster walking forward and backward as a model for multiplication of integers. The "product" of running the tape forward (+) with the student walking forward (+) is walking forward (+). The "product" of running it backward (−) with the student walking forward (+) is walking backwards (−). The other two combinations also work out correctly.

- Students use base ten blocks laid out in an array to show decimal multiplication. *How could the values of the blocks be changed to allow it to work? What new insights do we gain from the use of this model?*

- Students judge the reasonableness of the results of fraction addition and subtraction by "rounding off" the fractions involved to *0, 1/2, or 1*.

- Students explore the equivalence between fractions and repeated decimals by finding the decimal representations of various fractions and using the resulting patterns to find the fractional equivalents of some repeated decimals.

11. Develop, apply, and explain methods for solving problems involving proportions and percents.

- Students use *The Geometer's Sketchpad* software to draw a geometric figure on a computer screen, scale it larger or smaller, and then compare the lengths of the sides of the original with those of the scaled image. They also compare the areas of the two.

- Students are comfortable using a variety of approaches to the solution of proportion problems. Example: *If 8 pencils cost 40¢, how much do 10 pencils cost*? This problem can be solved by:

unit-rate method *8 pencils for 40¢ means 5¢/pencil or 10* x *5=50¢ for 10*

factor-of-change method *10 pencils is 10/8 of 8 pencils, so cost is (10/8)*x*40=50¢*

cross multiplication method *8/40 = 10/*x*, 8*x *= 400, so* x*=50¢.*

- Students set up a part/whole proportion as one method of solving percent problems.
- Students spend $100 by selecting items from a catalog. They must compute sales tax and consider it in deciding what they will buy.

12. Understand and apply the standard algebraic order of operations.

- Students bring in calculators from home to examine their differences. Among other activities, they each key in *"3 + 15 ÷ 3"* and then compare their calculator displays. Some of the displays show *6* and others show *8*. *Why? Which is right? Are the other calculators broken?* Students decide what key sequence would work for the calculators that do not use order of operations.
- Students play with the software *How the West was One + Three* x *Four*, which requires them to construct numerical expressions that use the standard order of operations.
- Students use the digits *1*, *2*, *3*, and *4* to find expressions for each of the numbers between *0* and *50*. For example, *7 = (3*x*4)/2 + 1*.

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Standard 8 — Numerical Operations — Grades 9-12

Overview

In the ninth through twelfth grades, the themes described in the K-12 Overview — **estimation, mental computation,** and **appropriate calculator and computer use**—become the focus of this standard. What is different about this standard at this level when compared to the traditional curriculum is its mere presence. In the traditional academic mathematics curriculum, work on numerical operations was basically finished by eighth grade and focus then shifted exclusively to the more abstract work in algebra and geometry. But, in the highly technological and data-driven world in which today's students will live and work, strong skills in numerical operations have perhaps even more importance than they once did. By giving older students a variety of approaches and strategies for the computation that they encounter in everyday life, approaches with which they can confidently approach numerical problems, they will be better prepared for their future.

The major work in this area, then, that will take place in the high school grades, is continued opportunity for **real-world applications of operations**, wise **choices of appropriate computational strategies**, and **integration of the numerical operations with other components of the mathematics curriculum**.

The new topics to be introduced in this standard for these grade levels involve factorials, matrices, operations with polynomials, and operations with irrational numbers as useful tools in problem solving situations.

Estimation, mental math, and **technology use** should fully mature in the high school years as students use these strategies in much the same way that they will as adults. If earlier instruction in these skills has been successful, students will be able to make appropriate choices about which computational strategies to use in given situations and will feel confident in using any of these in addition to paper-and-pencil techniques. Students need to continue to develop alternatives to paper-and-pencil as they learn about operations with matrices and other types of number, but the work here is almost exclusively on the continuing use of all of the strategies in rich, real-world, problem solving settings.

The topics that should comprise the numerical operations focus of the ninth through twelfth grade mathematics program are:

operations on real numbers
translation of arithmetic skills to algebraic operations
operations with factorials, exponents, and matrices

Standard 8 — Numerical Operations — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11 and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

6[*]. Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.

- Students frequently use all of these computational strategies in their ongoing mathematics work. Inclinations to over-use the calculator, in situations where other strategies would be more appropriate, are overcome with five minute "contests," speed drills, and warm-up exercises that keep the other skills sharp and point out their superiority in given situations.
- Numerical problems in class are almost always worked out in "rough" form before any precise calculation takes place so that everyone understands the "ballpark" in which the computed answer should lie and which answers would be considered unreasonable.
- Students use estimation in their work with irrational numbers, approximating the results of operations such as $\sqrt{15} + \sqrt{17}$ or $\sqrt{32}\,\sqrt{8}$, and developing general rules.
- Students discuss the advantages and disadvantages of using graphing calculators or computers to perform computations with matrices.
- Students demonstrate their ability to select and use appropriate computational methods by generating examples of situations in which they would choose to use a calculator, to estimate, or to use mental math.
- Students solve given computational problems using an assigned strategy and discuss the advantages and disadvantages of using that particular strategy with that particular problem.

13. Extend their understanding and use of operations to real numbers and algebraic procedures.

- Students work on the painted cube problem to enhance their skill in writing algebraic expressions: *A 3-inch cube is painted red. It is then cut into 1-inch cubes. How many of them have 3 red faces? 2 red faces? 1-red face? No red faces? Repeat the problem using an original 4-inch cube, then a five-inch cube, then an n-inch cube*.
- Students develop a procedure for binomial multiplication as an extension of their work with 2-digit whole number multiplication arrays. Using *algebra tiles*, they uncover the parallels between *23* x *14* (which can be thought of as $(20+3)(10+4)$) and $(2x+3)(x+4)$.
- Students work through the *Ice Cones* lesson that is described in the First Four Standards of

[*] Activities are included here for Indicator 6, which is also listed for grade 8, since the Standards specify that students demonstrate continued progress in this indicator.

this *Framework*. They discover that in order to graph the equation to determine the maximum volume of the cones, they need to use algebraic procedures to solve for h in terms of r.

- Students devise their own procedures and "rules" for operations on variables with exponents by performing trials of equivalent computations on whole numbers.
- Students use *algebra tiles* to develop procedures for adding and subtracting polynomials.
- Students use compasses and straightedges to construct a Golden Rectangle and find the ratio of the length to the width $(1 + \sqrt{5})/2$.
- Students consider the ratios of successive terms of the Fibonacci sequence *(1, 1, 2, 3, 5, 8, ...)*, where each term after the first two is the sum of the two preceding terms, finding that the ratios get closer and closer to the Golden Ratio $(1 + \sqrt{5})/2$.

14. Develop, apply, and explain methods for solving problems involving factorials, exponents, and matrices.

- Students work through the *Breaking the Mold* lesson that is described in the Introduction to this *Framework*. It uses a science experiment with growing mold to involve students in discussions and explorations of exponential growth.
- Students use their graphing calculators to find a curve that best fits the data from the population growth in the state of New Jersey over the past 200 years.
- Students discover the need for a factorial notation and later incorporate it into their problem solving strategies when solving simple combinatorics problems like: *How many different five card poker hands are there? In how many different orders can four students make their class presentations? In how many different orders can six packages be delivered by the letter carrier?*
- Students compare 2^{101}, $(2^{50})^2$, and $3x2^{100}$ to decide which is largest. They explain their reasoning.
- Students read "John Jones's Dollar" by Harry Keeler and discuss how it demonstrates exponential growth. They check the computations in the story, determining their accuracy.

References

Keeler, Harry Stephen. "John Jones's Dollar," in Clifton Fadiman, Ed. *Fantasia Mathematica.* New York: Simon and Schuster, 1958.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.