Reconnect '04
Introduction to Integer Programming

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Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under contract DE-AC04-94AL85000.

Integer programming (IP)

Min $c^T x$

Subject to:
$$Ax = b$$
$$\ell \le x \le u$$
$$x = (x_I, x_C)$$
$$x_I \in Z^n \quad \text{(integer values)}$$
$$x_C \in Q^n \quad \text{(rational values)}$$

- Can also have inequalities in either direction (slack variables):

$$a_i^T x \le b_i \Rightarrow a_i^T x + s_i = b_i \ , \ s_i \ge 0$$

- If $x_I \ne \varnothing$ and $x_C \ne \varnothing$ then this is a mixed-integer program (MIP)
- Linear programming (LP) has no integrality constraints $x_I = \varnothing$ (in P)
- IP (easily) expresses any NP-complete problem

Terminology

In this context **programming** means making decisions.
Leading terms say what kind:

- (Pure) Integer programming: all integer decisions
- Linear programming
- Quadratic programming: quadratic objective function
- Nonlinear programming: nonlinear constraints
- Stochastic programming: finite probability distribution of scenarios

Came from **operations research** (practical optimization discipline)

Computer programming (by someone) is required to solve these.

Decisions

The IPs I've encountered in practice involve either

- Allocation of scarce resources
- Study of a natural system
  - Computational biology
  - Mathematics

Maybe during or after this course, you can add to the list

Integer Variables

Use $x_i \in \{0,1\}$ (binary variables) to model:

- Yes/no decisions
- Disjunctions
- Logical conditions
- Piecewise linear functions (this not covered in this lecture)

General Integer Variables

Use general integer variables to choose a number of indivisible objects such as the number of planes to produce

Integer range should be small (e.g. 1-10)

- Computational tractability
- Larger ranges may be well approximated by rational variables (number of bags of potato chips to produce)

Example: Binary Knapsack

Given set of objects 1..n

total weight W, item weight/size w_i, value v_i

$$x_i = \begin{cases} 1 & \text{If we select item } i \\ 0 & \text{Otherwise} \end{cases}$$

$$\max \sum_{i=1}^{n} v_i x_i$$

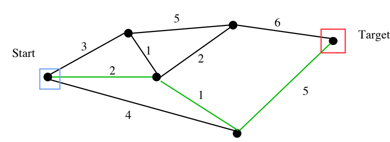
Subject to

$$\sum_{i=1}^{n} w_i x_i \le W$$

Example: Shortest-Path Network Interdiction

Delay an adversary moving through a network.

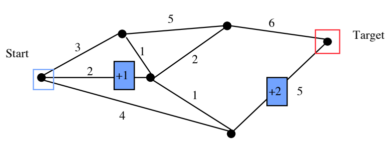
- Adversary moves start→target along a shortest path (in worst case)
- Path length = sum of edge lengths. Measure of time, exposure, etc.



Shortest Path Length: 8

Example: Shortest-Path Network Interdiction

Defender blocks the intruder by paying to increase edge lengths.
Goal: Maximize the resulting shortest path.



Shortest Path Length: 11

Path Interdiction Mixed-Integer Program

Graph G = (V,E)
Edge lengths ℓ_{uv} for edge (u,v)
Can increase length of (u,v) by λ_{uv} at cost c_{uv}
Budget B

Variables:

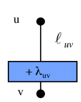
$$x_{uv} = \begin{cases} 1 & \text{if we pay to lengthen edge (u,v)} \\ 0 & \text{Otherwise} \end{cases}$$

d_u: shortest distance from start s to node u

Path Interdiction Integer Program

Objective: maximize the shortest path to the target
maximize d_t

Subject to:
Path to the start has length 0:
$d_s = 0$
Calculate a shortest path length:



$$d_u \le d_v + \ell_{uv} + \lambda_{uv} x_{uv} \text{ for all } (u,v) \in E$$
$$d_v \le d_u + \ell_{uv} + \lambda_{uv} x_{uv} \text{ for all } (u,v) \in E$$

Respect the budget:

$$\sum_{(u,v)\in E} c_{uv} x_{uv} \le B$$

Modeling Dependent Decisions

Suppose x, y are two binary variables that represent a decision (where 1 means "yes" and 0 means "no")

The constraint $x \le y$ allows x to be "yes" only if y is "yes"

Example: Unconstrained Facility Location

Given potential facility locations, n customers to be served
c_j = cost to build facility j
h_{ij} = cost to meet all of customer i's demand from facility j

$$x_j = \begin{cases} 1 \text{ if facility } j \text{ built} \\ 0 \text{ Otherwise} \end{cases} \qquad y_{ij} = \begin{cases} 1 \text{ if customer } i \text{ is served by facility } j \\ 0 \text{ Otherwise} \end{cases}$$

$$\min \sum_j c_j x_j + \sum_{i,j} h_{ij} y_{ij}$$

$$\text{st.} \quad \sum_j y_{ij} = 1 \qquad \forall i \text{ (each customer satisfied)}$$

$$y_{ij} \le x_j \qquad \forall i, j \text{ (facility built before use)}$$

$$x_j, y_i \in \{0,1\}$$

Sometimes it's OK to satisfy customers from multiple facilities:
y_{ij} becomes a percentage : $y_{ij} \in Q,\ 0 \le y_{ij} \le 1$

Formulation is really important in practice

Unconstrained facility location
Could sum constraints $y_{ij} \le x_i \quad \forall i, j$ over all customers i to get:

$$\sum_i y_{ij} \le n x_j \quad \forall j$$

Recall n is the number of customers.
Still requires a facility is built before use (IPs are equivalent at optimality)
But, for 40 customers, 40 facilities, random costs

- First formulation solves in 2 seconds
- Second formulation solves in 53,121 seconds (14.75 hours)

What makes one formulation so much better?

- Understanding this fully is an open problem.
- Some performance differences can be explained by the way IPs are solved in practice by branch-and-bound-like algorithms: the LP relaxation

Recall Integer Programming (IP)

Min $\quad c^T x$
Subject to:

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Linear programming (LP) relaxation of an IP

Min $c^T x$

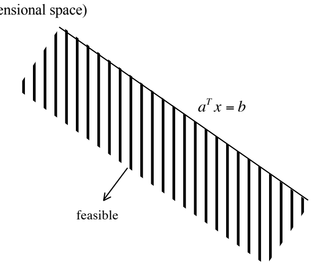
Subject to:

$$Ax = b$$
$$\ell \le x \le u$$
$$x = (x_I, x_C)$$
$$\cancel{x_I \in Z^n} \text{ (integer values)}$$
$$x_C \in Q^n \text{ (rational values)}$$

- LP can be solved efficiently (in theory and practice)
- Relaxation = removing constraints
  - All feasible IP solutions are feasible
  - LP gives a lower bound

Linear Programming Geometry

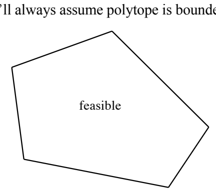
The solutions to a single inequality $a^T x \le b,\ x \in Q^n$ form a half space (in n-dimensional space)



Linear Programming Geometry

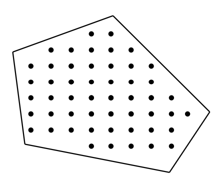
Intersection of all the linear (in)equalities form a convex polytope

- For simplicity, we'll always assume polytope is bounded



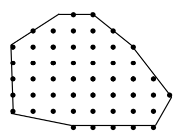
IP Geometry

Feasible integer points form a lattice inside the LP polytope



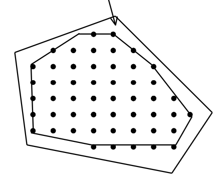
IP Geometry

The convex hull of this lattice forms the **integer polytope**



IP/LP Geometry

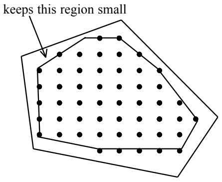
A "good" formulation keeps this region small



Every node for which the LP bound is lower than the integer optimal must be processed (e.g. expanded)

IP/LP Geometry

A "good" formulation keeps this region small



One measure of this is the **Integrality Gap**:
Integrality gap = $\max_{\text{instances } I}(IP(I))/(LP(I))$

Unconstrained Facility Location Revisited

Given potential facility locations, customers to be served
c_j = cost to build facility j
h_{ij} = cost to meet all of customer i's demand from facility j

$$x_j = \begin{cases} 1 \text{ if facility } j \text{ built} \\ 0 \text{ Otherwise} \end{cases} \qquad y_j = \begin{cases} 1 \text{ if customer } i \text{ is served by facility } j \\ 0 \text{ Otherwise} \end{cases}$$

$$\min \sum_j c_j x_j + \sum_{i,j} h_{ij} y_{ij}$$

$$\text{st.} \quad \sum_j y_{ij} = 1 \qquad \forall i \text{ (each customer satisfied)}$$

$$y_{ij} \le x_j \qquad \forall i, j \text{ (facility built before use)}$$

$$x_j, y_i \in \{0,1\}$$

How the weaker LP "cheats"

Using $\sum_i y_{ij} \leq n\, x_j \quad \forall j$

Allows the LP to completely satisfy customer *i* with facility *j* ($y_{ij} = 1$) even with $x_j = 1/n$.

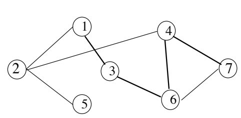
With these constraints: $y_{ij} \leq x_i \quad \forall i, j$

If $x_i = 1/n$, then $y_{ij} <= 1/n$

Can't we just round the LP Solution?

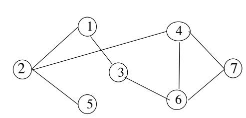
- Not generally feasible
- If (miraculously) it is feasible, it's not generally good

Example: Maximum Independent Set



- Find a maximum-size set of vertices that have no edges between any pair

Example: Maximum Independent Set



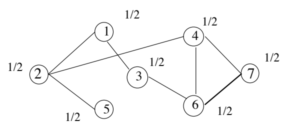
$$v_i = \begin{cases} 1 \text{ if vertex i is in the MIS} \\ 0 \text{ otherwise} \end{cases}$$

$$\max \sum v_i$$

$$\text{s.t. } v_i + v_j \leq 1 \quad \forall (i, j) \in E$$

$$v_i \in \{0,1\}$$

Example: Maximum Independent Set

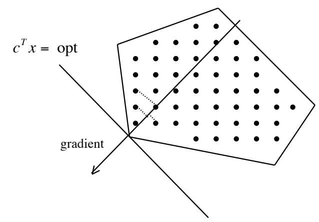


$$\max \sum v_i$$
$$\text{s.t. } v_i + v_j \le 1 \quad \forall (i,j) \in E$$

The zero-information solution ($v_i = .5$ for all i) is feasible and it's optimal if the optimal MIS has size at most |V|/2.

Rounding everything (up) is infeasible.

Can't we project the lattice onto the objective gradient?

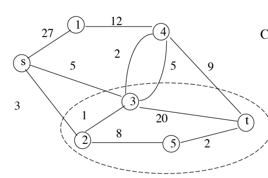


- Hard to find a feasible solution to project (NP-complete!)
  - Make the objective a constraint and do binary search
- This is a lot harder in *n* dimensions than it looks like in 2

Perfect formulations

- Sometimes solving an LP is guaranteed to give an integer solution
  - All polytope corners have integer coefficients (naturally integer)
  - Sometimes only for specific objectives (e.g. $c \ge 0$)

Perfect Formulation Example: Minimum Cut



Capacity u_e

- Special nodes *s* and *t*
- Each edge *e* has capacity u_e. Set of edges S has capacity $\sum_{e \in S} u_e$
- Partition vertex set V into S,T where $s \in S$ and $t \in T$
- A cut is the edges (u,v) such that $u \in S$ and $v \in T$

Find a cut with minimum capacity

Perfect Formulation Example: Minimum Cut IP

$$v_i = \begin{cases} 0 \text{ if node } v \text{ is on the } s \text{ side} \\ 1 \text{ if node } v \text{ is on the } t \text{ side} \end{cases}$$

Helper variables $y_e = 1$ if e is in the cut and 0 otherwise

$$\min \sum u_e y_e$$

$$\begin{aligned} \text{st} \quad & y_e \geq v_i - v_j \quad \forall e = (i,j) \\ & y_e \geq v_j - v_i \quad \forall e = (i,j) \\ & v_s = 0,\ v_t = 1 \\ & v_e \in \{0,1\} \end{aligned}$$

The y variables will be integral if the v variables are.

Total Unimodularity

The minimum cut matrix (possibly with slack variables) is **totally unimodular (TU)**: all subdeterminants (including the matrix entries) have value 0, 1, or -1.

- All corner solutions x satisfy $Ax=b$
- By Kramer's rule x will be integral

Network matrices (adjacency matrices of graphs) are TU.

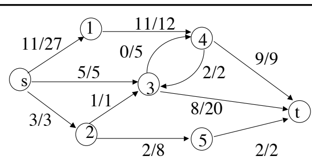
Nemhauser and Wolsey (Integer and Combinatorial Optimization, Wiley, 1988) give some sufficient conditions for a matrix to be TU.

Note: if a matrix is TU, there is always an efficient combinatorial algorithm to solve the problem (not necessarily obvious)

Total Unimodularity is Fragile

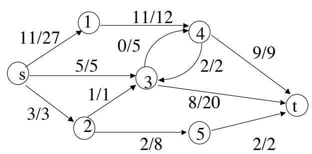
- Example: Network Interdiction
  - Expend a limited budget to maximally damage the transport capacity of a network

Network Flow



- Source(s) s, sink (consumers) t
- Capacity (bottom number)
- Flow (top number)
- Maximize flow from s to t obeying
  - Capacity constraints on edges
  - Conservation constraints on all nodes other than s,t

Network Interdiction

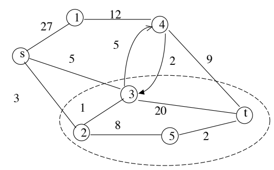


- Each edge e now has a destruction cost d_e (cost to remove e; assume linear)
- Budget B

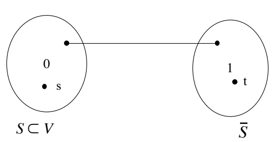
Expend at most B removing (pieces of) edges in the network so resulting max flow is minimized

Network Interdiction

By LP duality (we'll see later)
value of max flow = value of min cut
So
$\min_{\text{attacks}}$ max flow = $\min_{\text{attacks}}$ min cut
Pay to knock out transport capacity from *s* to *t*



A Mixed Integer Program for Network Inhibition



- Based on min-cut LP
- Find best cut to attack
- Decision variables place vertices on the s or t side as before
- All edges going across the cut must be destroyed (consume budget) or contribute to residual cut capacity

Network Inhibition IP

$$v_i = \begin{cases} 0 \text{ if node } v \text{ is on the } s \text{ side} \\ 1 \text{ if node } v \text{ is on the } t \text{ side} \end{cases}$$

Helper variables y_e = percent of an edge in cut that is not removed
z_e = percent of an edge in the cut that is destroyed

$$\min \sum u_e y_e$$

$$\begin{aligned} \text{st} \quad & y_e + z_e \geq v_i - v_j \quad \forall e = (i,j) \\ & y_e + z_e \geq v_j - v_i \quad \forall e = (i,j) \\ & v_s = 0,\ v_t = 1 \\ & \sum_e d_e z_e \leq B \\ & v_e \in \{0,1\} \end{aligned}$$

Total Unimodularity is Fragile

$$\begin{aligned} \min \ & \sum u_e y_e \\ \text{st} \ & y_e + z_e \geq v_i - v_j \quad \forall e = (i, j) \\ & y_e + z_e \geq v_j - v_i \quad \forall e = (i, j) \\ & v_s = 0,\ v_t = 1 \\ & \textstyle\sum_e d_e z_e \leq B \\ & v_e \in \{0,1\} \end{aligned}$$

The matrix is still TU without the budget constraint

Adding the budget constraint makes the problem strongly NP-complete

- No known polynomial-time approximation algorithms
- Still has some very nice structure that gives a pseudo-approximation
  - Pseudo-approximation might give a superoptimal solution that slightly exceeds the budget or it could give a true approximation

Modeling Sets

Given a set T,

- $\sum_{i \in T} x_i \geq 1$ means select at least 1 element of T
  - Making sure at least one local warehouse has inventory for each customer
- $\sum_{i \in T} x_i \leq 1$ means select at most 1 element of T
  - Conflicts (e.g. modeled by a maximum independent set problem)
  - Resource constraints
- $\sum_{i \in T} x_i = 1$ means select exactly 1 element of T
  - Time indexed scheduling variables x_{jt}, schedule job j at time t. This picks a single time for job j.

Modeling Disjunctive Constraints

Let $a_1^T x \geq b_1$ and $a_2^T x \geq b_2$ be two constraints with nonnegative coefficients $(a_i \geq 0,\ i = 1,2)$

To force satisfaction of **at least one** of these constraints:

$$\begin{aligned} & a_1^T x \geq y b_1 \\ & a_2^T x \geq (1 - y) b_2 \\ & y \in \{0,1\} \end{aligned}$$

Modeling Disjunctive Constraints - General Number

Let $a_i^T x \geq b_i,\ i = 1 \dots m$ be m constraints with nonnegative coefficients $(a_i \geq 0)$

To force satisfaction of **at least k** of these constraints:

$$\begin{aligned} & a_i^T x \geq b_i y_i \quad i = 1 \dots m \\ & \textstyle\sum_{i=1}^m y_i \geq k \\ & y \in \{0,1\} \end{aligned}$$

Modeling a Restricted Set of Values

- Variable x can take on only values in $\{v_1, v_2, \ldots v_m\}$
  - Frequently the v_i are sorted
  - Example: capacity of an airplane assigned to a flight

$$x = \sum_{i=1}^{m} v_i y_i$$
$$\sum_{i=1}^{m} y_i = 1$$
$$y \in \{0,1\}$$

  - The y_i's are a **special ordered set**.

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Some simple logical constraints

Want $\quad y = x_1 \vee x_2 \quad$ (logical or)

$$y \geq x_1$$
$$y \geq x_2$$

Suffices if there is pressure in the objective function to keep *y* low.

- Saw this in minimum cut

Similarly if we want $\quad y = x_1 \wedge x_2 \quad$ (logical and)

$$y \leq x_1$$
$$y \leq x_2$$

Suffices if there is pressure in the objective function to keep *y* high.

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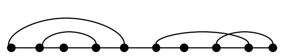
Example: Protein Structure Comparison



Contact Map

- 2 nonadjacent amino acids share an edge if they're physically close when folded

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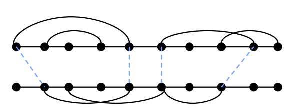
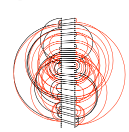
Example: Protein Structure Comparison



- 2 nonadjacent amino acids share an edge if they're physically close folded
- Noncrossing alignment of two proteins to maximize shared contacts
- Measure of similarity



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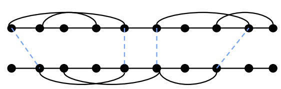
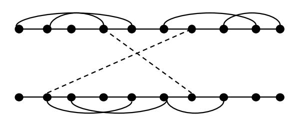
Protein Structure Comparison

- Variables x_{ij} = 1 if amino acid in position i of the top protein is matched to amino acid in position j of the bottom protein, 0 otherwise
- Helper variables $y_{ijkl} = x_{ij} \wedge x_{kl}$



Non-crossing alignment

- For any pair of edges, we can tell if they cross



$$y_{ijkl} = 0$$

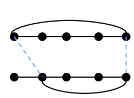
if the pair is forbidden (simply don't create this variable).

- There are more clever ways to do this (e.g. using Ramsey theory). See what you can come up with.

Protein Structure Comparison

Only consider y_{ijkl} if this is a shared contact ((i,k) a contact, (j,l) a contact)

$$\begin{aligned}
\max \quad & \sum_{y_{ijkl} \text{ exists}} y_{ijkl} \\
\text{st} \quad & \mathrm{y}_{\mathrm{ijkl}} = 0 \ \text{ if } (i,j) \text{ and } (k,l) \text{ cross (doesn't exist)} \\
& \mathrm{y}_{\mathrm{ijkl}} \le x_{ij} \\
& \mathrm{y}_{\mathrm{ijkl}} \le x_{kl} \\
& x_{ij} \in \{0,1\}
\end{aligned}$$



MIP Applications (Small Sample)

- Logistics
  - Capacity planning, scheduling, workforce planning, military spares management
- Infrastructure/network security
  - Vulnerability analysis, reinforcement, reliability, design, integrity of physical transport media
  - Sensor placement (water systems, roadways)
- Waste remediation
- Vehicle routing, fleet planning
- Bioinformatics: protein structure prediction/comparison, drug docking
- VLSI, robot design
- Tools for high-performance computing (scheduling, node allocation, domain decomposition, meshing)

Solving Integer Programs

- NP-hard
- Many special cases have efficient solutions or provably-good approximation bounds
  - Need time to explore structure
- General IPs can be hard due to size and/or structure

(Sufficiently) optimal solution is important
- When lives or big $ at stake
- For rigorous benchmarking of heuristic/approximation methods
- To gain structural insight for better algorithms/proofs.