PORTA, NEOS, and Knapsack Covers

Cover Inequalities

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Today's Outline

- Knapsack Cover inequalities
  - Facets
  - Lifting
- Why would we care?

Valid Inequalities for the Knapsack Problem

- We are interested in valid inequalities for the knapsack set KNAP

$$\text{KNAP} = \{x \in \mathbb{B}^n | \sum_{j \in N} a_j x_j \leq b\}$$

- $N = \{1, 2, \ldots n\}$
- A set $C \subseteq N$ is a *cover* if $\sum_{j \in C} a_j > b$
- A cover C is a *minimal cover* if $C \setminus j$ is not a cover $\forall j \in C$
- If $C \subseteq N$ is a cover, then the *cover inequality*

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a valid inequality for S

Using Valid Inequalities for a Relaxation

- I want to solve MIPs, why do I care about strong inequalities for the knapsack problem?
- If $P = \{x \in \mathbb{B}^n \mid Ax \leq b\}$, then for any row i, $P_i = \{x \in \mathbb{B}^n \mid a_i^T x \leq b_i\}$ is a relaxation of P.
  - $P \subseteq P_i \; \forall i = 1, 2, \ldots m$
  - $P \subseteq \bigcap_{i=1}^m P_i$
- Any inequality valid for a relaxation of an IP is valid for the IP itself.
- Generating valid inequalities for a relaxation is often easier.
- If the intersection of the relaxations is a good approximation to the true problem, then the inequalities will be quite useful.
- Crowder, Johnson, and Padberg is the seminal paper that shows this to be true.

Example

$$\text{MYKNAP} = \{x \in \mathbb{B}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$$

- Some minimal covers are the following:

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 2 \\ x_1 + x_2 + x_6 &\leq 2 \\ x_1 + x_5 + x_6 &\leq 2 \\ x_3 + x_4 + x_5 + x_6 &\leq 3 \end{aligned}$$

Back to the Knapsack

- If $C \subseteq N$ is a cover, the *extended cover* $E(C)$ is defined as
  - ◇ $E(C) = C \cup \{j \in N \mid a_j \geq a_i \; \forall i \in C\}$
- If $E(C)$ is an extended cover for S, then the *extended cover inequality*
$$\sum_{j \in E(C)} x_j \leq |C| - 1,$$
is a valid inequality for S
- Note this inequality dominates the cover inequality if $E(C) \setminus C \neq \emptyset$
- **(Example, cont.)** The cover inequality $x_3 + x_4 + x_5 + x_6 \leq 3$ is dominated by the extended cover inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$

In General...

- Order the variables so that $a_1 \geq a_2 \ldots \geq a_n$
- Let C be a cover with $C = \{j_1, j_2, \ldots j_r\}$ $(j_1 < j_2 < \ldots < j_r)$ so that $a_{j_1} \geq a_{j_2} \geq \ldots \geq a_{j_r}$. Let $p = \min\{j \mid j \in N \setminus E(C)\}$.
- If any of the following conditions hold, then

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

gives a facet of conv(KNAP)

  - ◇ $C = N$
  - ◇ $E(C) = N$ and (*) $\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b$
  - ◇ $C = E(C)$ and (**) $\sum_{j \in C \setminus j_1} a_j + a_p \leq b$
  - ◇ $C \subset E(C) \subset N$ and (*) and (**).

Examples

- $C = \{1, 2, 6\}$. $E(C) = C$.
  - ◇ If $a_2 + a_6 + a_3 \leq b$, then $x_1 + x_2 + x_6 \leq 2$ is a facet of conv(MYKNAP)
  - ◇ $16 \leq 19$. It is a facet!
- $C = \{3, 4, 5, 6\}$. $E(C) = \{1, 2, 3, 4, 5, 6\}$. $C \subset E(C) \subset N$. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$ is a facet of conv(MYKNAP) if...
  - ◇ $a_4 + a_5 + a_6 + a_7 \leq b$? **(Yes!)**
  - ◇ $a_5 + a_6 + a_1 \leq b$ **(No!)**,
- So $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$ is **not** facet-defining for conv(MYKNAP)

$\mathbf{conv}(\text{MYKNAP})$

$$\begin{aligned}
x_j &\geq 0 & \forall j = 1, 2, \ldots, 7 \\
x_j &\leq 1 & \forall j = 1, 2, \ldots, 7 \\
x_1 + x_5 + x_6 &\leq 2 \\
x_1 + x_4 + x_6 &\leq 2 \\
x_1 + x_4 + x_5 &\leq 2 \\
x_1 + x_3 + x_6 &\leq 2 \\
x_1 + x_3 + x_5 &\leq 2 \\
x_1 + x_3 + x_4 &\leq 2 \\
x_1 + x_2 + x_6 &\leq 2 \\
x_1 + x_2 + x_5 &\leq 2 \\
x_1 + x_2 + x_4 &\leq 2 \\
x_1 + x_2 + x_3 &\leq 2 \\
2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &\leq 3
\end{aligned}$$

Covers and Lifting

- Let $P_{1,2,7} = \text{MYKNAP} \cap \{x \in \Re^7 \mid x_1 = x_2 = x_7 = 0\}$
- Consider the cover inequality arising from $C = \{3, 4, 5, 6\}$.
- $\sum_{j \in C} x_j \leq 3$ is facet defining for $P_{1,2,7}$
- If x_1 is not fixed at 0, can we strengthen the inequality?
- For what values of α_1 is the inequality

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

valid for

$$P_{2,7} = \{x \in \text{MYKNAP} \mid x_2 = x_7 = 0\}?$$

  - ◇ If $x_1 = 0$ then the inequality is valid for all values of α_1

The Other Case

- If $x_1 = 1$, the inequality is valid if and only if

$$\alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

is valid for all $x \in \mathbb{B}^4$ satisfying

$$6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11$$

- Equivalently, if and only if

$$\alpha_1 + \max_{x \in \mathbb{B}^4} \{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\} \leq 3$$

- Equivalently if and only if $\alpha_1 \leq 3 - \gamma$, where

$$\gamma = \max_{x \in \mathbb{B}^4} \{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}.$$

Solving the Knapsack Problem

- In this case, we can "solve" the knapsack problem to see that $\gamma = 1$. Therefore $\alpha_1 \leq 2$.
- The inequality

$$2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

  is a valid inequality for P_{27}

  - ◇ Is it facet-defining?

An "Uplifting" Experience

- What we've done is called lifting. Lifting is a process in which a valid (and facet defining) inequality for $S \cap \{x \in \mathbb{B}^n \mid x_k = 0\}$ is turned into a facet defining inequality for S.

- **Theorem.** Let $S \subseteq \mathbb{B}^n$, for $\delta \in \{0, 1\}, S^\delta = S \cap \{x \in \mathbb{B}^n \mid x_1 = \delta\}$. Suppose

$$\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

is valid for S^0.

Lifting Thm. (2)

- If $S^1 = \emptyset$, then $x_1 \leq 0$ is valid for S
- If $S^1 \neq \emptyset$, then

$$\alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

is valid for S for any $\alpha_1 \leq \pi_0 - \gamma$, where

$$\gamma - \max\{\sum_{j=2}^{n} \pi_j x_j \mid x \in S^1\}.$$

Lifting Thm. (3)

- If $\alpha_1 = \pi_0 - \gamma$ and $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a face of dimension k of $\text{conv}(S^0)$, then

$$\alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

defines a face of dimension *at least* $k + 1$ of $\text{conv}(S)$.

You Can Also "DownLift"

- Let $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ be valid for S^1.
- If $S^0 = \emptyset$, $x_1 \geq 1$ is valid for S, otherwise

$$\xi_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0 + \xi_1$$

  is valid for S, for $\xi_i \geq \gamma - \pi_0$
  - ◇ $\gamma = \max\{\sum_{j=2}^{n} \pi_j x_j \mid x \in S^0\}$.
- Similar facet/dimension results to uplifting if the lifting is maximum.

Group Exercise

- Group exercise
- Find facets of the polyhedron:

$$35x_1+27x_2+23x_3+19x_4+15x_5+15x_6+12x_7+8x_8+6x_9+3x_{10} \leq 39$$

Quiz Problem

```
( 12) +  x1                            + x9      <=  1
( 13) +  x1                        + x8          <=  1
( 14) +  x1                    + x7              <=  1
( 15) +  x1+ x2            + x6                  <=  1
( 16) +  x1+ x2        + x5                      <=  1
( 17) +  x1+ x2+ x3+ x4                          <=  1
( 18) + 2x1+ x2                    + x8+ x9      <=  2
( 19) +  x1+ x2                + x7        +x10  <=  2
( 20) + 2x1+ x2+ x3            + x7    + x9      <=  2
( 21) + 2x1+ x2+ x3            + x7+ x8          <=  2
( 22) + 2x1+2x2+ x3+ x4+ x5+ x6                  <=  2
( 23) +  x1+ x2+ x3        + x6            +x10  <=  2
( 24) +  x1+ x2+ x3    + x5                +x10  <=  2
( 25) + 2x1+ x2+ x3+ x4    + x6        + x9      <=  2
( 26) + 2x1+ x2+ x3+ x4    + x6    + x8          <=  2
( 27) + 2x1+ x2+ x3+ x4+ x5            + x9      <=  2
( 28) + 2x1+ x2+ x3+ x4+ x5        + x8          <=  2
( 29) + 2x1+ x2+ x3+ x4+ x5+ x6+ x7              <=  2
( 30) + 3x1+2x2+2x3+2x4+ x5+ x6        + x9      <=  3
( 31) + 3x1+2x2+2x3+2x4+ x5+ x6    + x8          <=  3
( 32) + 2x1+2x2+2x3+ x4+ x5+ x6            +x10  <=  3
( 33) + 2x1+ x2+ x3                + x8+ x9+x10  <=  3
( 34) + 3x1+2x2+2x3+ x4+ x5+ x6+ x7    + x9      <=  3
( 35) + 3x1+2x2+2x3+ x4+ x5+ x6+ x7+ x8          <=  3
( 36) + 2x1+ x2+ x3+ x4        + x7    + x9+x10  <=  3
( 37) + 2x1+ x2+ x3+ x4        + x7+ x8    +x10  <=  3
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( 38) + 2x1+2x2+ x3+ x4+ x5+ x6+ x7        +x10 <=  3
( 39) + 2x1+ x2+ x3+ x4+ x5+ x6    + x8    +x10 <=  3
( 40) + 3x1+2x2+ x3+ x4+ x5+ x6+ x7+ x8+ x9     <=  3
( 41) + 3x1+2x2+ x3+ x4        +2x7    + x9+x10 <=  4
( 42) + 3x1+2x2+ x3+ x4        +2x7+ x8    +x10 <=  4
( 43) + 3x1+3x2+2x3+ x4+ x5+2x6+ x7        +x10 <=  4
( 44) + 3x1+3x2+2x3+ x4+2x5+ x6+ x7        +x10 <=  4
( 45) + 3x1+2x2+2x3+ x4+ x5+2x6    + x8    +x10 <=  4
( 46) + 3x1+2x2+2x3+ x4+2x5+ x6    + x8    +x10 <=  4
( 47) + 4x1+3x2+2x3+2x4+ x5+2x6+ x7+ x8+ x9     <=  4
( 48) + 4x1+3x2+2x3+2x4+2x5+ x6+ x7+ x8+ x9     <=  4
( 49) + 4x1+2x2+2x3+ x4+ x5+ x6+2x7+ x8+ x9     <=  4
( 50) + 3x1+2x2+2x3+2x4+ x5+ x6+ x7    + x9+x10 <=  4
( 51) + 3x1+2x2+2x3+2x4+ x5+ x6+ x7+ x8    +x10 <=  4
( 52) + 3x1+2x2+2x3+ x4+ x5+ x6+ x7+ x8+ x9+x10 <=  4
( 53) + 4x1+4x2+3x3+2x4+2x5+2x6+ x7        +x10 <=  5
( 54) + 5x1+3x2+3x3+2x4+2x5+2x6+2x7+ x8+ x9     <=  5
( 55) + 5x1+4x2+3x3+3x4+2x5+2x6+ x7+ x8+ x9     <=  5
( 56) + 4x1+3x2+3x3+2x4+2x5+2x6+ x7+ x8    +x10 <=  5
( 57) + 4x1+3x2+3x3+2x4+ x5+2x6+ x7+ x8+ x9+x10 <=  5
( 58) + 4x1+3x2+3x3+2x4+2x5+ x6+ x7+ x8+ x9+x10 <=  5
( 59) + 4x1+3x2+2x3+2x4+ x5+ x6+2x7+ x8+ x9+x10 <=  5
( 60) + 5x1+3x2+3x3+3x4+2x5+2x6+ x7+2x8    +x10 <=  6
( 61) + 5x1+4x2+3x3+3x4+2x5+2x6+2x7+ x8+ x9+x10 <=  6
( 62) + 5x1+3x2+3x3+2x4+2x5+2x6+ x7+2x8+ x9+x10 <=  6
( 63) + 5x1+4x2+4x3+3x4+2x5+2x6+ x7+ x8+ x9+x10 <=  6
( 64) + 5x1+3x2+3x3+2x4+ x5+ x6+2x7+ x8+2x9+x10 <=  6
( 65) + 5x1+3x2+3x3+2x4+ x5+ x6+2x7+2x8+ x9+x10 <=  6
( 66) + 6x1+5x2+4x3+3x4+3x5+3x6+2x7+ x8    +x10 <=  7
( 67) + 6x1+4x2+4x3+3x4+2x5+2x6+2x7+ x8+2x9+x10 <=  7
( 68) + 6x1+4x2+4x3+3x4+2x5+3x6+ x7+2x8+ x9+x10 <=  7
( 69) + 6x1+4x2+4x3+3x4+3x5+2x6+ x7+2x8+ x9+x10 <=  7
( 70) + 7x1+5x2+4x3+3x4+2x5+2x6+3x7+2x8+2x9+x10 <=  8
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( 71) + 7x1+5x2+5x3+4x4+3x5+3x6+2x7+2x8+ x9+x10 <=  8
( 72) + 7x1+5x2+5x3+4x4+2x5+3x6+2x7+ x8+2x9+x10 <=  8
( 73) + 7x1+5x2+5x3+4x4+3x5+2x6+2x7+ x8+2x9+x10 <=  8
( 74) + 8x1+6x2+5x3+4x4+3x5+3x6+3x7+2x8+2x9+x10 <=  9
( 75) + 8x1+6x2+6x3+5x4+3x5+3x6+2x7+ x8+2x9+x10 <=  9
( 76) + 9x1+7x2+6x3+5x4+3x5+4x6+3x7+2x8+2x9+x10 <= 10
( 77) + 9x1+7x2+6x3+5x4+4x5+3x6+3x7+2x8+2x9+x10 <= 10
( 78) + 9x1+7x2+6x3+5x4+4x5+4x6+3x7+2x8+ x9+x10 <= 10
( 79) +10x1+8x2+7x3+6x4+4x5+4x6+3x7+2x8+2x9+x10 <= 11
( 80) +12x1+9x2+8x3+6x4+5x5+5x6+4x7+3x8+2x9+x10 <= 13
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Knapsack Separation

- So there are *lots* of inequalities. How do I find one that might be useful?
- First note that $\sum_{j \in C} x_j \leq |C| - 1$ can be rewritten as

$$\sum_{j \in C} (1 - x_j) \geq 1.$$

- Separation Problem: Given a "fractional" LP solution $\hat{x}$, does $\exists\ C \subseteq N$ such that $\sum_{j \in C} a_j > b$ and $\sum_{j \in C} (1 - \hat{x}_j) < 1$?
- Is $\gamma = \min_{C \subseteq N} \{\sum_{j \in C} (1 - \hat{x}_j \mid \sum_{j \in C} a_j > b\} < 1$
- Let $z_j \in \{0, 1\}$, $z_j = 1$ if $j \in C$, $z_j = 0$ if $j \notin C$.
- Is $\gamma = \min\{\sum_{j \in N} (1 - \hat{x}_j) z_j \mid \sum_{j \in N} a_j z_j > b, z \in \mathbb{B}^n\} < 1$?
- If $\gamma \geq 1, \hat{x}$ satisfies all cover inequalities

- If $\gamma < 1$ with optimal solution z_R, then $\sum_{j \in R} x_j \leq |R| - 1$ is a violated cover inequality.

Example

$\textsc{myknap} = \{x \in \mathbb{B}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$

- $\hat{x} = (0, 2/3, 0, 1, 1, 1, 1)$

$$\gamma = \min_{z \in \mathbb{B}^7} \{z_1 + 1/3 z_2 + z_3 \mid 11z_1 + 6z_2 + 6z_3 + 5z_4 + 5z_5 + 4z_6 + z_7 \geq 20\}.$$

- $\gamma = 1/3$
- $z = (0, 1, 0, 1, 1, 1, 1)$
- $x_2 + x_4 + x_5 + x_6 + x_7 \leq 4$
- Minimal Cover: $x_2 + x_4 + x_5 + x_6 \leq 3$
- Extended Cover: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$
- To get the facet, you would have to start lifting from the

minimal cover, with x_1, x_2, x_7 fixed at 0.

General Lifting and SuperAdditivity

- $K = \text{conv}(\{x \in \mathbb{Z}_+^{|N|}, y \in \Re_+^{|M|} \mid a^T x + g^T y \leq b, x \leq u\})$
- Partition N into $[L, U, R]$
  - ◇ $L = \{i \in N \mid x_i = 0\}$
  - ◇ $U = \{i \in N \mid x_i = u_i\}$
  - ◇ $R = N \setminus L \setminus U$
- We will use the notation: x_R to mean the vector of variables that are in the set R.
  - ◇ $a_R^T x_R = \sum_{j \in R} a_j x_j$
- $K(L, U) = \text{conv}(\{x \in \mathbb{Z}_+^{|N|}, y \in \Re_+^{|M|} \mid a_R^T x + g^T y \leq d, x_R \leq u_R, x_i = 0 \ \forall i \in L, x_i = u_i \ \forall i \in U.\})$
  - ◇ So $d = b - a_U^T x_U$

Lifting

- Let $\pi^T x_R - \sigma^T y \leq \pi_0$ be a valid inequality for $K(L, U)$.
- Consider the *lifting function* $\Phi : \Re \rightarrow \Re \cup \{\infty\}$
  - ◇ (∞) if lifting problem is infeasible

$$\Phi(\alpha) = \pi_0 - \max\{\pi_R^T x_R + \sigma^T y \mid a_R^T x_R + g^T y \leq d - \alpha, x_R \leq u_R, x_R \in \mathbb{Z}_+^{|R|}, y \in \Re_+^{|M|}\}$$

- In words, $\Phi(\alpha)$ is the maximum value of the LHS of the valid inequality if the RHS in K is reduced by α.

Φ, Schmi

- Why do we care about Φ?

$$\pi_R^T x_R + \pi_L^T x_L + \pi_U^T (u_U - x_U) + \sigma^T y \le \pi_0$$

is a valid inequality for K if and only if

$$\pi_L^T x_L + \pi_U^T (u_U - x_U) \le \Phi(a_L^T x_L + a_U^T (x_U - u_U)) \ \forall (x, y) \in K.$$

Proof.?

Example—Sequential Lifting

- Lifting one variable (at a time) in 0-1 IP (like we have done so far)...
- $\alpha x_k + \pi_R^T x_R \leq \pi_0$ is valid for $P \Leftrightarrow \alpha x_k \leq \Phi(a_k x_k)\ \forall x \in P$
  - ◇ $x_k = 0, \quad 0 \leq \Phi(0)$ is always true.
  - ◇ $x_k = 1, \quad \Rightarrow \alpha \leq \Phi(a_l)$
- If I "know" $\Phi(q)(\forall q \in \Re)$, I can just "lookup" the value of the lifting coefficient for variable x_k

⋆ Note that if I have restricted more than one variable, then this "lookup" logic is not necessarily true

  - ◇ For lifting two (0-1) variables, I would have to look at four possible values.
  - ◇ In general, the lifting function changes with each new variable "lifted".

Superadditivity

- A function $\phi : \Re \rightarrow \Re$ is *superadditive* if

$$\phi(q_1) + \phi(q_2) \leq \phi(q_1 + q_2)$$

- Superadditive functions play a significant role in the theory of integer programming. (See N&W page 229). (We'll probably revisit them later).

- Superadditive Fact:

$$\sum_{j \in N} \phi(a_j) x_j \leq \sum_{j \in N} \phi(a_j x_j) \leq \phi\left(\sum_{j \in N} a_j x_j\right).$$

"Multiple Lookup"—Superadditivity

- Suppose that ϕ is a superadditive lower bound on Φ that satisfies $\pi_i = \phi(a_i)\ \forall i \in L$ and $\pi_i = \phi(-a_i)\ \forall i \in U$

$$\begin{aligned}
\sum_{i \in L} \phi(a_i) x_i + \sum_{i \in U} \phi(-a_i)(u_i - x_i) &\leq \phi(a_L^T x_L + a_U^T (x_U - u_U)) \\
&\leq \Phi(a_L^T x_L + a_U^T (x_U - u_U))
\end{aligned}$$

- So

$$\pi_R^T x_R + \pi_L^T x_L + \pi_U^T (u_U - x_U) + \sigma^T y \leq \pi_0$$

is a valid inequality for K

The Main Result

- If ϕ is a superadditive lower bound on Φ, any inequality of the form $\pi_R^T x_R - \sigma^T y \leq \pi_0$, which is valid for $K(L, U)$, can be extended to the inequality

$$\pi_R^T x_R + \sum_{j \in L} \phi(a_j) x_j + \sum_{j \in U} \phi(-a_j)(u_j - x_j) + \sigma^T y \leq \pi_0$$

  which is valid for K.

- If $\pi_i = \phi(a_i)\ \forall i \in L$ and $\pi_i = \phi(-a_i)\ \forall i \in U$ and $\pi^T x_R - \sigma^T y = \pi_0$ defines a k-dimensional face of $K(L, U)$, then the lifted inequality defines a face of dimension at least $k + |L| + |U|$.