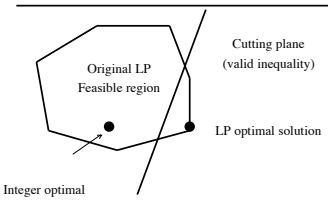
Reconnect '04
A Few Topics in Polyhedral Combinatorics

Cynthia Phillips, Sandia National Laboratories

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under contract DE-AC04-94AL85000.

Sandia National Laboratories

Strengthen Linear Program with Cutting Planes



- Make LP polytope closer to integer polytope
- Use families of constraints too large to explicitly list
  - Exponential, pseudopolynomial, polynomial (n^4, n^5)

Sandia National Laboratories

Separation

- Consider a family of cutting planes $a_i^T x \le b_i$
  - Abbreviate as (a_i, b_i)
- A separation algorithm takes this family and an x^* and in polynomial time either
  - Returns a member of the family (a_i, b_i) such that $a_i^T x^* > b_i$
    - x^* violates (a_i, b_i)
  - Says (truthfully) that x^* violates no member of the family
- If we iteratively add the cuts returned by the separation algorithm, in polynomial time, we will have an optimal LP solution that satisfies the whole family (Ellipsoid algorithm)

Sandia National Laboratories

Example: Traveling Salesman Problem

- Input: a set of n cities, $1 \ldots n$, distance d_{ij} between cities i and j
  - Can travel between any pair (automatic with the triangle inequality)
- Goal: Visit each city exactly once so as to minimize the total distance

Variable $x_{ij} = 1$ if edge (i,j) in the tour, 0 otherwise

$$\min \sum_{i<j} c_{ij} x_{ij}$$

$$\text{st} \quad \sum_i x_{ij} = 2 \quad \forall j$$

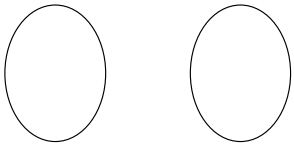
$$x_{ij} = x_{ji}$$

Undirected formulation
Enforce all nodes have degree 2 in the tour

Sandia National Laboratories

Subtours

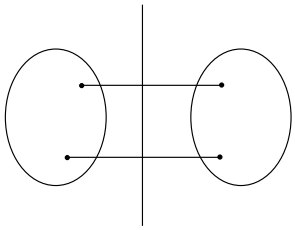
- Degree constraints aren't sufficient for an IP formulation because we have have disconnected cycles



Sandia National Laboratories

Eliminate Subtours

- Force 2 edges to cross every (nontrivial) cut



Sandia National Laboratories

Subtour Elimination Constraints

$$\sum_{i \in S,\ j \notin S,\ j > i} x_{ij} \geq 2 \quad \forall S \subseteq N,\ 2 \leq |S| \leq n-2$$

- If we give each edge e weight x_e^*, then the separation algorithm is looking for a cut of capacity less than 2
- Just run a standard minimum cut code

Sandia National Laboratories

The Power of Separation

For 300 cities, there are over 10^{90} subtour elimination constraints!
But we can enforce them all for instances with thousands of cities.

Adding classes of cutting planes can provably reduce the integrality gap (ratio between best IP solution and best LP solution)

Sandia National Laboratories

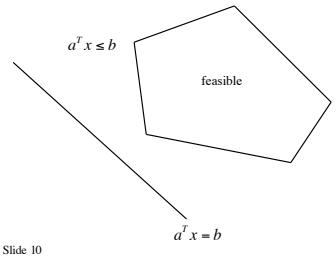
Compact Formulations

- When the separation algorithm is itself an LP can sometimes represent the entire separation process as a single LP (with polynomially more constraints)

Sandia National Laboratories

Valid Inequality

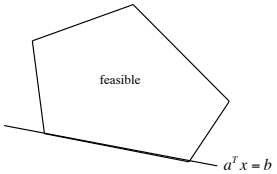
An inequality is valid for a polytope if it contains the whole polytope



Sandia National Laboratories

Facet

Let $a^T x \le b$ be a valid inequality for polyhedron P
Then $F = \{x \in P \mid a^T x = b\}$ is a face of the polyhedron
If $F \neq \varnothing$, then F **supports** P
If F is exactly one dimension smaller than P, then it is a **facet**
Families of facet-defining inequalities are optimal in a sense

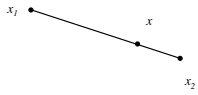


Sandia National Laboratories

Convex Combinations

- A point x is a convex combination of two others x_1 and x_2 if (componentwise)

$$x = \lambda_1 x_1 + \lambda_2 x_2, \text{ with}$$
$$\lambda_1 + \lambda_2 = 1$$
$$\lambda_1, \lambda_2 \ge 0$$



Sandia National Laboratories

Extreme Points

Another definition of an extreme point (corner of a polyhedron):
$x \in P$ is an extreme point if and only if there are no $x_1, x_2 \in P$ such that x is a convex combination of x_1 and x_2

Sandia National Laboratories

Convex Decomposition

x = feasible solution to the LP relaxation

Find feasible *integer* solutions

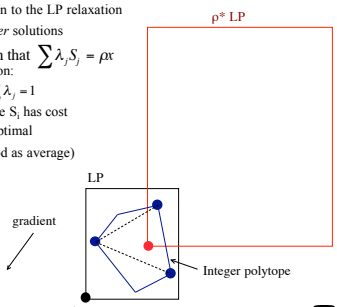
$S_0, S_1, \ldots, S_m$ such that $\sum \lambda_j S_j = \rho x$

Convex combination:

$0 \le \lambda_j \le 1, \sum \lambda_j = 1$

- Implies one of the S_i has cost at most ρ * LP optimal

(something's a good as average)



LP opt ≠ 0

Sandia National Laboratories

Decomposition Precisely Defines Integrality Gap

IP has a solution within ρ times the LP bound if and only if ρx^* can be decomposed into a convex combination of feasible solutions.

Definition: A ρ-approximation algorithm for a minimization problem guarantees a solution no more than ρ times the optimal solution for all instances.

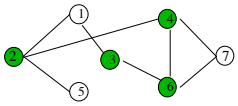
Sandia National Laboratories

LP-Relaxation-Based Approximation for IP

- Compute LP relaxation (lower bound).
- Common technique:
  - Use structural information from LP solution to find feasible IP solution (use parallelism if possible)
  - Bound quality using LP bound
- Integrality gap = $\max_I$(IP (I))/(LP(I))
- This technique cannot prove anything better than integrality gap

Sandia National Laboratories

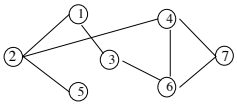
Example: Vertex Cover



Find a minimum-size set of vertices such that each edge has at least one endpoint in the set.

Sandia National Laboratories

Example: Vertex Cover



$$v_i = \begin{cases} 1 \text{ if vertex i is in the VC} \\ 0 \text{ otherwise} \end{cases}$$

$$\min \sum v_i$$

$$\text{s.t. } v_i + v_j \geq 1 \quad \forall (i, j) \in E$$

$$v_i \in \{0, 1\}$$

Sandia National Laboratories

2-Approximation algorithm for Vertex Cover

- Solve the LP relaxation for vertex cover:

$$\min \sum v_i$$

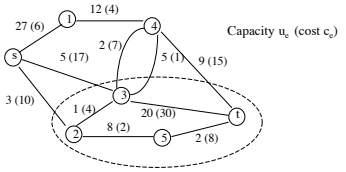
$$\text{s.t. } v_i + v_j \geq 1 \quad \forall (i, j) \in E$$

$$0 \leq v_i \leq 1$$

- Select all vertices i such that $v_i \geq 1/2$.
- This covers all edges: at least one endpoint will have value at least 1/2.
- Each such vertex contributed as least 1/2 to the optimal LP solution, so rounding to 1 at most doubles cost.

Sandia National Laboratories

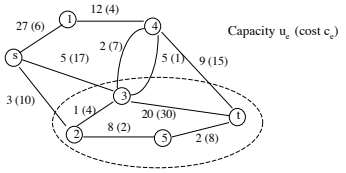
Capacitated Network Design



- Each pair (v_i, v_j) has a demand (required connectivity) d_{ij}
  - Min cut separating v_i and v_j is at least d_{ij}
- Choose min-cost subgraph s.t. all pairwise demands satisified
- All/none decision for each edge.

Sandia National Laboratories

Network Reinforcement - Communication Network



message packets take "all" paths, must capture all packets to compromise (Franklin)

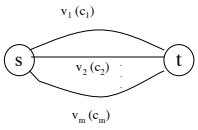
- Capacity = attacker cost to compromise edge
- Min cut = attacker cost to eavesdrop

Pay to protect all communication at desired level.

Sandia National Laboratories

Special Case - Minimum Knapsack Problem

Given: Set of objects: Object i has cost c_i, value v_i
Required value V
Find: minimum-cost set of objects with total value at least V



Sandia National Laboratories

Generalization - Capacitated Covering

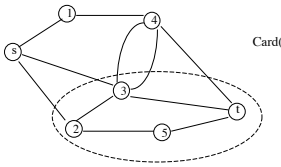
$$\min c^T x$$
$$st.\ Ux \geq d$$
$$0 \leq x \leq b$$
$$x \in Z^+$$

- All entries of c,U,d are nonnegative.

Sandia National Laboratories

Definition: Bond

A bond is a minimal set of edges whose removal disconnects a pair with positive demand. Count multiedges as 1.



Max bond of graph G, β(G) is max cardinality of any bond in G

Sandia National Laboratories

Integer Program (IP) for capacitated network design

Solving a linear objective function subject to linear constraints. Variables must take integer values.

- Models any NP-complete problem

A simple IP for capacitated network design:

$$\min \sum_{e \in C} c_e x_e$$

$$\sum_{e \in C} u_e x_e \geq d(C) \quad \forall \text{ cutset } C$$

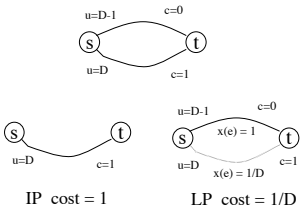
$$x_e \in \{0,1\}$$

Where d(C) is the maximum demand d_i for any pair that crosses cut C

$x_e = 1$ if edge e is selected

Sandia National Laboratories

Simple Network Reinforcement IP has Bad Integrality Gap

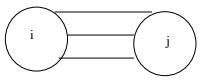


IP cost = 1 LP cost = 1/D

Ratio OPT(IP)/OPT(LP) = D

Sandia National Laboratories

Effective Capacities



Can assume

$$u_e \leq \max_{e \in C} D(C)$$

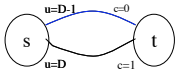
C is a cut,

$$D(C) = \max_{i,j \text{ split by } C} d_{ij}$$

Sandia National Laboratories

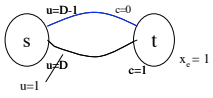
Inhibiting One Form of Cheating

Demand D



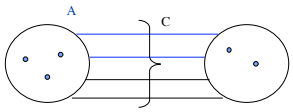
- New problem with remaining edges and residual Demand D - (D-1) = 1

Residual Demand 1



Sandia National Laboratories

Knapsack Cover (KC) Inequalities



$$u(A) = \sum_{e \in A} u_A < D(C)$$

residual $D(A) = D - u(A)$

$u_A(e) = \min(u_e, D(A))$

$$\text{KC}: \sum_{e \in C-A} u_A(e) x_e \geq D(A)$$

Sandia National Laboratories

Knapsack Cover (KC) Cuts for General Graphs

$$\min \sum_{e \in E} c_e x_e$$

$$\text{st} \sum_{e \in E-A} u_A(e) x_e \geq D(A) \text{ for } A \subset E, u(A) < D$$

$$x_e \in \{0,1\}$$

Sandia National Laboratories

New Integrality Gaps

- 2 for Knapsack
- β (G) + 1 for general graphs

Proof: Find feasible integer solution with cost 2 (or β (G) + 1) times LP optimal

Sandia National Laboratories