Problems to Explore

Rutgers Young Scholars Program

Summer 2020

PROBLEM 1: TEMPUS FUGIT

In a day of 24 hours, 5/7 of the time that has already passed in the day is left in the day. How much time has passed in the day?

PROBLEM 2: RUNNING LAPS

Two runners run at constant speed around an oval track. The first runner goes around the track every 60 seconds. The second runner, going in the opposite direction, meets the first runner every 35 seconds. How long does it take the second runner to go around the track?

PROBLEM 3: THE AGES OF WOMAN

The ages of a woman, her mother, and her daughter are all two digit integers. Her mother's age is five times her daughter's age. The woman's age can be found from her daughter's age by reversing the digits. When her daughter was born the woman was in her thirties.

What are the ages of the woman, her mother, and her daughter?

PROBLEM 4: A VISIT TO THE SODA SHOP

A group consisting of several students visited a soda shop. Each student had a soda that cost the same amount. The total bill was \$6.23.

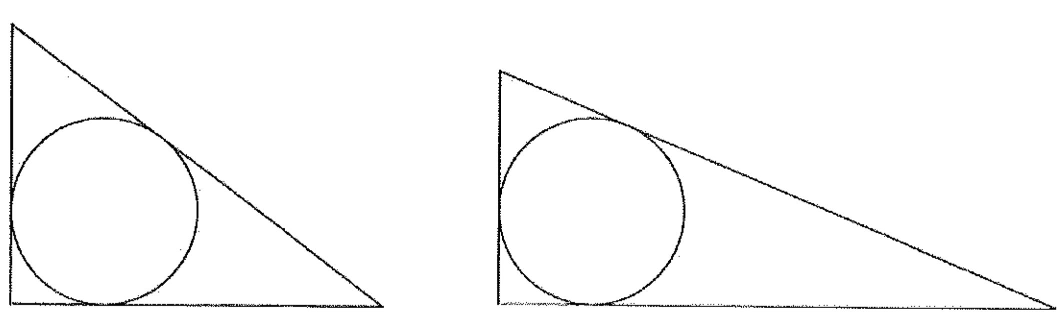
The students were somewhat surprised, since the cost had slightly increased. During the same group of students' previous visit, the total cost was 70¢ less.

(a) How many students were there?

(b) How much did each soda cost?

PROBLEM 5: A CIRCLE INSCRIBED IN A RIGHT TRIANGLE

A circle is inscribed in a right triangle with legs of lengths x and y and hypotenuse of length z, so that the circle is tangent to each of the legs and the hypotenuse. Two examples are shown.



What is the radius of the circle in terms of x, y, and z?

PROBLEM 6: SURFACE AREA OF A BOX

A rectangular box has edges of integer length. Find the dimensions of all boxes with total surface area of 288 square units.

PROBLEM 7: ATTACKING QUEENS

Recall that in chess a queen attacks any square that is on a straight line — horizontally, vertically, or diagonally — from the square on which the queen stands. Regard the queen as also attacking the square which she occupies. What is the smallest number of queens required to attack all of the squares of a 6×6 chessboard?

PROBLEM 8: BLOW THE PLAY

In the addition at the right, each letter stands for one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Assume that different letters correspond to different digits. In addition, to avoid confusion, assume that the letter O corresponds to the digit 0. Find all additions having this pattern.

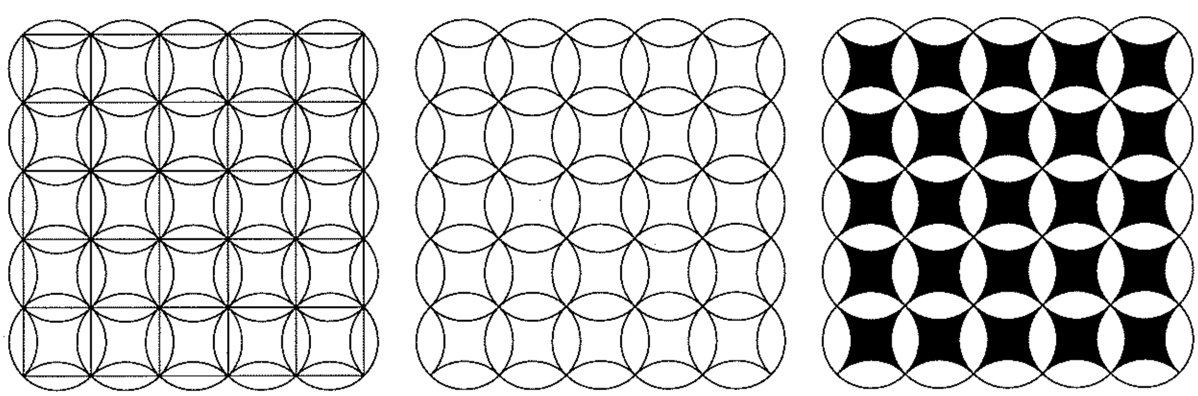
$$\begin{array}{r} \mathrm{D\,R\,O\,P} \\ \mathrm{T\,H\,E} \\ \hline \mathrm{B\,A\,L\,L} \end{array}$$

Once a solution is obtained, an additional solution can be obtained by interchanging P and E. Thus, assume that P < E. Likewise, suppose that R < T.

PROBLEM 9: SHADED AREAS

A grid in the plane has successive horizontal and vertical lines, respectively, at distance 1 from each other. Each 1×1 grid square is inscribed in a circle. This is shown in the left hand picture below. Then the grid is removed as in the central picture. Lastly, the area within each circle but outside each of its four neighbors is shaded as in the picture at the right.

If the pattern is extended to the entire plane, what fraction of the plane is shaded? In other words, what fraction of one of the original 1×1 squares is shaded? Give an exact answer.



PROBLEM 10: ISOSCELES TRIANGLE FROM ITS PERIMETER AND AREA

An isosceles triangle has perimeter 256 units and area 2640 square units. Find the lengths of the sides of the triangle.

Include all answers, not just those in which the lengths are integers.

PROBLEM 11: KNIGHTS ERRANT, VISITING ALL POSITIONS EXACTLY ONCE

Recall the knight's move in chess: either two squares vertically and one horizontally or two squares horizontally and one square vertically.

The diagrams below show ways in which a knight can start at the lower left hand corner of a 5×6 chessboard and visit every square exactly once.

					C
	D				
		E			
A					B

In all cases, the knight starts at A. In the first case, the knight ends at B, and, in the second case, the knight finishes at C. In the third case, the knight finishes at D. In the next move, the knight can then return to the starting point at A. The same would also be true if the knight had ended at E instead. The knight visits the positions in order $1, \ldots, 30$.

21	26	19	10	5	14
24	9	22	13	18	11
27	20	25	6	15	4
8	23	2	29	12	17
1	28	7	16	3	30

15	24	19	10	5	30
18	9	16	13	20	11
25	14	23	6	29	4
8	17	2	27	12	21
1	26	7	22	3	28

29	20	25	10	5	14
26	9	28	13	24	11
19	30	21	6	15	4
8	27	2	17	12	23
1	18	7	22	3	16

Then, the question conerns whether the knight can do the same things (a) on the 5×5 and (b) on the 6×6 board.

(*i*) Start at A and end at B.
(*ii*) Start at A and end at C.
(*iii*) Start at A and end at D or E.

(a)

				C
	D			
		E		
A				B

(b)

					C
	D				
		E			
A					B

PROBLEM 12: GEOMETRIC ADDITION

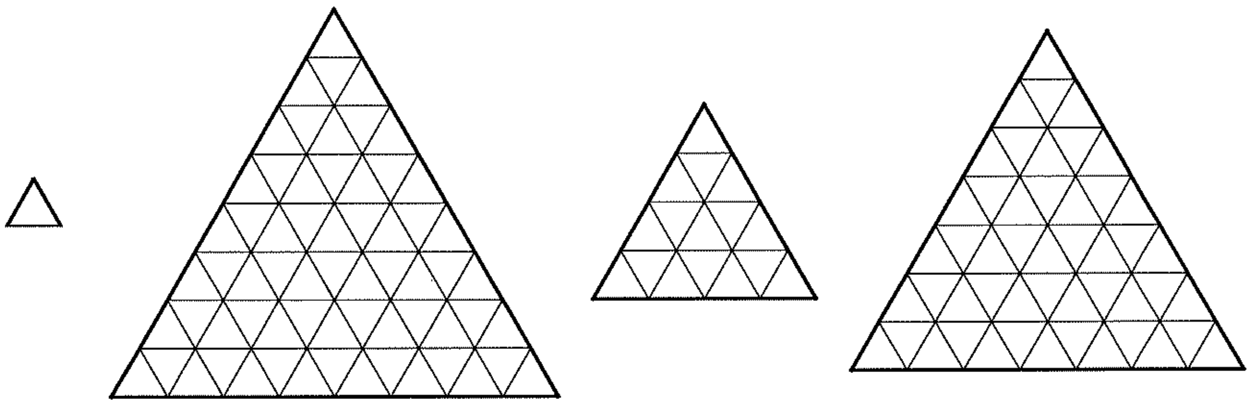
Shown below are equilateral triangles with sides of length 1, 8, 4, 7, decomposed respectively into $1^2, 8^2, 4^2, 7^2$ equilateral triangles with sides of length 1.

Verify geometrically that $1^2 + 8^2 = 4^2 + 7^2$ by cutting the equilateral triangle with sides of length 8 into four pieces that together with the triangle with sides of length 1 can be rearranged to give the equilateral triangles with sides of length 4 and 7.

Do this two different ways:

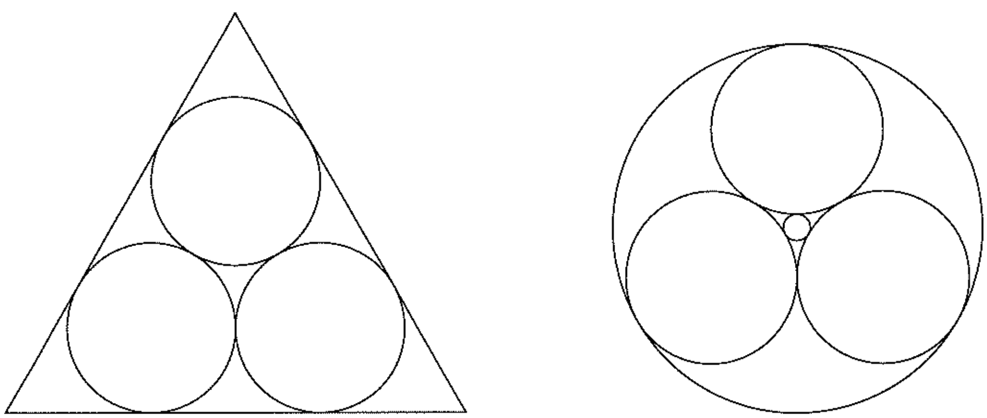
(a) In the first dissection one of the pieces into which the equilateral triangle with sides of length 8 is divided is an equilateral triangle with sides of length 7;

(b) In the second dissection one of the pieces into which the equilateral triangle with sides of length 8 is divided is an equilateral triangle with sides of length 4.



PROBLEM 13: THREE CIRCLES

Three circles of radius 1 are inside of an equilateral triangle, tangent to the triangle at the two point of intersection and tangent to each other.



(a) What is the length of a side of the triangle?

(b) A circle is placed outside the three circles so that it is tangent to each. What is its radius?

(c) Likewise, a circle placed in the hole between the three circles is tangent to each. What is its radius?

Give exact answers, not numerical approximations.

PROBLEM 14: SETS OF CONSECUTIVE PERFECT SQUARES SPLIT INTO TWO SETS WITH EQUAL SUMS

As usual, a perfect square means the square of a positive integer.

As the following examples show, sometimes a set of consecutive perfect squares can be split into two subsets with equal sums, to wit, $3^2 + 4^2 = 5^2$ and $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.

(a) If k is a positive integer, show that there is a set of $2k+1$ consecutive perfect squares so that the sum of the $k+1$ smaller squares equals the sum of the k larger squares.

Give a formula for the largest square in terms of the integer k.

In doing this problem, it may be useful to recall the formula for the sum of the first k positive integers, $1 + 2 + 3 + \cdots + k = k(k+1)/2$.

(b) Give another example of a set of five consecutive squares so that the sum of three of the squares equals the sum of the other two squares.

(c) Show, on the other hand, that there is no set of four consecutive squares that can be split into two disjoint sets with equal sums.

(d) Show that any set of eight consecutive integers can be split into two disjoint subsets of four members, so that the sums of the members of each set are equal and the sums of the squares of the members of each set are equal. (Hint: Try this first with {1,2,3,4,5,6,7,8}.)

PROBLEM 15: THREADING A MAZE

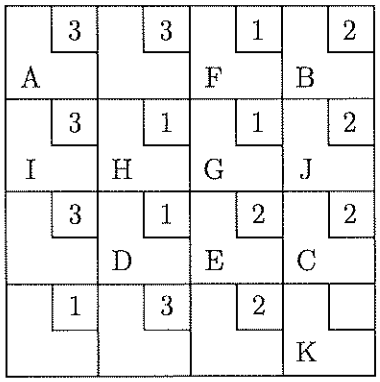
The 4×4 grid at the right represents a maze. The objective is start in the upper left hand corner with the goal of reaching the lower right hand corner. Note that in the corner of each grid square, except for the square that is the goal, there is a number. This number is the number of squares that can be jumped, horizontally or vertically, to get to the next grid square.

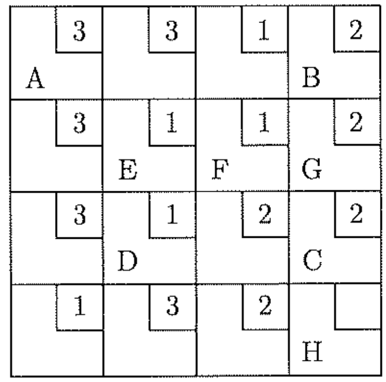
3	3	1	2
3	1	1	2
3	1	2	2
1	3	2	

For example, starting at A, one can jump 3 squares, either to B or C. Then, from B one can jump two squares, either to D or E, from C one square, to F or G.

3 A	3 D	1	2 B
3	1	1	2
3 F	1	2	2 E
1 C	3 G	2	

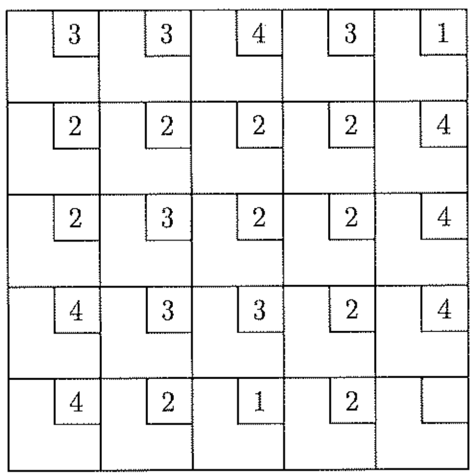
Next are shown two paths through the maze. The first requires ten jumps. The second requires only seven jumps and is the shortest possible path.





Shown below are similar 5 × 5 and 6 × 6 mazes. Find the shortest path through each. Some credit will be given for a longer path, but for full credit find the shortest.

(a)



(b)