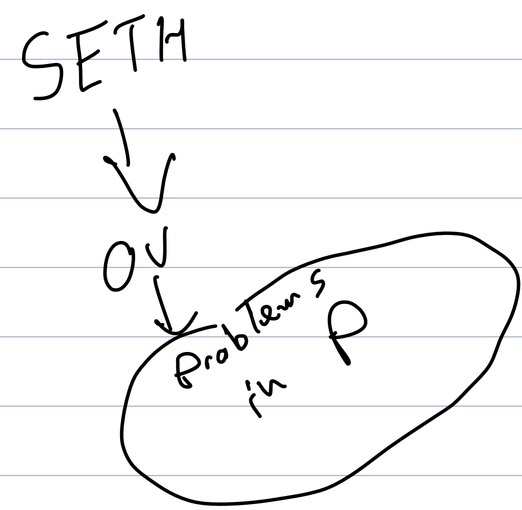
Last talk: SAT algs, ETH, SETH

this talk:
- ETH lower bounds
- The Sparsification Lemma

Next talk:



Recall: k-CNF formula is a conjunction (AND) of clauses, where each clause is a disjunction (OR) of up to k literals, where a literal is a variable or its negation.

ETH: $\exists\ \delta > 0$ s.t. 3-SAT is <u>not</u> in $O(2^{\delta n})$.

i.e. $2^{\Omega(n)}$ for 3-SAT.

SETH: $\forall\ \varepsilon > 0\ \exists\ k \geq 3$ s.t. k-SAT is not in $O(2^{(1-\varepsilon)\cdot n})$ time.

Dominating-Set: Given a graph $G=(V,E)$, and a parameter q, is there a set $S \subseteq V$ of size $|S|=q$ s.t. $\forall v \in V$ either $v \in S$ or there is a node $u \in S$ and $\{u,v\} \in E$.

Alg: $\binom{n}{q} \cdot qn$ exponential time for large q.

NP-Hardness proof:

3-SAT $\longrightarrow$ Dom. Set

formula φ graph G

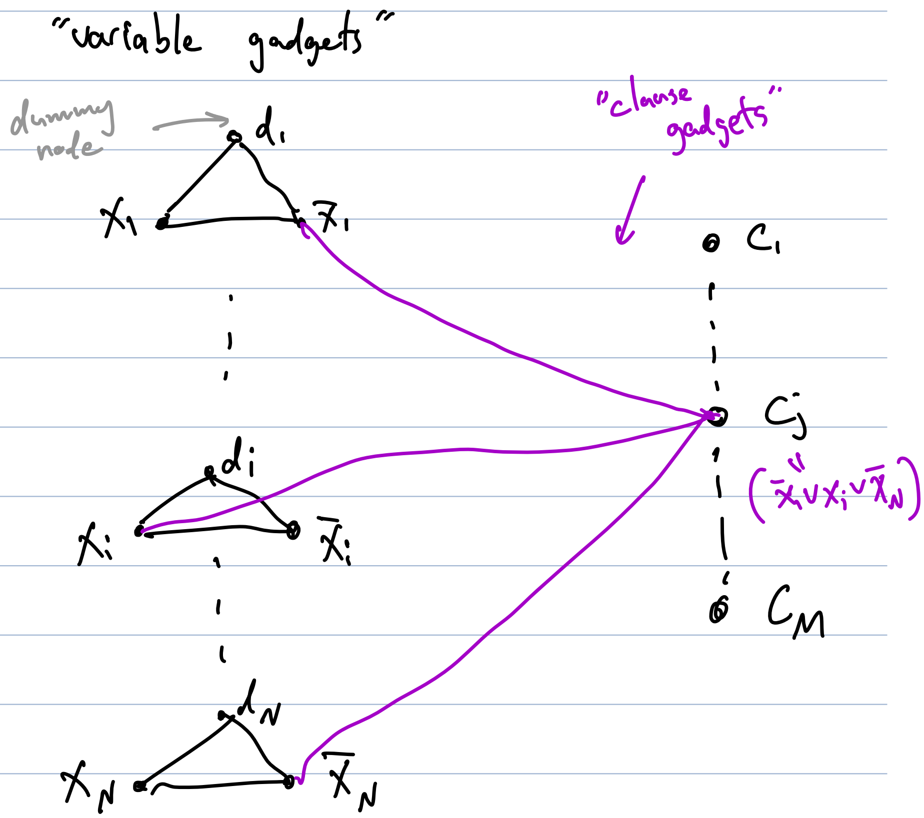
N vars $O(N+M)$ time $n = 3N+M$ nodes

M clauses $q = N$

$P=NP \Leftarrow N^{O(1)} \Leftarrow M^{O(1)} \Longleftarrow n^{O(1)}$

(**Claim:** φ is satisfiable iff G has a dom-set of size q.)

Construction:



Thm (weak): ETH $\Rightarrow$ $2^{\Omega(n^{1/3})}$ for Dom-Set.

pf: assume $\forall \delta > 0$ Dom-Set is in $O(2^{\delta n^{1/3}})$.
Let $\delta' > 0$. To solve 3-SAT in $O(2^{\delta' N})$ time, use the reduction:

$$n = 3N + M \leq 3N + \binom{N}{3} \cdot 2^3 \leq 11 N^3$$

and get an alg: $O(2^{\delta (11 N^3)^{1/3}}) = O(2^{\delta' \cdot N})$
for $\delta = \delta' / \sqrt[3]{11}$.

Want: $2^{\Omega(n)}$ for Dom-Set.
How? Can we assume that $M = O(N)$? Yes!

Now: Can we assume that $M = O(n)$? Yes:

The Sparsification Lemma (Impagliazzo - Paturi - Zane 2001):

$\forall k \geq 3$ and $\varepsilon > 0$ there is a constant $C = C(k, \varepsilon)$ and an alg s.t.

1. Given a k-CNF φ on N vars, alg computes $\varphi_1, \dots, \varphi_t$. (k-CNF formulas)
2. φ is satisfiable $\iff$ $\exists i$ s.t. φ_i is satisfiable.
3. $t \leq 2^{\varepsilon N}$ and alg takes $O(2^{\varepsilon N} \mathrm{poly}(N))$ time.
4. Each φ_i is on N vars and $M_i \leq C \cdot N$ clauses.

$$\varphi \longrightarrow \mathrm{OR}(\varphi_1, \dots, \varphi_t)$$

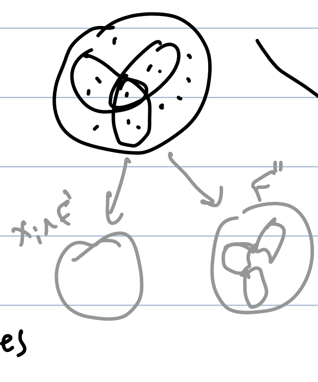
only $M = O(n)$ clauses

$t = 2^{o(n)}$

$C(k, \varepsilon) = \left(\frac{k}{\varepsilon}\right)^{O(k)}$

High level idea of proof:

find the var (or more generally, a sub-clause) that appears most often.



set it to true
$\Downarrow$
remove all clauses

set it to false
$\Downarrow$
all these clauses become

that contain it smaller

— repeat εN times...

Thm: ETH $\Rightarrow 2^{\Omega(n)}$ for Dom-Set.

pf: Suppose $\forall \delta' > 0$, Dom-Set is in $O(2^{\delta' N})$.

Let $\delta > 0$, we will show: 3-SAT in $O(2^{\delta N})$.

Given φ on N vars:

- set $\varepsilon = \delta/2$ and use sparsification Lemma:
  $\varphi \longrightarrow \varphi_1, \dots, \varphi_t$ $\quad t \le 2^{\varepsilon N}$, $M_i \le C(3,\varepsilon) \cdot N$.

- set $\delta' = \frac{\delta}{2} \cdot \frac{1}{(3 + C(3,\varepsilon))}$.

- for $i = 1 \dots t$:
  - Reduce φ_i to G_i on $n \le 3N + M_i$ nodes
  - Solve Dom-Set on G_i in $O(2^{\delta' n})$.

$\Rightarrow$ time per i:

$$2^{\delta' n} \le 2^{\delta' \cdot (3 + C(3,\varepsilon)) \cdot N} = 2^{\frac{\delta}{2} \cdot \frac{(3 + C(3,\varepsilon))}{(3 + C(3,\varepsilon))} \cdot N} = 2^{\frac{\delta}{2} \cdot N}$$

$\Rightarrow$ for all i:

$$t \cdot 2^{\frac{\delta}{2}\cdot N} = 2^{\varepsilon N}\cdot 2^{\frac{\delta}{2}\cdot N} = 2^{\delta N}.$$

$\blacksquare$

Dom-Set $O(1.4969^n)$

Max Indepent Set 1.19^n

Subset Sum $2^{n/2}$

Set Cover 2

3-Coloring $1{,}33^n$

10-Coloring 2^n

Traveling Salesman 2^n

- $\Omega(c^n)$ for some $c>0$, assuming ETH.

$\boxed{\text{Open: tight bounds under SETH.}}$

Next: SETH $\longrightarrow$ Problems in P