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3SUM 1

Problem 1 (3SUM Variants). Show that 3SUM is subquadratic equivalent to the following variations of 3SUM:

1. *3SUM with Duplicates:* Given $S \subseteq \{-n^c, \dots, n^c\}$ of size n (for some constant c), decide if there are elements $a, b, c \in S$ (which are not necessarily distinct) and that satisfy $a + b + c = 0$.
2. *3SUM with Target:* Given $S \subseteq \{-n^c, \dots, n^c\}$ of size n (for some constant c) and some *target* $t \in \{-n^c, \dots, n^c\}$, decide if there are elements $a, b, c \in S$ that satisfy $a + b + c = t$.
3. *Positive 3SUM:* Given $S \subseteq \{1, \dots, n^c\}$ of size n (for some constant c), decide if there are elements $a, b, c \in S$ that satisfy $a + b = c$.

Problem 2 (Universe Reduction). In the *Cubic-Universe 3SUM* problem the input consists of a set $S \subseteq \{-n^3, \dots, n^3\}$ of size n, and the goal is to decide if there are elements $a, b, c \in S$ with $a+b+c = 0$. Show that 3SUM $\longleftrightarrow_2$ Cubic-Universe 3SUM.

Hint: In the reduction from 3SUM, use additive hashing to hash all numbers to a universe of size $\approx n^3$.

Problem 3 (Zero Triangle). Consider an undirected graph $G = (V, E)$ with edge weights $w : E \to \mathbb{Z}$. We call a triple of vertices $(x, y, z) \in V^3$ a *triangle* if there are edges $(x, y), (y, z), (x, z) \in E$. We further say that (x, y, z) is a *zero-weight triangle* if

$$w(x, y) + w(y, z) + w(x, z) = 0.$$

In the *Zero Triangle* problem the goal is to decide whether a given graph contains a zero-weight triangle. The goal of this exercise is to pinpoint the complexity of Zero Triangle.

1. Convince yourself that Zero Triangle can be solved in time $O(n^3)$.
2. Let a, b, c be length-n vectors, where for simplicity we assume that n is a square number. We construct a complete tripartite graph with node parts X, Y, Z where $X = Y = \{0, \dots, \sqrt{n} - 1\}$ and $Z = \{-2n, \dots, 2n\}$. We assign the edge weights to be

$$\begin{aligned} w(x, z) &= a[x + z] && (\text{for } x \in X, z \in Z),\\ w(y, z) &= b[y\sqrt{n} - z] && (\text{for } y \in Y, z \in Z),\\ w(x, y) &= -c[x + y\sqrt{n}] && (\text{for } x \in X, y \in Y), \end{aligned}$$

   and delete all edges for which this definition is out-of-bounds (e.g. if $x+z < 0$). Show that (a, b, c) is a YES instance of Convolution 3SUM *if and only if* there is a zero-weight triangle in this graph.
3. Conclude that 3SUM ${}_2\!\longrightarrow_3$ Zero Triangle.

Bonus Problem 4 (3SUM with Small Numbers). Recall that the 3SUM hypothesis postulates that there is no truly subquadratic-time algorithm for 3SUM in the case where the input numbers have size up to n^c for some constant c, and in light of Problem 2 we can take $c = 3$. In this exercise we show that for much smaller numbers, namely $c < 2$, 3SUM does have a subquadratic-time algorithm. Concretely, show that the 3SUM problem over the universe $\{-u, \ldots, u\}$ can be solved in time $O(u \log u)$.

In your solution you may find the following algorithmic tools useful: A degree-d polynomial $p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_d x^d$ can be represented on a machine by storing its $d+1$ coefficients $p_0, p_1, \ldots, p_d$ in an array of length $d+1$. A very useful fact is that the product of two degree-d polynomials can be computed in time $O(d \log d)$ by the so-called *Fast Fourier Transform*[1].

[1] *The Fast Fourier Transform (FFT) is an elegant algebraic algorithm. Unfortunately it is beyond the scope of the tutorial to cover it in detail, but we recommend that you check it out it if you have not seen it before. See e.g.* `https://en.wikipedia.org/wiki/Cooley%E2%80%93Tukey_FFT_algorithm`, *or various online lecture notes.*