# Tutorial on Fine-Grained Complexity

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WEIZMANN INSTITUTE OF SCIENCE

DIMACS
Center for Discrete Mathematics and Theoretical Computer Science
Founded as a National Science Foundation Science and Technology Center

NSF

INSAIT
Institute for Computer Science, Artificial Intelligence and Technology

# Lecture 0: Welcome and Overview

- Welcome & Logistics
- Intro and quick overview
- Remarks about the tutorial

# Complexity Theory

| NP-hard | in P |
|---|---|
| k-SAT | Linear Programming |
| Travelling Salesman | All Pairs Shortest Paths |
| Independent Set | Fourier Transform |
| Subset Sum | Edit-Distance |
| Protein Folding | Context-Free Grammar Parsing |
| … | … |

*What about the problems inside P?*

# Complexity Theory

Popular since 1970's
(Traditional) Complexity:

"Polynomial Time = Efficient"

*Polynomial* vs. *exponential?*

*A theory for Small Data*

Popular since 2010's
Fine-Grained Complexity:

"Near-linear Time = Efficient"

$O(n), O(n^{1.5}), O(n^2), \ldots ?$

*A theory for Big Data*

## Longest Common Subsequence (LCS)

Input: two sequences of length n

S = cddcabbabcbaa

T = adbdbbcabacdd

Output: the length of the longest common subsequence

Dynamic Programming: $O(n^2)$

$$M[i,j] = \max \begin{cases} M[i-1,j-1] + (S[i] == T[j]), \\ M[i-1,j], \\ M[i,j-1] \end{cases}$$

[Masek - Paterson ’80]
$O(n^2 / \log^2 n)$

Can we do better?

## *Why care about $n$ vs. $n^2$ vs. $n^3$...?*

*Here's one example where it matters...*

### Local Alignment

Input: two (DNA) sequences of length n and a scoring matrix.

AGCCCGTCTACGTGCAACCGGGGAAAGTATA
AAACGTGACGAGAGAGAGAACCCATTACGAA

Output: The optimal alignment of two substrings.

C C G – T C T A C G
C C C A T – T A C G
+1 +1 -0.5 -1 +1 -1 +1 +1 +1 +1 = +4.5

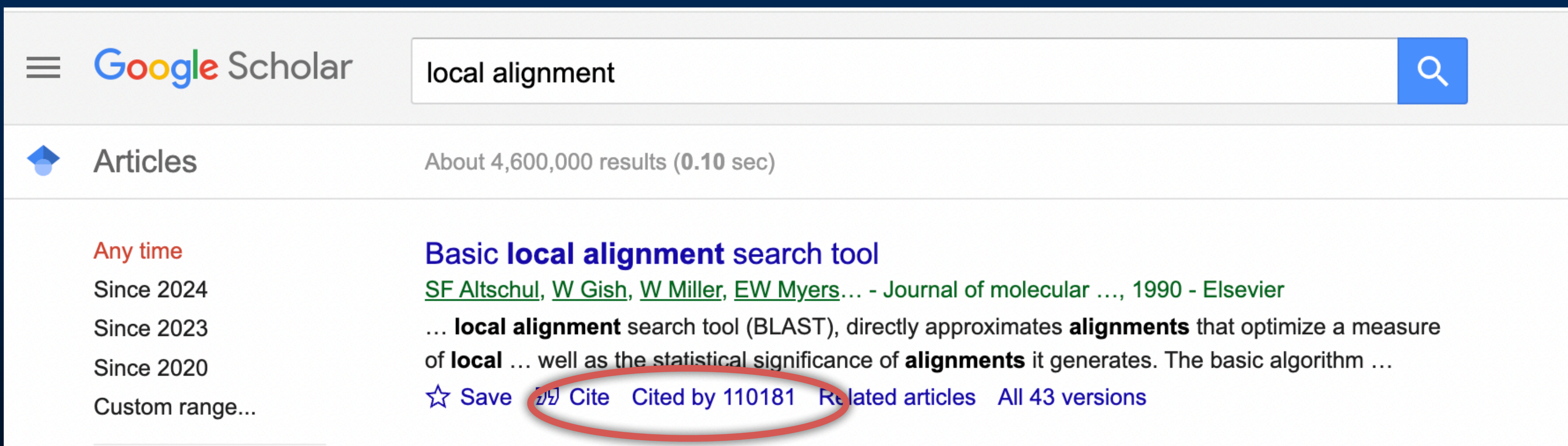
| | A | C | G | T | – |
|---|---|---|---|---|---|
| A | +1 | -1.4 | -1.8 | -0.7 | -1 |
| C | -1.4 | +1 | -0.5 | -1 | -1 |
| G | -1.8 | -0.5 | +1 | -1.9 | -1 |
| T | -0.7 | -1 | -1.9 | +1 | -1 |
| - | -1 | -1 | -1 | -1 | $-\infty$ |

Typically: ***$n \gg 10^6$***

[Smith-Waterman '81] $O(n^2)$ with dynamic programming - too slow!

# *Why care about $n$ vs. $n^2$ vs. $n^3$…?*

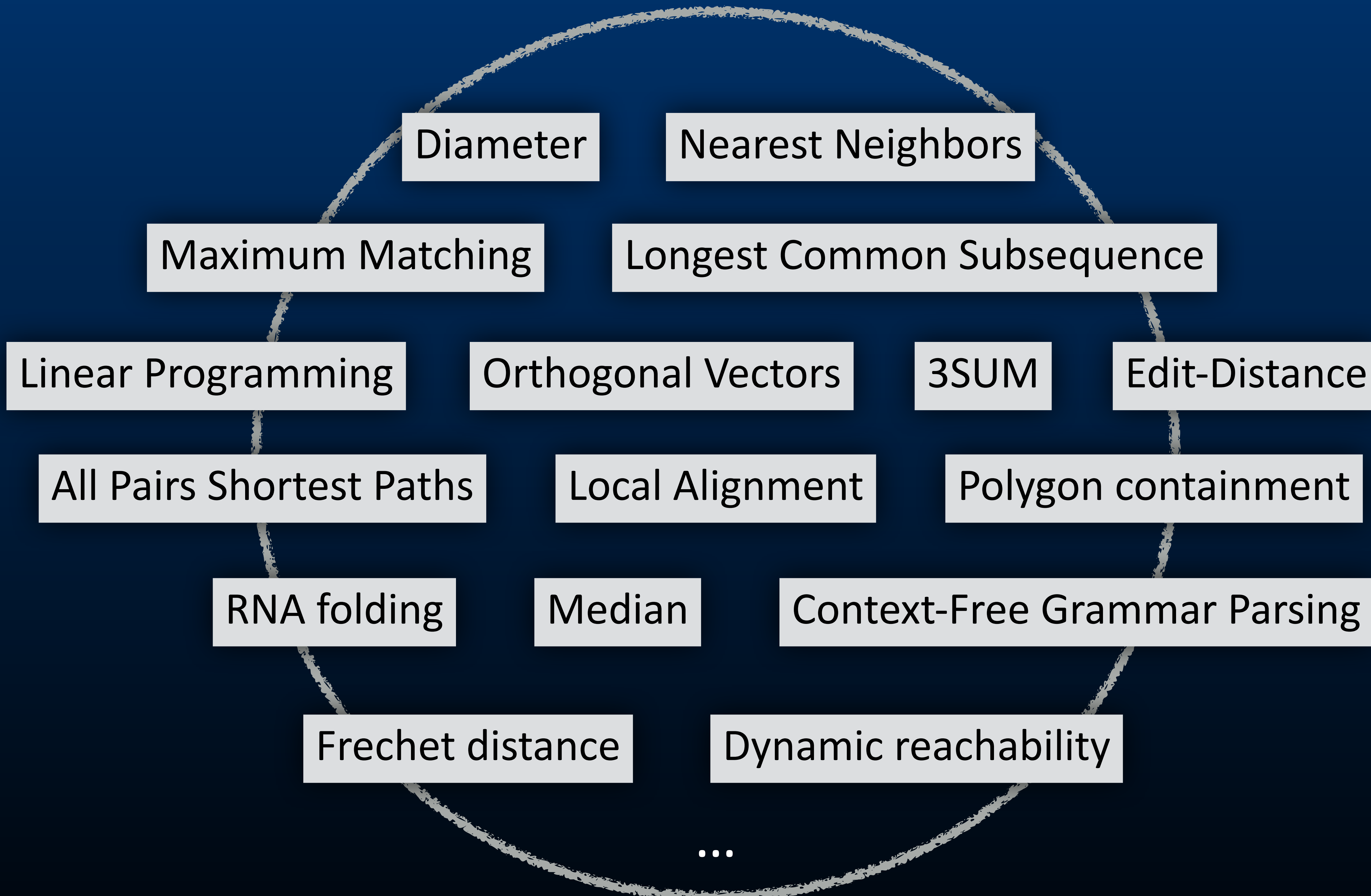
**BLAST**: A *heuristic, linear time* algorithm for **Local Alignment**.




110k citations!

Are there fast algorithms with optimality guarantees?

# The Class P




*Goal:* *Understand the time complexity of important problems.*

# Fine-Grained Complexity or: Hardness in P

Take a problem X in P, say in ***$O(n^2)$*** time.

And prove that:

“ X *probably* *cannot* be solved in $O(n^{2-\varepsilon})$ time. ”

*But how?*

# How do we get $n^2$ and $n^3$ lower bounds?

## Unconditional polynomial lower bounds?

*"Any Turing Machine has to spend $\Omega(n^2)$ time…"*

<u>Time Hierarchy Thm (1965):</u> Some (artificial) problems require $\Omega(n^2)$ time.

But $\Omega(n^2)$ for natural problems, even for SAT, is far out of reach of current techniques. Best lower bound is $3.1n$.

## Lower bounds for restricted algorithms?

e.g. $\Omega(n \log n)$ for sorting in the comparisons-only model.

Not general enough, and only gives partial answers.

NP-hardness is not fine-grained enough…

# Complexity Theory

*How do we prove hardness results?*

Popular since 1970's
(Traditional) Complexity:

*Polynomial* vs. *exponential?*

Reductions!

**P ≠ NP:**
"k-SAT cannot be solved in polynomial time."

Popular since 2010's
Fine-Grained Complexity:

$O(n), O(n^{1.5}), O(n^2), \ldots ?$

# Complexity Theory

*How do we prove hardness results?*

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My problem is in P
$\downarrow$
$P = NP$

Popular since 2010's
Fine-Grained Complexity:

$O(n), O(n^{1.5}), O(n^2), \ldots ?$

Fine-Grained Reductions!

# Complexity Theory

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Fine-Grained Reductions!

**SETH:**
"k-SAT cannot be solved even in $O(1.99^n)$ time."

# Complexity Theory

*How do we prove hardness results?*

| Popular since 1970's<br>(Traditional) Complexity: | Popular since 2010's<br>Fine-Grained Complexity: |
|---|---|
| *Polynomial* vs. *exponential?* | $O(n), O(n^{1.5}), O(n^2), \ldots ?$ |
| Reductions! | Fine-Grained Reductions! |
| My problem is in P<br>↓<br>$P = NP$ | My problem is linear<br>↓<br>SETH is false |

# An Example of a Fine-Grained Lower Bound

**Theorem** [AVW'14]:

"If for some $\varepsilon > 0$, we can solve Local Alignment in $O(n^{2-\varepsilon})$ time, then we can solve k-SAT in $O((2-\delta)^n)$ time for some $\delta > 0$ and all $k > 0$."

Faster Local Alignment
e.g. $O(n^{1.99})$
$\longrightarrow$
Faster k-SAT
e.g. $O(1.99^n)$
$\longrightarrow$
SETH is false

$P \neq NP$: "k-SAT cannot be solved in polynomial time."

**ETH:** "k-SAT cannot be solved even in $2^{o(n)}$ time."

**SETH (The Strong Exponential Time Hypothesis):**

"k-SAT cannot be solved even in $O(1.99^n)$ time."

# SETH

**k-SAT**: given a k-CNF formula on $n$ variables and $m$ clauses, is it satisfiable?

$$\phi = (x_1 \vee x_2 \vee \bar{x}_3 \vee x_{10}) \wedge \cdots \wedge (x_2 \vee \bar{x}_1 \vee x_4)$$

Fastest algorithms:

$$O\left(2^{\left(1-\frac{1}{ck}\right)\cdot n}\right)$$

k=3: $1.308^n$

k=4: $1.504^n$

k=5: $1.592^n$

... $k \to \infty : 2^n$

**The Strong Exponential Time Hypothesis (SETH):**

[Impagliazzo-Paturi'01]

There is no $\varepsilon > 0$ such that for all $k > 2$,

**k-SAT** can be solved in $O((2-\varepsilon)^n)$ time.

SETH: "k-SAT cannot be solved in ***O(1.99^n)*** time."

# The Class P (before)

Diameter

Nearest Neighbors

Maximum Matching

Longest Common Subsequence

Linear Programming

Orthogonal Vectors

3SUM

Edit-Distance

All Pairs Shortest Paths

Local Alignment

Polygon containment

RNA folding

Median

Context-Free Grammar Parsing

Frechet distance

Dynamic reachability

…

# The Class P (after)

**SETH**

Diameter
Closest Pair
Local Alignment
Dynamic Reachability
Single-Source Max-Flow
Subtree Isomorphism
Stable Matching
Edit-Distance
Frechet
LCS
…

*Problem domains:*
Graph Algorithms
Pattern Matching
Bioinformatics
Computational Geometry
Data Structures
Machine Learning
Formal Languages
…

Many problems remain unclassified…

# The Class P (after)

**SETH**

SAT on $n$ variable formulas requires $\Omega((2-\varepsilon)^n)$ time.

Closest Pair
Local Alignment
Dynamic Reachability
Single-Source Max-Flow
Subtree Isomorphism
Stable Matching
Edit-Distance
Frechet
LCS
…

**3SUM**

Finding 3 that sum to zero among $n$ numbers requires $\Omega(n^{2-\varepsilon})$ time.

Polygon Containment
Strips Cover Rectangle
Triangle Enumeration
Compressed Inner Product
Dynamic Max Matching
Set Intersection
…

**APSP**

Computing all distances in $n$ node graphs requires $\Omega(n^{3-\varepsilon})$ time.

Dynamic Max Matching
Stochastic Context-Free Grammar Parsing
Negative Triangle
Dynamic Max Flow
Replacement Paths
Median
…

Many problems remain unclassified…

# 3SUM

**3SUM:** Given *n* integers, are there 3 that sum to 0?

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| -15 | -6 | 33 | 8 | 1 | -21 | 4 | -30 | 7 | ... | 107 |

*A famous conjecture in computational geometry:*

<u>The 3-SUM Conjecture:</u>

"3-SUM cannot be solved in $O(n^{1.99})$ time."

# The Class P

**SETH**

↓

- Diameter
- Closest Pair
- Local Alignment
- Dynamic Reachability
- Single-Source Max-Flow
- Subtree Isomorphism
- Stable Matching
- Edit-Distance
- Frechet
- LCS
- …

**3SUM**

↓

- Colinearity
- Polygon Containment
- Strips Cover Rectangle
- Triangle Enumeration
- Compressed Inner Product
- Dynamic Max Matching
- Set Intersection
- …

# All Pairs Shortest Paths

APSP: Given a weighted graph on n nodes and $n^2$ edges, compute the distance between every pair of nodes.

| Author | Runtime | Year |
| --- | --- | --- |
| *Fredman* | $n^3 \log \log^{1/3} n / \log^{1/3}$ | 1976 |
| *Takaoka* | $n^3 \log \log^{1/2} n / \log^{1/2}$ | 1992 |
| *Dobosiewicz* | $n^3 / \log^{1/2} n$ | 1992 |
| *Han* | $n^3 \log \log^{5/7} n / \log^{5/7}$ | 2004 |
| *Takaoka* | $n^3 \log \log^2 n / \log n$ | 2004 |
| *Zwick* | $n^3 \log \log^{1/2} n / \log n$ | 2004 |
| *Chan* | $n^3 / \log n$ | 2005 |
| *Han* | $n^3 \log \log^{5/4} n / \log^{5/4}$ | 2006 |
| *Chan* | $n^3 \log \log^3 n / \log^2 n$ | 2007 |
| *Han, Takaoka* | $n^3 \log \log n / \log^2 n$ | 2012 |
| *Williams* | $n^3 / 2^{\Omega(\sqrt{\log n})}$ | 2014 |

Classical Algs: $O(n^3)$

*Bellman-Ford, Dijkstra,…*

Conjecture:
APSP cannot be solved in $O(n^{3-\epsilon})$ time.

# The Class P (before)

Diameter

Nearest Neighbors

Maximum Matching

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# The Class P (after)

**SETH**

- Diameter
- Closest Pair
- Local Alignment
- Dynamic Reachability
- Single-Source Max-Flow
- Subtree Isomorphism
- Stable Matching
- Edit-Distance
- Frechet
- LCS
- …

**3SUM**

- Colinearity
- Polygon Containment
- Strips Cover Rectangle
- Triangle Enumeration
- Compressed Inner Product
- Dynamic Max Matching
- Set Intersection
- …

**APSP**

- Radius
- Dynamic Max Matching
- Stochastic Context-Free Grammar Parsing
- Negative Triangle
- Dynamic Max Flow
- Replacement Paths
- Median
- …

Many problems remain unclassified…

# Technical Remarks

- We will ignore $\log n,\ \log^{O(1)} n,\ 2^{\sqrt{\log n}}$ or any $n^{o(1)}$ factors.
  - Many reductions have such overheads.
- We allow randomness.
  - The conjectures are assumed to holds against randomized algorithms too.
  - Many reductions use randomness.
- We use the (standard) Word RAM model with $w = O(\log n)$.
  - You can do any operations on words in constant time: addition, multiplication, random access, hashing, etc.
  - Since we allow log factors and randomness, this is not too important.
- Numbers are assumed to be in a polynomial range.
  - Integers in $\{-n^{O(1)}, \ldots, +n^{O(1)}\}$, real numbers with precision $1/n^{O(1)}$.

# Fine-Grained Reductions

Thm: If $A \;{}_a\!\!\rightarrow_b\; B$ and B is in $O(n^{b-\varepsilon})$ time then A is in $O(n^{a-\delta})$ time.

Thm: If $A \;{}_a\!\!\rightarrow_b\; B$ and $B \;{}_b\!\!\rightarrow_c\; C$ then also $A \;{}_a\!\!\rightarrow_c\; C$.

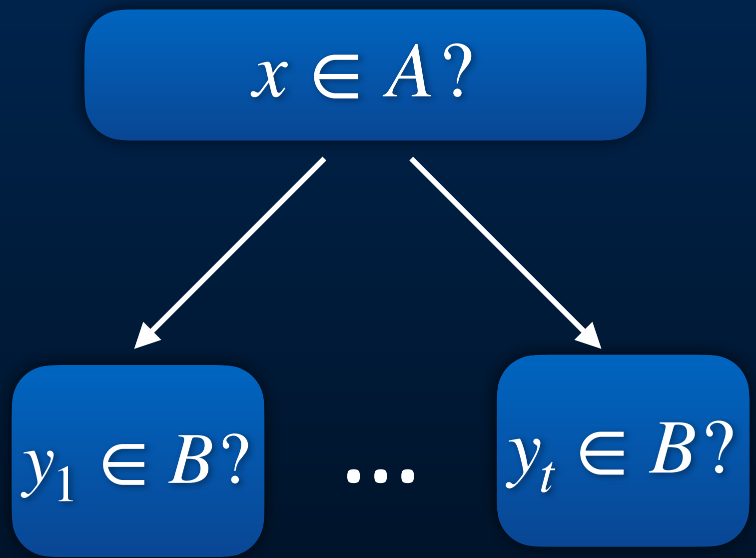
Definition: $A \;{}_a\!\!\rightarrow_b\; B$

A fine-grained $(a, b)$-reduction from A to B is an algorithm $\mathscr{A}^B$ for A with oracle access to B, such that:

$\forall \varepsilon > 0 : \;\; \exists \delta > 0 :$ for all input x of size n:

1. $\mathscr{A}^B(x)$ is correct w.p. $\geq 1 - 1/n^{10}$
2. $\mathscr{A}^B(x)$ runs in $O(n^{a-\delta})$ time.
3. Let $y_1, \ldots, y_t$ be the oracle calls, then:

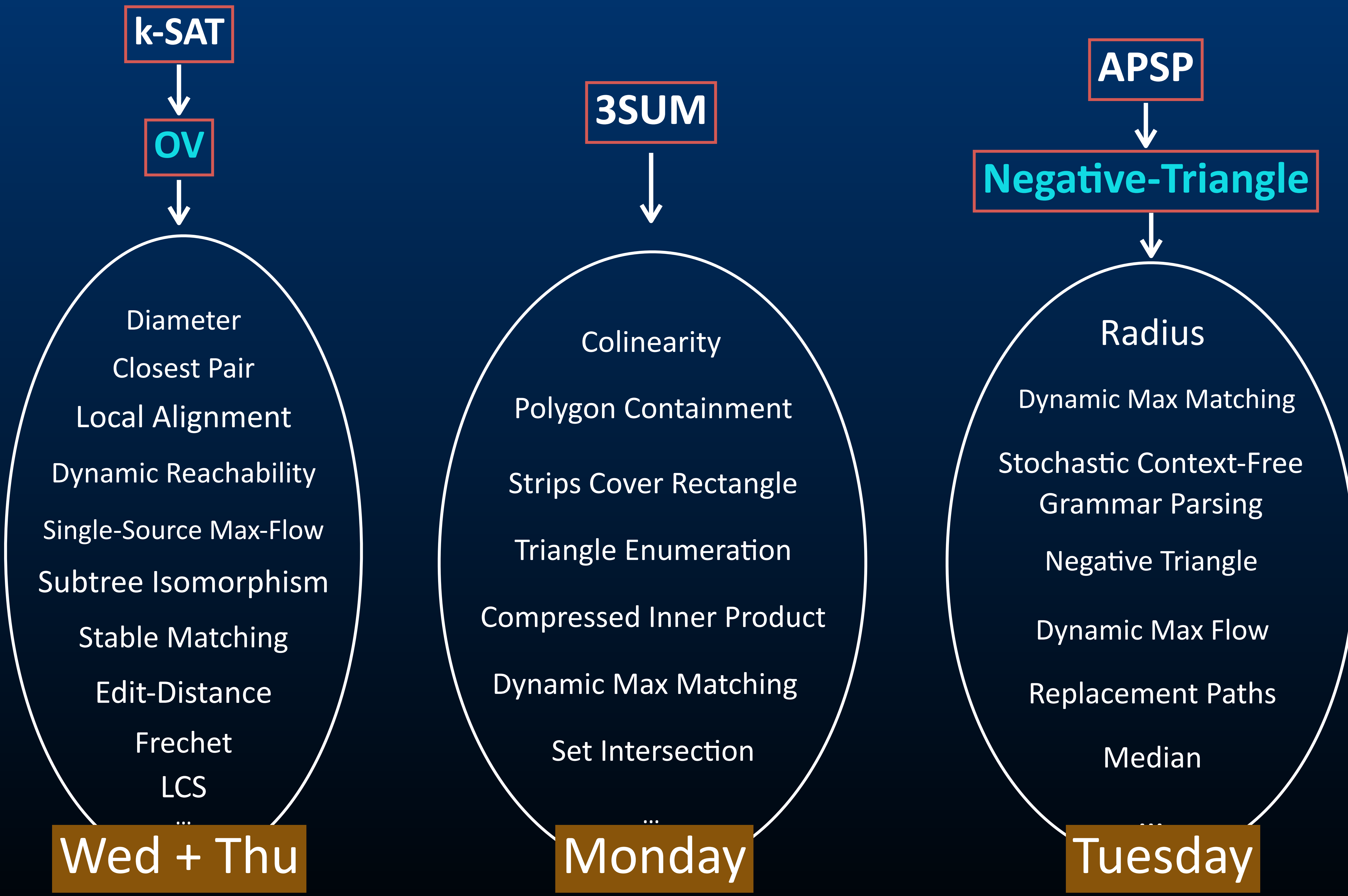
$$\sum_{i=1}^{t} |y_i|^{b-\varepsilon} = O(n^{a-\delta})$$




[Vassilevska & Williams '10]

# Tutorial Objectives

- Goal 0: The ability to understand FGC results.
- Goal 1: The ability to prove your own FGC results.
    - We will highlight the simplest hard problems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...
Wed + Thu
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...
Monday
APSP
Negative-Triangle
Radius
Dynamic Max Matching
Stochastic Context-Free Grammar Parsing
Negative Triangle
Dynamic Max Flow
Replacement Paths
Median
...
Tuesday

# Tutorial Objectives

- Goal 0: The ability to understand FGC results.
- Goal 1: The ability to prove your own FGC results.
  - We will highlight the simplest hard problems
- Goal 2: Intimacy with the theory and with current research.
  - This is the purpose of the afternoon lectures (and Friday).
- *Most importantly: To have fun thinking about basic problems!*