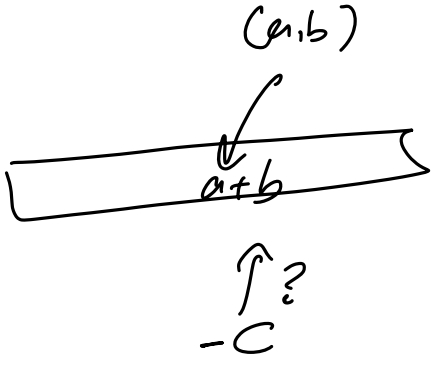
Plan:

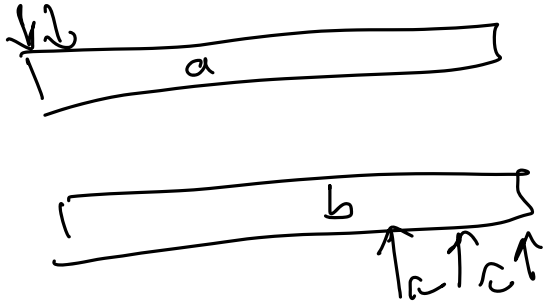
- 3SUM intro
- Equivalent variants
- Reductions via additive hashing
- Reductions to geometric problems.

} next parts

3SUM: given a set of n integers $S \subseteq [-U, +U]$, is there $a, b, c \in S$ s.t. $a + b + c = 0$?

- $O(n^2)$ algs.

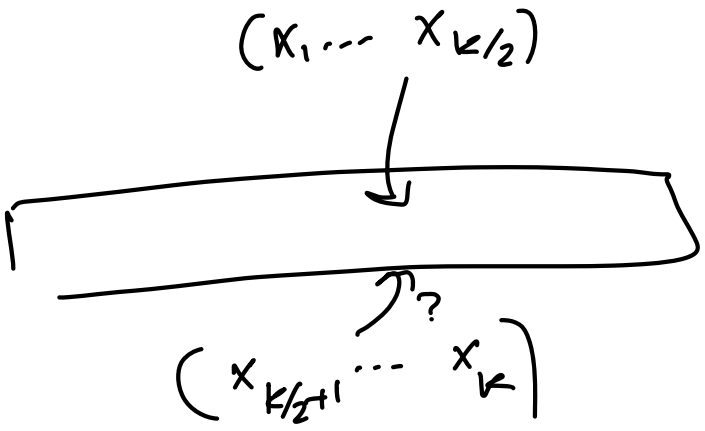




Conj: no $O(n^{2-\epsilon})$ alg.

k-SUM: Given $S \subseteq [-U, U]$ are there $x_1, \ldots, x_k \in S$ s.t. $\sum_{i=1}^{k} x_i = 0$?

- $O(n^{\lceil k/2 \rceil})$ alg.



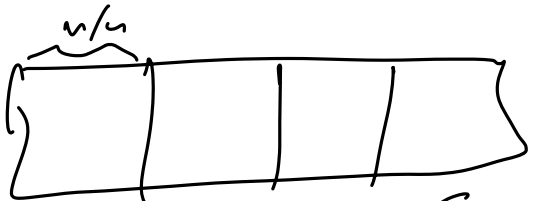
- $n^{o(k)}$ for k-SUM
  $\Rightarrow$ SAT is in $2^{o(n)}$ time (refutes ETH) [Patrascu Williams '10]

3SUM-Finding: given S ... return (a,b,c) s.t. $a+b+c=0$, if exists.

[Recall: "Finding → Decision" in NP]

Thm: If 3SUM is in $T(n)$ time, then 3SUM-Finding is in $\tilde{O}(T(n))$ time.

pf:



S_1 S_2 S_3 S_4

- let $S_i' = S \setminus S_i$. note: $|S_i'| = \frac{3}{4} \cdot n$
- call 3SUM(S_i') $\forall i \in [4]$.
- Recurse on first S_i' that is yes, or return fail.
- if $|S| = O(1)$, solve in $|S|^3$ time.

* Correctness: easy. ✓

* time: $T_{Find}(n) \leq 4 \cdot T_{Dec}(n) + T_{Find}\left(\frac{3}{4}n\right)$

easy: $O(T_{Dec}(n) \cdot \log n)$

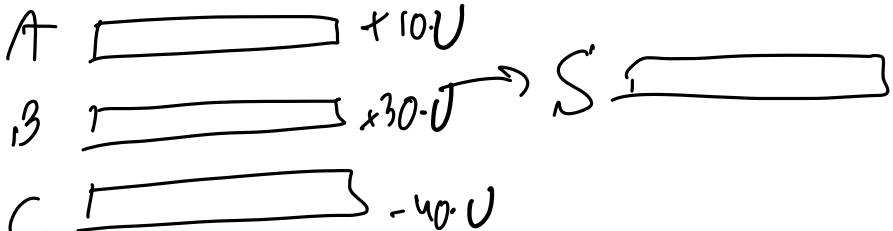
* (almost) never worry about finding vs. Decision!

3SUM $\underset{\text{recursing}}{\overset{\text{by def}}{\leftrightarrows}}$ 3SUM-Finding

---

Colorful-3SUM: given 3 sets of n numbers $A, B, C \subseteq [-U, +U]$; is there $a \in A, b \in B, c \in C$ s.t. $a+b+c = 0$?

Thm 1: 3SUM in $T(n)$ time $\Rightarrow$ Colorful 3SUM in $O(T(n))$.



$$S = \{a + 10 \cdot U \mid a \in A\}$$
$$\cup \{b + 30 \cdot U \mid b \in B\}$$
$$\cup \{c - 40 \cdot U \mid c \in C\}$$

Time: ✓

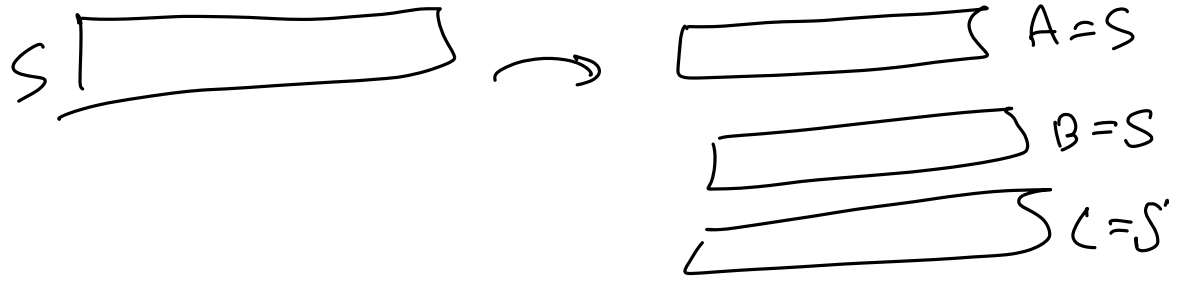
Correctness: sum of any 3 with reps in $\{10, 30, -40\}$ is >10 away from 0.

Colorful-3SUM $\xrightarrow{\text{add } U}$ 3SUM

$\overset{?}{\leftarrow}$

Thm 2: If Col-3SUM in $T(n)$ time, then 3SUM is in $\tilde{O}(T(n))$ time.

pf:



only proves 3SUM-with-Duplicates

↓

Col-3SUM.

trick: Color-Coding (Alon-Yuster-Zwick '94)

Reduction:

Log many times:

- repeat $200 \cdot \lg n$ times:
  - $\forall x \in S$ let $col(x) = \begin{cases} 1 & \text{w.p. } 1/3 \\ 2 & \text{w.p. } 1/3 \\ 3 & \text{w.p. } 1/3 \end{cases}$
  - let $A = \{x \in S \mid col(x) = 1\}$
    $B = \{x \in S \mid col(x) = 2\}$
    $C = \{x \in S \mid col(x) = 3\}$
  - if Col-3SUM(A,B,C) = yes, return yes
- return no.

time: ✓

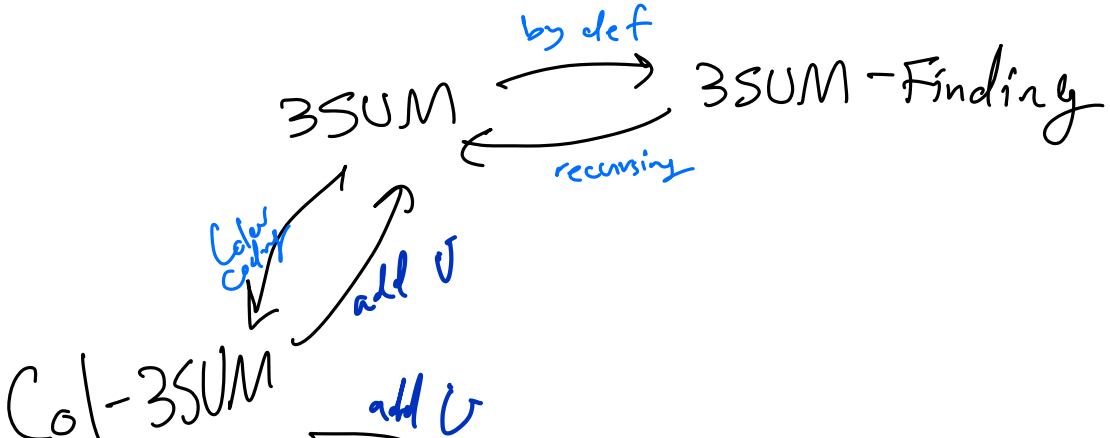
correctness: $\forall a,b,c$: $\Pr\left[\begin{matrix} col(a)=1 \\ col(b)=2 \\ col(c)=3 \end{matrix}\right] = \left(\frac{1}{3}\right)^3$

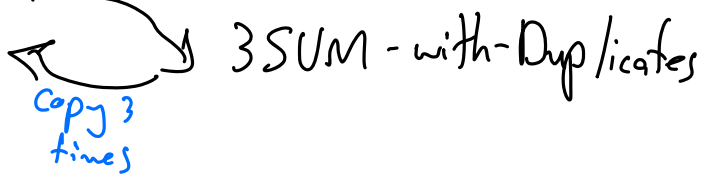
$\Rightarrow \Pr[\text{never}] \leq \left(1-\left(\frac{1}{3}\right)^3\right)^{200 \lg n}$

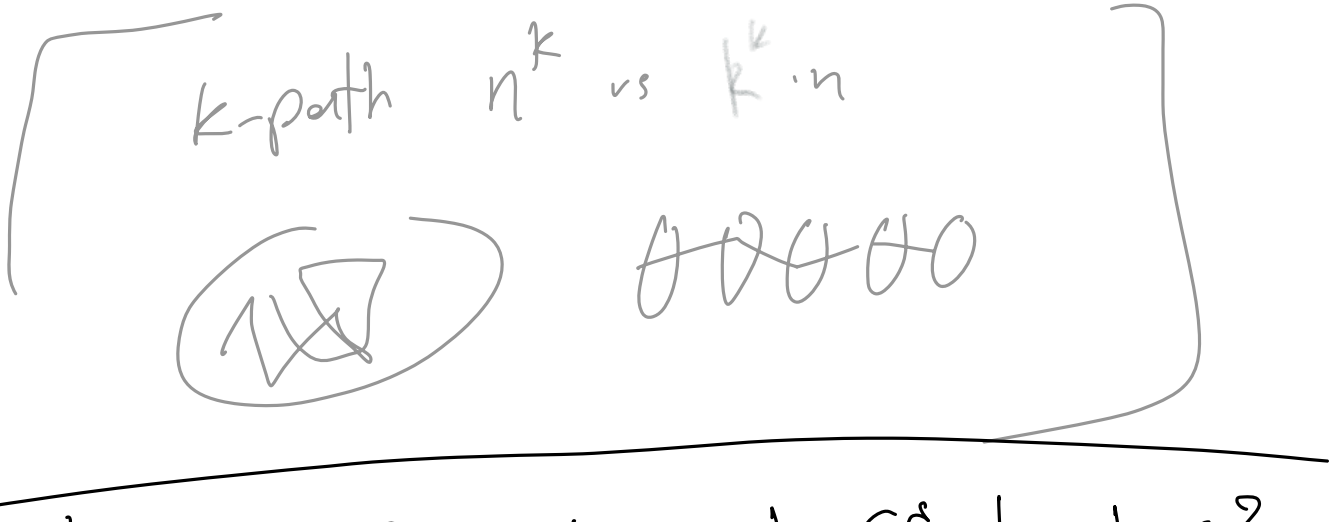
$\leq \left(\frac{1}{2}\right)^{10 \lg n} \leq \frac{1}{n^{10}}$.

"no" $\Rightarrow$ "no"

"yes" $\Rightarrow$ "yes" w.p. $\geq 1 - \frac{1}{n^{10}}$.







3SUM': given S are there $a,b,c \in S$ s.t. $a+b=c$?

Col-3SUM': given A,B,C ... $a \in A, b \in B, c \in C$ s.t. $a+b=c$?