SETH: $\forall \varepsilon > 0 \ \exists k \geq 3$ s.t. k-SAT is not in $2^{(1-\varepsilon)\cdot N}$ time.

(*) Can assume only $M = O(N)$ clauses.

i.e. $\forall \varepsilon > 0 \ \exists C, k \geq 3$ s.t. k-SAT on N vars and $C \cdot N$ clauses is not in $2^{(1-\varepsilon)\cdot N}$ time

SAT $\rightarrow$ OV $\rightarrow$ Problems in P

OV:

Given two sets $A, B \subseteq \{0,1\}^d$ of binary vectors where $|A| = |B| = n$, is there $a \in A, b \in B$ s.t.

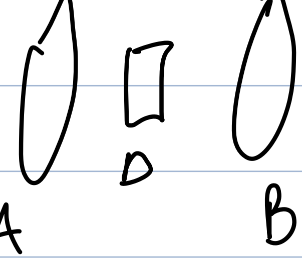
$$\langle a, b \rangle = 0 \quad \text{i.e.} \quad \forall j: \big(a[j] = 0 \text{ OR } b[j] = 0\big)$$

ex.

0010	1010
1100	0110
✓	✗

Graph-OV:

Given graph $V = A \cup B \cup D$, $E \subseteq A \times D \cup B \times D$
where $|A| = |B| = n$, $|D| = d$,
is there $a \in A$, $b \in B$, s.t. $d(a,b) > 2$?



- $O(n^2 d)$ trivial
- $O(2^d \cdot n)$ subquadratic for $d < \log n$
- $n^{2 - \frac{d}{\Theta(\log n)}}$ subquadratic for $d = c \log n$ for all c [A-Williams-Yu '15]

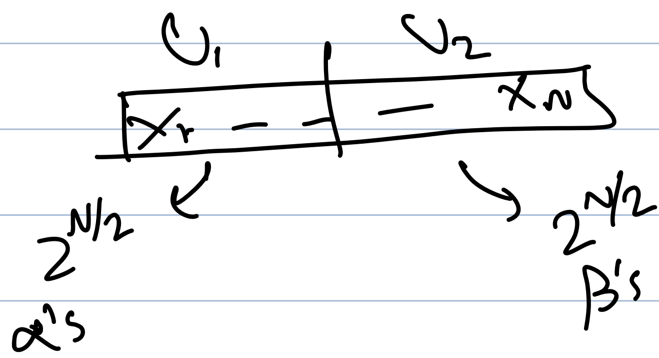
Thm: SETH $\Rightarrow$ "$\Omega(n^{2-\varepsilon})$ for OV even when $d = O(\log n)$"

i.e. $\forall \varepsilon > 0$, $\exists c > 0$ s.t. OV with $d = c \log n$

[Williams '05] cannot be solved in $O(n^{2-\varepsilon})$ time.

pf: Given a k-CNF formula φ on N vars.

1. Split the vars into $U_1 = \{x_1, \dots, x_{\frac{N}{2}}\}$, $U_2 = \{x_{\frac{N}{2}+1}, \dots, x_N\}$



2. $\forall$ assignment α to U_1:

create vector a of dim $d = M$ s.t.

$$a[j] = \begin{cases} 0 & \alpha \text{ satisfies } C_j \\ 1 & \alpha \text{ doesn't satisfy } C_j \text{ by itself} \end{cases}$$

3. $\forall$ assignment β to U_2:

create vector b of dim $d = M$ s.t.

$$b[j] = \begin{cases} 0 & \beta \text{ sat. } C_j \\ 1 & \text{o.w.} \end{cases}$$

Claim: $\langle a, b \rangle = 0$ iff α & β together satisfy φ.

(Cor.: $\exists$ orthogonal pair $a \in A$, $b \in B$ iff φ is sat.)

pf: $\langle \alpha, \beta \rangle = 0 \iff \forall j \ (a[j] = 0 \text{ OR } b[j] = 0)$

$\iff \forall j: (\alpha \text{ sat } C_j \text{ OR } \beta \text{ sat } C_j)$

$\iff \alpha$ & β satisfy φ

Correctness: ✓

Efficiency:

\# vectors $n = |A| = |B| = 2^{N/2}$.

$\Rightarrow n^{2-\varepsilon} = (2^{N/2})^{2-\varepsilon} = 2^{(1-\varepsilon/2)\cdot N}$.

dimension: $d = M = c \cdot N = c' \cdot \lg n$.

Remark: SETH $\Rightarrow \Omega(n^{k-\varepsilon})$ for k-OV

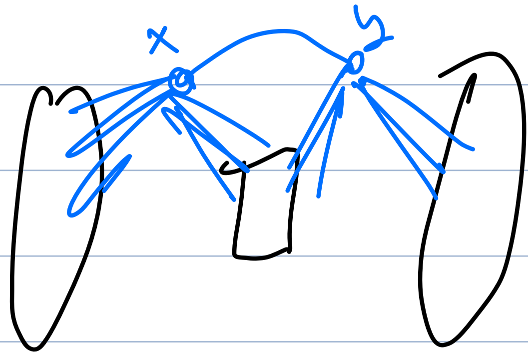
for all $k \geq 2$.

3-OV: given A, B, C ... are there

$a \in A, b \in B, c \in C$ s.t. $\forall j\ (a[j] \cdot b[j] \cdot c[j] = 0)$

Diameter: compute $\max d(u,v)$.

Thm: SETH $\Rightarrow \Omega(n^{2-\varepsilon})$ for Diameter when $m = \tilde{O}(n)$.



[Roditty-Vassilevska W. '13]

- $1.5-\delta$ apx $\Omega(n^{2-\varepsilon})$
- exact: $O(n^2)$
- 2-apx $\tilde{O}(n)$
- 1.5-apx $\tilde{O}(n^{1.5})$ — best possible in subq. time
- Very recently: [DLV '21]
  No $(2-\delta)$-apx in $\tilde{O}(n)$ time.

U.B.	L.B.
$n^{1+\frac{1}{k}}$ time	$n^{1+\frac{1}{k}}$ time
$\sim 2 - 1/2^k$ apx	$\sim 2 - \frac{1}{k}$ apx

Structure of OV reductions:

- "vector gadget"

- "OR gadget"

* will see more examples today & tomorrow,