Master Document

Fair Division

Fair Division – February 24-25, 2009

I did this workshop today for the first time. There were 28 participants of the NJ program on Tuesday and xx on Wednesday. The workshop went well both days.

The instructor notes are badly in need of revision. My experience of the workshop, including the changes that I made, were as follows:

1. They didn't seem to have a clue about how to divide the brownie other than the use of more accurate instruments. TSP #3 was useful as a way of introducing the importance of perception, but is something that should have been discussed more fully later in the workshop, after some of the methods were introduced. That also applies to TSP #4. I skipped TSP #5-6 altogether. All four of these TSPs should be better written, but I didn't focus on that this time around.

2. I went through TSP #7-13 pretty much as is. However, after discussing divider / chooser, I discussed the possibility of two people dividing a cake that is half chocolate and half vanilla. If the divider likes chocolate and the chooser doesn't, then the divider divides it between chocolate and vanilla. If the divider likes chocolate and the chooser does too, then the divider divides both sections in half. If the divider likes chocolate and the chooser doesn't care, then the divider perhaps makes one piece that is only ¾ of the chocolate and the other piece that is all of the vanilla and ¼ of the chocolate in the hope that the other person will go for quantity … but maybe he was lying. Although I thought that perhaps we should discuss the four methods on Handout #1 one at a time, I realized that they were able to traverse this page pretty well. Someone suggested that in the Howe's method, changing Bob and Alice's roles in the final step would improve matters – it does, but it is still unfair to Alice.

3. Rewrote the activity on the 24-acre field so that there is a handout with more specific goals and a resource page which gives one solution to the problem of dividing the field. This activity reinforced the idea that it is better to be the chooser than the divider in a divider-chooser situation, and was also used to introduce the idea of envy-free fair division where not only does each person feel that s/he has received a fair share, but is not envious of what the other person (people) have received, as is the case in this example. This example should also be used to underscore the distinction between continuous and discrete fair division (and then between continuous and discrete mathematics), since the next activity is the method of markers (when the distinction can be repeated). A good segue would have been to divided the field with diagonal lines – I didn't do this the first day, but did on the second day.

4. I used a TSP based on the 20 items in Tenenbaum's discussion on Tuesday, but found that his discussion was incorrect if the sets of markers are not disjoint (as might happen if someone places his third marker before someone else's second marker). There is also no handout or resource material on the method of markers, so I added a TSP and a handout (that will also go in the resource book). There seem to be three ways of doing this activity – as a whole class activity with hundreds of candies, as a small group activity, with each table dividing its 34 candies, or (as Val suggested) with two tables dividing 68 candies. Since I

had five tables, I had each table do it by itself. That worked very well, although on the first day I used Tenenbaum's method and that caused problems at some of the tables.

5. There should have been a summary at the end of the workshop.

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised October 31, 1999

Follow-up – Fair Division

While participants arrive, place TSP #1 on the overhead.

Before you begin the first activity, mention to participants that each group will be receiving a brownie and the group needs to decide how to fairly divide it according to each person's own value system. It is important to emphasize, before the conversations begin, that "fair division" does not necessarily imply "divide into equal parts" as so many elementary teachers automatically infer. For example, one player may decide that a fair share for her is a smaller piece with bigger nuts.

Activity #1: Dividing a brownie: the continuous case
(Allocated time = 20 minutes; 10 minutes for A, 10 minutes for B)

A. Distribute brownies, one per table, and ask the participants to devise some strategies for fairly dividing the brownie among the 5 or 6 people in the group. Note, the answer "divide it into sixths and pick one each" is not satisfactory, because it yet remains to be decided who decides when the sixths are fairly made and who gets which of the "equal" sixths.

We've liked using "Little Debbie's Brownies with nuts" because the size of the brownie is good for a group of 5 or 6 people and the "nuts" add lively conversation to the issue of fairness.

B. Use a blank transparency placed on top of TSP #2 to review the various strategies each group used to fairly divide their brownie. Keep in mind that you are guiding them to discover the goal of fair division (see TSP #3) and to feel the value of the three criteria in the table below, though you shouldn't make them explicit yet. (There is a slide for these -- TSP #4 -- if you wish to put them up at any time during the workshop.)

Fair	Meaning each player feels that they got 1/N of the whole
Internal	Meaning that the players divide it up by themselves with their own judgement.

Decisive	Meaning that if the rules are followed, then the procedure will always yield a fair division.

Show TSP #3 through TSP #6 to summarize your points about the goal of fair division (TSP #3), what a fair division scheme should consist of (TSP #4), what we assume about the players (TSP #5), and what our assumptions save us from (TSP #6). When showing TSP #3, it is important to emphasize that the goal in fair division is that each person can be guaranteed of getting what he/she "perceives" to be at least 1/N of the object regardless of how other people divide and choose (as long as the rules are followed). Again, you will need to emphasize that different value systems should not interfere with a fair scheme.

Goal of fair division: Devise a scheme for dividing an object among N people so that if the scheme is followed precisely, each person can be guaranteed of getting at least 1/N of the object regardless of how the other people divide and choose, as long as they follow the rules.

On TSP #5, many participants have had difficulty with the idea of "quantitative". What is important for them to realize is that they should be treating the brownie as a quantifiable object, of which they each want 1/Nth according to their own value system. For example, a player would not say, "I'm allergic to chocolate therefore I only want nuts" or another person can't say, "I don't like brownies at all, so I shouldn't get any." Each player always wants at least 1/N regardless of these type of emotional preferences. It is important to make this assumption about the players because if we try to take into account such emotional preferences, we won't be able to devise a fair scheme since we wouldn't be able to quantify all possible emotional preferences.

Activity #2: Determining if a scheme is fair: the 2-player case
(Allocated time = 10 minutes)

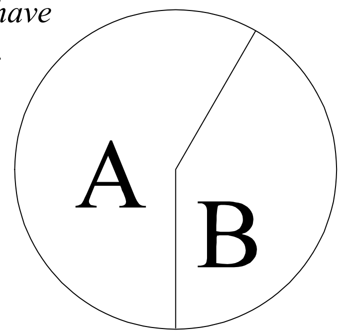
A. Tell participants that the brownie situation is difficult to analyze because there are many people in each group and because there is a mixture of things on the brownie (i.e., there is a brownie part to the cake and a nut part to the cake). Ask participants if they can think of a way to make the analysis simpler. Someone will suggest use fewer people and someone else will suggest to only use cake (no icing, no nuts, just cake). This will help introduce the simpler case of just two people wishing to fairly divide a homogeneous cake. Show TSP #7 and discuss whether the scheme is fair or not. Participants will easily explain that Bob can get gypped when Alice has control of the cutting and the choosing.

B. Show TSP #9, but cover the bottom portion of the slide so participants do not see the box with the words "Divider-Chooser" in it. Ask if this 2-player scheme is fair and if

not, what could go wrong.

Sometimes when you introduce the idea of a simpler problem (i.e., the two person case), participants will automatically suggest a scheme which is essentially the divider-chooser method (TSP #9). If they do, introduce & discuss TSP #9, using a blank transparency on top of TSP #8 as necessary to demonstrate examples raised by participants, and then return to TSP #7. Otherwise, begin a guided discussion for how to determine if a scheme is fair by first showing and discussing TSP #7 and then TSP #9. Model the different cases that a person needs to consider before determining whether a scheme is fair on a blank transparency on top of TSP #8.

Try out an example assuming (outrageously, if necessary) any action on the part of the various players to see if the method can always leave each person feeling like he got a fair share. See the next paragraph for an illustration.



One thing to keep in mind is that the various players may have different notions of what half or 1/3 of an object looks like. But for our purposes, it is the person's perception of value that matters. For example, if the object to be divided is a pie, as shown to the right, suppose the divider is B and he divides the pie as shown. Then A, seizing her big chance, chooses the indicated piece, and feels like she got more. Perhaps she did. But in B's estimation, if he followed the rules and divided the cake in such a way that he would be satisfied with either piece (which is the divider's role in the divider-chooser method) then he would perceive that he got half, despite what A might think about her share.

Activity #3: Determining if a scheme is fair: the 3-player case
(Allocated time = 45 minutes; 30 minutes for A. and 15 minutes for B.)

A. Distribute Handout #1 with the 4 methods of three-person fair division. Also distribute Handouts #2 and #3 with the circles sheets (two pages per person). Have participants work in groups to try out the 4 methods on the sheet. The first and last are no good (for Boyd's method, if Alice's first piece is too big, then Carl will choose it, and Bob will get gypped. Alice will perceive a fair division, however. For Howe's method, if Bob has an opinion about division which leaves a much larger piece in each half, according to their way of thinking, then when Carl chooses it, and Bob chooses, Alice ends up getting quite gypped. Bob however will perceive a fair division.) Dewey's method is the "Lone Divider" method, and Cheetham's method is the "Lone Chooser" method. All four of these are illustrated on the TSP #10 - 13.

B. When they have all had sufficient time to analyze the different schemes, review them with the participants, asking volunteers to explain why the bogus ones don't work and why the good ones do work.

If you are running behind schedule, delete Activity #4 from the workshop to be sure that you have enough time to complete Activity #5 (since it has proven quite popular among K-8 teachers particularly because it is readily used in the classroom).

Activity #4: Dividing a plot of land: the continuous case with variety
(Allocated time = 20 minutes)

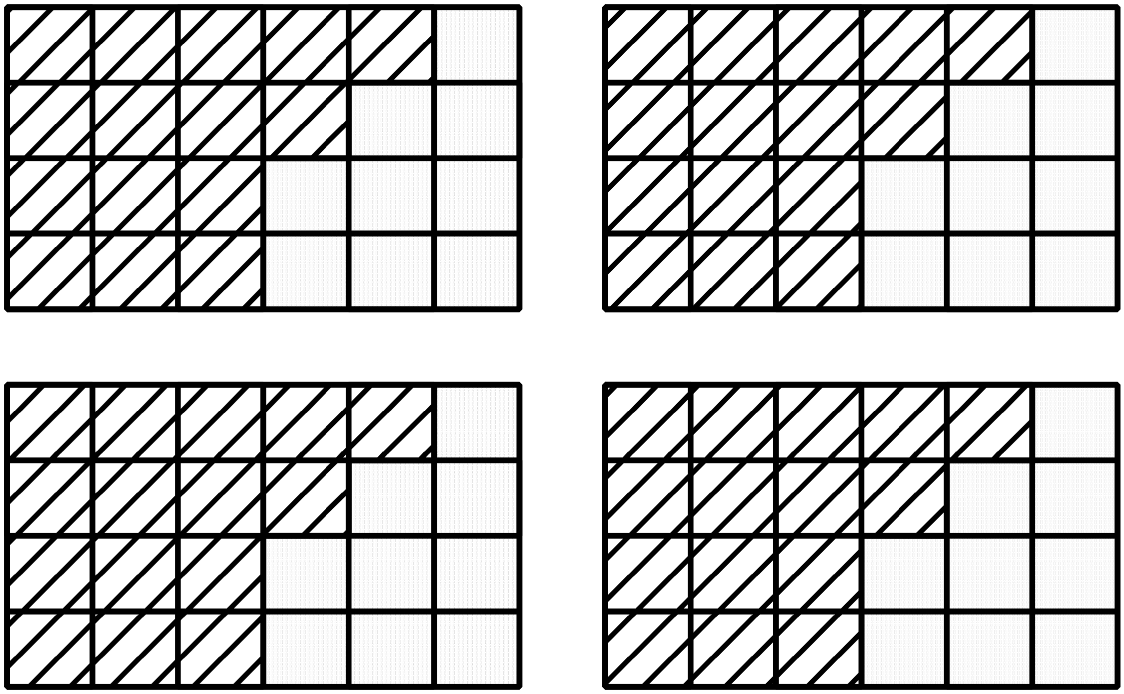
A. Show TSP #14 and explain that sometimes a continuous object has parts to it that have different value to different people (i.e., in the case of Neapolitan ice cream). In the next example, we look at using the Divider-Chooser method to partition a 24-acre plot of land consisting of woodland (diagonal lines) and grassland (lightly shaded – will appear as unshaded when xeroxed). Each block represents a one acre plot. Invite a participant to act as "your partner" to investigate what happens using the following procedure:

You and a partner will use the Divider-Chooser method to fairly divide the plot of land below. (4 copies of it are shown.) This exercise will help to emphasize just what is so fair about fair division.

1. *You and your partner should each secretly write down how much you feel an acre of each type of land is worth (pick a whole number from 1 to 20 representing that many hundred dollars per acre.)*
2. *You and your partner should then decide the total value of the 24 acre plot, based on 1.*
3. *Use the divider-chooser method four times, each time the divider should choose a different way to divide up the land. You can decide who plays which role, on each of the four trials.*
4. *When this is done 4 times, compare what each person got to what each person expected to get. Is there a correlation between being the divider or chooser and how much land you got?*

Ask the participant what value he/she would assign to each of the property types. Whatever the participant chooses (say, woods $4, grassland $5) you take the opposite (say, grassland $5 and woods $4).

- Figure total amount
- Instructor cut so that get = to me and allow participant to choose



- have class figure value & pick
- explain I got ½ because of the set up and participant (chooser) got much more ½ (or has the potential to get much more)
- Next example allow participant to divide and instructor choose
- Explain the upshot of the activity. explain that a worksheet is in resource book if want to try this with a friend.

B. Summarize this activity with TSP #15. Explain that the upshot of this activity is that, when there is a mixture of items at stake, which different players place different values on, then not only can each player be guaranteed a fair (1/2) share of the total value, but the chooser will typically get *more* than his fair share! And the more unequal the values they place on the items and the more asymmetrical the divider's division, the better the chooser will make out.

NOTE: We have found this activity to be quite complicated for some K-8 groups. If the participants in your workshop seem to be having difficulty up to this point (or if you're running short on time), we advise you to omit Activity #4 and move directly to Activity #5.

Activity #5: Dividing a bag of candy: the discrete case
(Allocated time = 40 minutes; 10 minutes for A., 10 minutes for B., and 20 minutes for C.)

A. Distribute 34 pieces of variety candy to each of the tables, and arrange that there be 5 or 6 at each table, if this wasn't already the case. Ask them to think of ways that they could fairly divide these objects among everyone at the table. Explain that this problem is different from the brownie problem because we can't cut the objects in pieces to resolve

problems.

B. After participant groups have found a fair scheme, let them discuss the pros and cons of the various methods as a large group. Use a blank overlay on TSP #16 to record their schemes and whether people think they are fair. Then ask, "What do you think if I told you that a method exists where everyone would be guaranteed to receive their fair share and MORE!" Ask participants to put their candy back into the middle of the table.

If some participants hesitate to return their candy because they don't want to let go of their "fair share" ask each person to write down what pieces of candy they have received so later they can reconstruct their portions. At this point they don't realize that they WILL be happy with their new allocation and some may express a desire to keep their initial allocations of candy. The point here is to move the workshop along and not get tripped up over participants wanting to hold onto their candy.

C. Now model the Method of Markers on a blank transparency to demonstrate one way to fairly divide a bag of discrete objects. Be explicit that this method only works when the objects are of relatively similar worth. For example, a bag of candy could be divided among N people using this method, but an estate -- where the discrete objects often have a wide range in value -- would not be a good method to divide among the N people. Distribute N-1 markers to each of N people in a group.

In the past, we've used wooden stirrers with the participant's initials written on them so when all the player place their markers on the table you can easily tell which marker belongs to which participant. We've also used unifix cubes as markers where each player received a different color cube.

Ask the participants to randomly place the candy in a row across the table as quickly as possible. (We ask them to place the candy quickly so that they are not placing the candy in any particular order.) After this is completed, ask each player to partition the candy line by placing their markers in such a way that they would be happy receiving any section of candy between any two of their own markers. (Obviously in placing the first marker, they should be happy with receiving the candy located to the left of the marker. And in placing the last marker, they should be happy with receiving the candy located to the right of the marker.)

Allocations of candy are made beginning from left to right in the candy line. Assign to the player owning the first marker all the candy to the left of his marker. Remove this player's markers from the candy line. Scan the candy line for the next left-most marker. Identify the player who owns that marker and scan the candy line for this player's second left-most marker. Assign the candy between his two markers to this player. Remove this player's markers from the candy line. Continue in this fashion until the last player receives his or her last segment.

Typically, there will be some leftover candy. They can be allocated to the players by lottery, or, the same method of markers can be repeated – provided there are many more pieces leftover than players.

Fair Division

Strategies to Fairly Divide a Brownie

Fair Division

Goal:

To devise a scheme for dividing an object among N people so that if the scheme is followed precisely, each person can be guaranteed of getting what he/she ***perceives*** to be at least 1/N of the object regardless of how the other people divide and choose, as long as they follow the rules.

Fair Division Scheme

Should be:

- **Fair**: If there are 8 players, for example, then each player can guarantee that he receives what he perceives to be 1/8 of the whole.

- **Internal**: The scheme should not require the intervention of any outside judge or arbiter.

- **Decisive**: If the rules are followed, the scheme should always yield a fair division.

Assumptions

Our players are:

- **Quantitative**: The players view the goal of fair division as a numerical one, based on their own value system not on whimsical considerations.

- **Isolated**: They have no knowledge of each other's value systems.

- **Accurate**: The players can divide the goods into any fractional part that they wish and detect fair shares with perfect accuracy.

These assumptions save us from

- **Regrets**: A player, having declared a share to be fair to him, doesn't later feel that it was too small.

- **Despair**: Without an outside judge, the players were in control.

A fair division scheme avoids

- **Arguments**: Everyone will receive what he perceives to be a fair share.

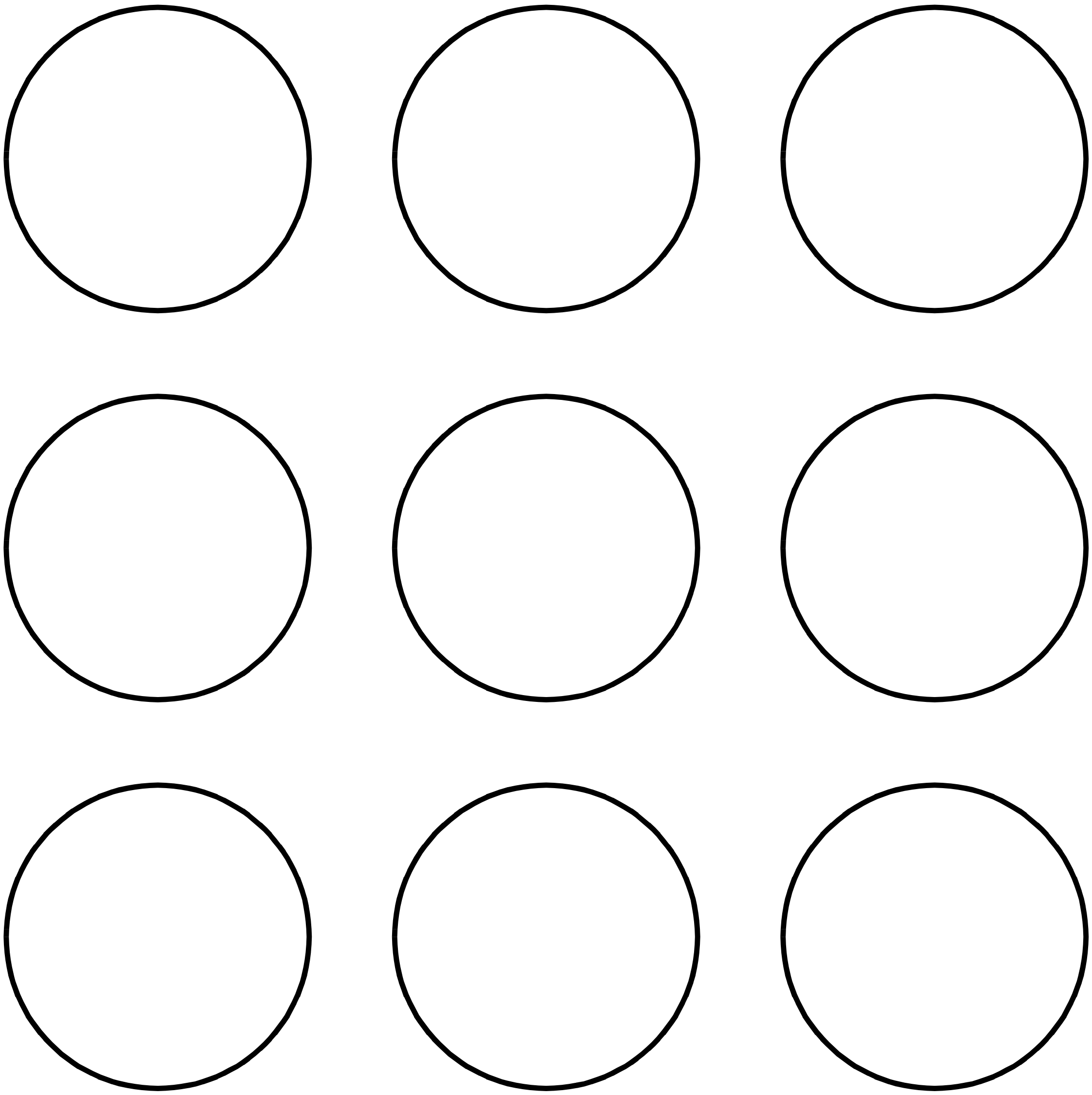
Is this a good 2-player scheme?

- Alice and Bob have a cake to divide.
- Alice cuts.
- Then Alice chooses.

Is it Fair?

What, if anything, could go wrong?

Is it Fair?



Is this a good 2-player scheme?

- Alice and Bob have a cake to divide.
- Alice divides.
- Bob chooses.

Is it Fair?

What could go wrong?

This scheme is called

$\boxed{\textbf{Divider-Chooser}}$

3 Player Fair Division
Boyd's Method

Is this scheme fair?

1. Alice cuts a piece, (1/3), called A

2. Bob cuts the remainder into two pieces, B and C.

3. Carl picks one of the pieces, then Bob picks a piece, then Alice gets the last.

Is it Fair?

What could go wrong?

Lone Divider Method
Dewey's Method

Three people, Alice, Bob and Carl, wish to divide a cake between them.

1. Alice (the lone divider) cuts the cake into what she considers equal thirds.
2. Bob and Carl each secretly write down which of the three pieces(at least one) they would be happy with.
3. If this results in a satisfactory division, then they take their choices and Alice gets the remainder.
4. If both Bob and Carl will each only accept the same piece, say P, then they give one of the *other* pieces to Alice, put the remaining two pieces together, and use divider-chooser.

The Lone Chooser Method
Cheetham's Method

Three people, Alice, Bob and Carl, wish to divide a cake between them.

1. Alice and Bob use the divider-chooser method so that each gets what he or she believes is at least half.
2. Alice and Bob then each divide their halves into three pieces which they believe to be of equal sizes.
3. Carl then picks one of Alice's three pieces and one of Bob's three pieces, and that forms his share. Alice and Bob are left with the remaining two pieces from their halves.

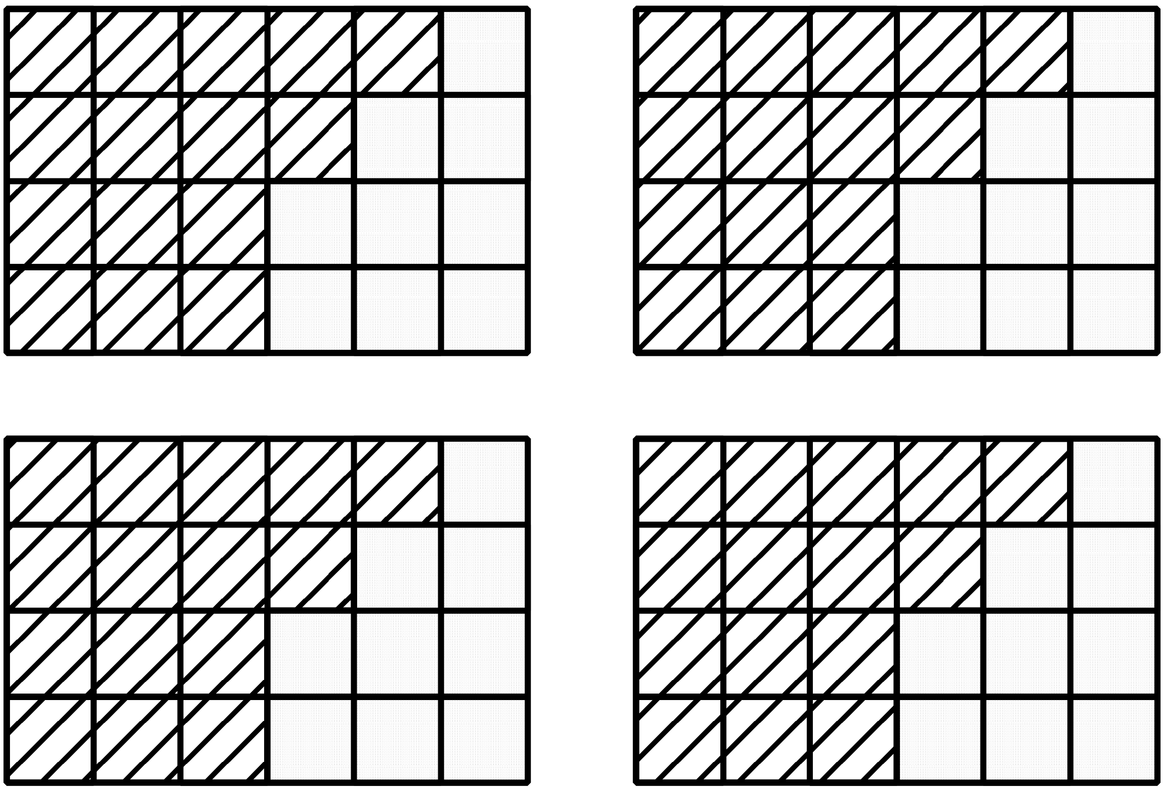
3 Player Fair Division
Howe's Method

Is this scheme fair?

1. Alice cuts the cake into two pieces which she believes to be equal.
2. Bob then cuts each of these halves into three pieces which he believes to be equal.
3. Carl then picks one piece from each half.
4. Bob picks one piece from the two remaining pieces in each half, and Alice gets the last pieces in each half.

What could go wrong?

Which is better ...



to be the divider or chooser?

$$: Divide: Diagonal____Unshaded____

$$: Choose: Diagonal____Unshaded____

Choosy Choosers Choose Chooser

When there is a mixture of items at stake, which different players place different values on, then not only can each player be guaranteed a fair (1/2) share of the total value, but the chooser can often get more than his fair share!

Note: The more the value systems differ, the more potential the chooser has for getting more than his fair share.

Strategies to Fairly Divide a Bag of Candy

Handout #1 — Fair Division

Here are four methods for dividing a cake among three people. Try them out in your groups. Analyze them to see which ones can fail to be fair and which ones will always yield a fair division. Remember that people may have their own ideas of fair which differ from yours, and so may do some strange things which seem strange to you but are considered fair by the person who does them.

Assume that Alice, Bob and Carl are the three players.

Boyd's Method:

1. Alice cuts a piece (called A) which she believes to be 1/3 of the pie.
2. Bob cuts the remainder into two pieces, B and C, which he believes to be equal.
3. Carl picks one of the three pieces, then Bob picks a piece from the remaining two, and Alice gets the last.

Dewey's Method:

1. Alice cuts the cake into what she believes to be equal thirds.
2. Bob and Carl each secretly write down which of the three pieces he believes to be at least a fair share.
3. If there are two different pieces, one that Bob liked and one that Carl liked, then these are given to them and Alice gets the remaining piece. Otherwise if Bob and Carl both liked *only* one piece, and it was the *same* piece, then Alice is given one of the other two pieces, which neither Bob nor Carl liked, and the remaining two pieces are put together. Bob and Carl then use the divider-chooser method to divide it up.

Cheetham's Method:

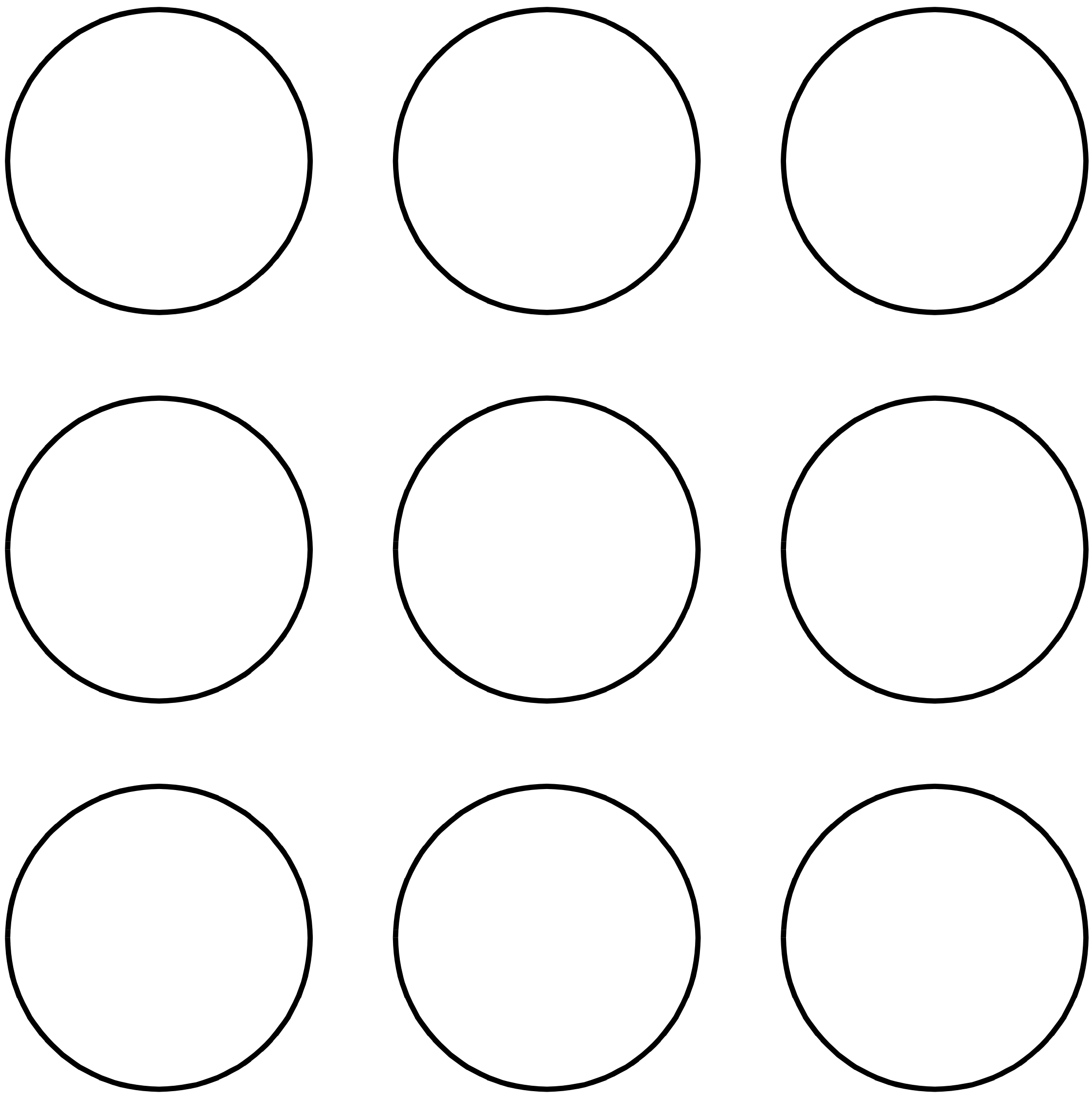
1. Alice and Bob use the divider-chooser method so that each one gets what he or she perceives to be at least half of the cake.
2. Alice and Bob then each divide their halves into three pieces which they believe to be of equal sizes.
3. Carl then picks one of Alice's three pieces and one of Bob's three pieces, and that forms his share. Alice and Bob are left with the remaining two pieces from their halves.

Howe's Method:

1. Alice cuts the cake in two pieces which she believes to be equal.
2. Bob then cuts each of these halves into three pieces which he believes to be equal.
3. Carl chooses one of the three pieces from each half.
4. Bob chooses one of the two remaining pieces from each half, and Alice gets the remainder.

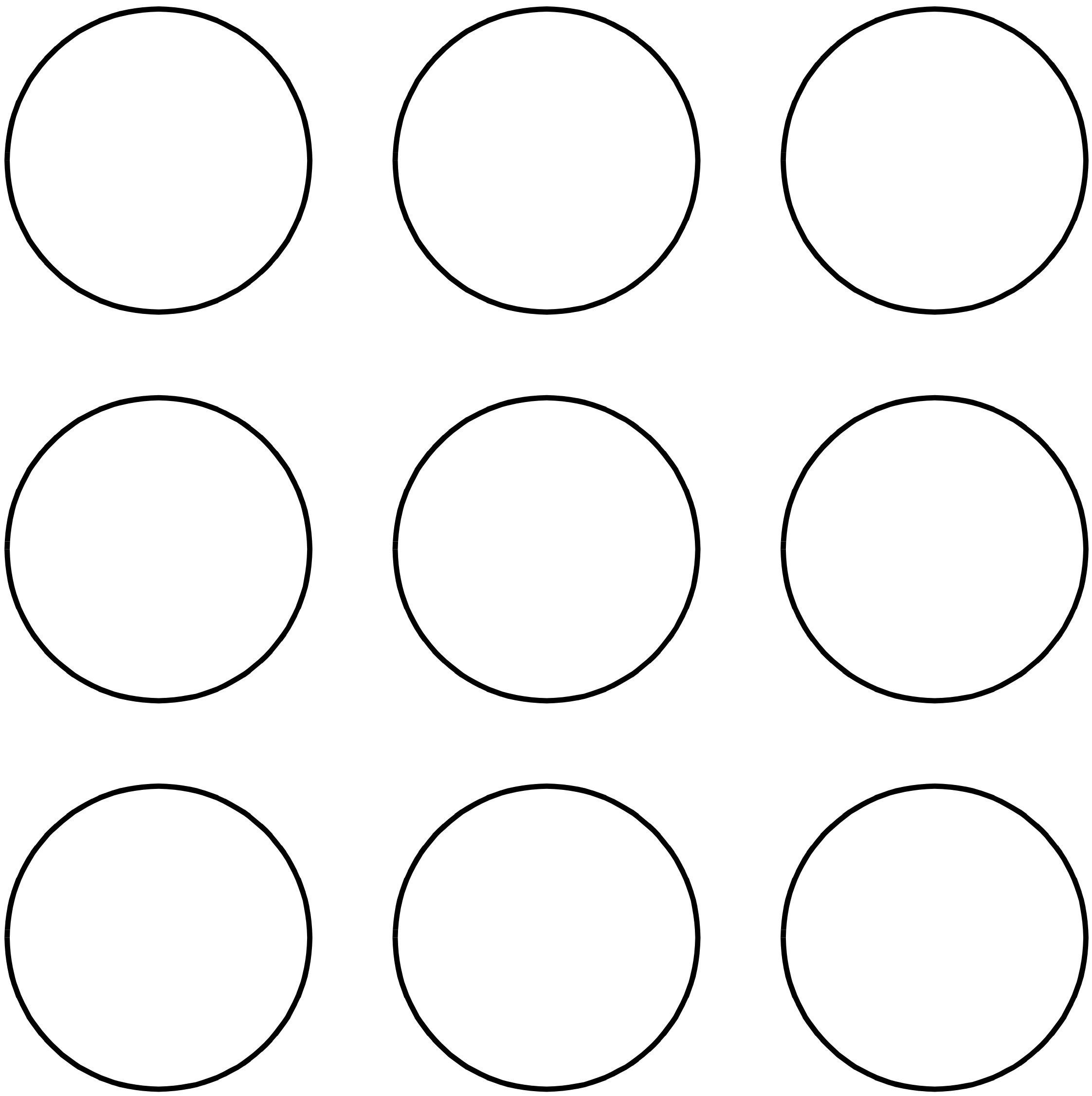
Handout #2 — Fair Division

Here are some circles for you to fairly divide.



Handout #3 — Fair Division

Here are some more circles for you to fairly divide.

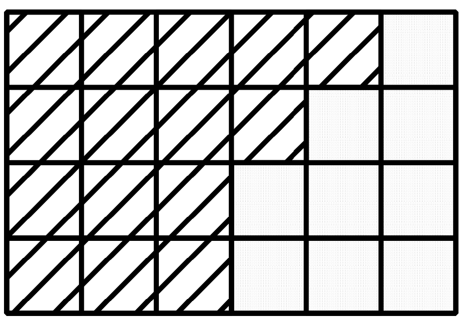


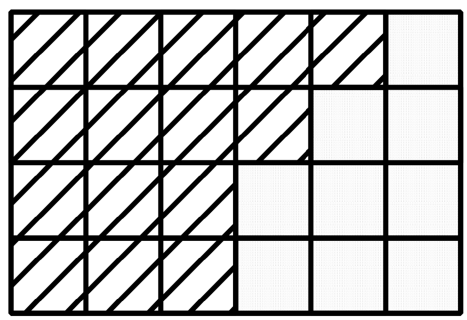
Handout #4 — Fair Division.

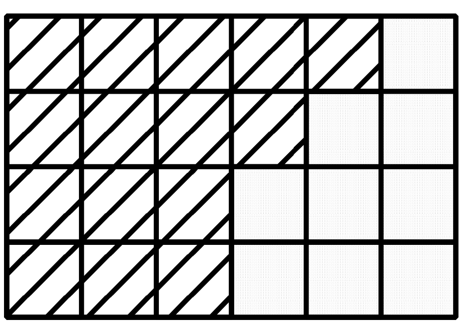
You and a partner will use the Divider-Chooser method to fairly divide the plot of land below. (4 copies of it are shown.) This exercise will help to emphasize just what is so fair about fair division.

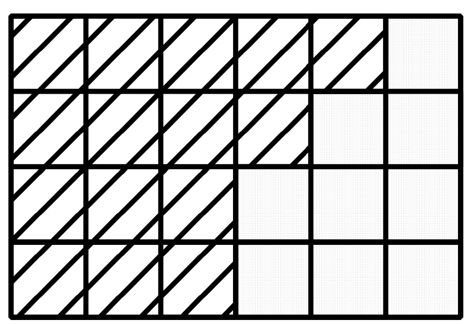
The figures below represent a plot of 24 acres of land consisting of woodland (diagonal lines) and grassland (unshaded).

1. You and your partner should each secretly write down how much you feel an acre of each type of land is worth (pick a whole number from 1 to 20 representing that many hundred dollars per acre.)
2. You and your partner should then decide the total value of the 24 acre plot, based on 1.
3. Use the divider-chooser method four times, each time the divider should choose a different way to divide up the land. You can decide who plays which role, on each of the four trials.
4. When this is done 4 times, compare what each person got to what each person expected to get. Is there a correlation between being the divider or chooser and how much land you got?







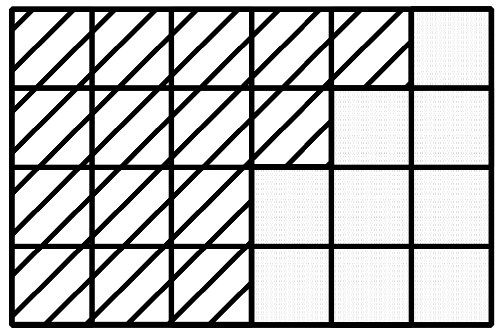


Handout #4a: Would you rather be divider or chooser?

Consider the following example: Bob and Carol jointly own the 24 acres of land pictured below. The striped area is woodland and the shaded area is grassland. They decide to divide the land. Now Bob is a woodsman and prefers woodland and Carol is a farmer and prefers grassland.

Bob assigns a value of $7,000 to each acre of woodland but only $5,000 to each acre of grassland.

How much does Bob think the land is worth? _______
How much does he think his fair share to be worth? _______



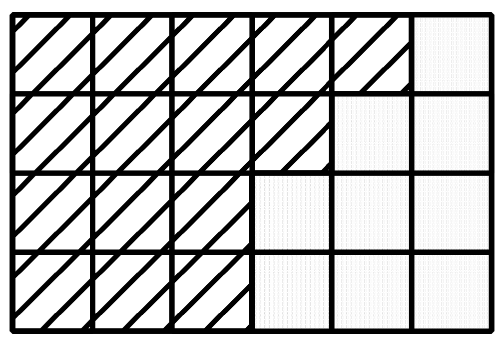
Carol assigns a value of $5,000 to each acre of woodland and $7,000 to each acre of grassland.

How much does Carol think the land is worth? ______
How much does she think her fair share to be worth? ______

Suppose that Bob is the divider. He wants to divide the land into two parts, each of equal value. Show on the first diagram a way of dividing the land so that the two parts have equal value to Bob. What is that value? _______

Now imagine that you are Carol. How much are each of those parts worth to you? ______ and _____ Which part will you choose? _______ Did you get more than, less than, or equal to what you considered to be your fair share? ___________ Who benefited most, the divider or the chooser? ____

Suppose, on the other hand, that Carol is the divider. She wants to divide the land into two parts, each of equal value. Show on the second diagram a way of dividing the land so that the two parts have equal value to Carol. What is that value? _______



Now imagine that you are Bob. How much are each of those parts worth to you? ______ and _____ Which part will you choose? _______ Did you get more than, less than, or equal to what you considered to be your fair share? ___________ Who benefited most, the divider or the chooser? ____

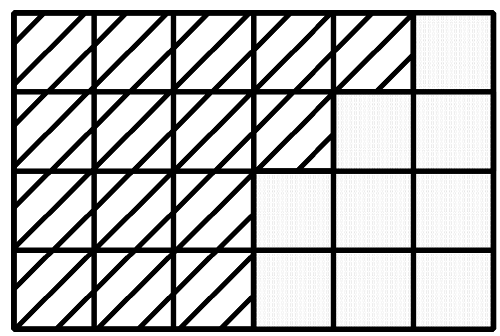
Would you rather be divider or chooser?

Handout 4b: Would you rather be divider or chooser?

Consider the following example: Bob and Carol jointly own the 24 acres of land pictured below. The striped area is woodland and the shaded area is grassland. They decide to divide the land. Now Bob is a woodsman and prefers woodland and Carol is a farmer and prefers grassland.

Bob assigns a value of $7,000 to each acre of woodland but only $5,000 to each acre of grassland. Since there are 15 acres of woodland and 9 acres of grassland, Bob believes that the total value of the property is 15x7 + 5x9 = 105 + 45 = $150,000. So when the land is divided, Bob wants to make sure that he gets $75,000 worth of land.

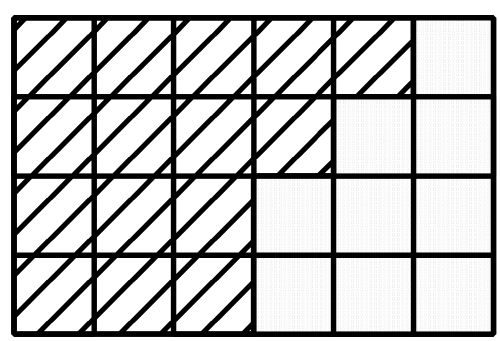
Carol assigns a value of $5,000 to each acre of woodland and $7,000 to each acre of grassland. She believes that the total value of the property is 15x5 + 9x7 = 75 + 63 = $138,000. So when the land is divide, Carol wants to make sure that she gets $69,000 worth of land.



Suppose that Bob is the divider. He wants to divide the land into two parts, each of which is worth $75,000. Then, whichever part Carol chooses, the remaining part will be worth $75,000 to Bob. For example, he can put into one part 5 acres of woodland ($35,000) and 8 acres of grassland ($40,000). The other part will have 10 acres of woodland ($70,000) and one acre of grassland ($5,000). This division is pictured above.

Which part will Carol choose? The part with 10 acres of woodland and 1 acre of grassland is worth 10x5 + 1x7 = $57,000 to Carol, whereas the part with 5 acres of woodland and 8 acres of grassland is worth 5x5 + 8x7 = $81,000 to Carol, so she will certainly take that part of the land. So the land that Bob gets is worth exactly half of the total value that the land had for him, but the part that Carol gets is worth more than half of the total value that the land had for her.

Suppose on the other hand that Carol is the divider. She wants to divide the land into two parts, each of which is worth $69,000. Then, whichever part Bob chooses, the remaining part will be worth $69,000 to Carol. For example, she can put into one part 4 acres of woodland ($20,000) and 7 acres of grassland ($49,000). The other part will have 11 acres of woodland ($55,000) and 2 acres of grassland ($14,000). This division is pictured at the right.



Which part will Bob choose? The part with 4 acres of woodland and 7 acres of grassland is worth 4x7 + 7x5 $63,000 to Bob, whereas the part with 11 acres of woodland and 2 acres of grassland is worth 11x7 + 2x5 = $87,000 to Bob, so he will certainly take that part of the land. So the land that Carol gets is worth exactly half of the total value tha the land had for her, but the part that Bob gets is wroth more than half of the total value that the land had for him.

Would you rather be divider or chooser?

Method of Markers

The items to be divided by the four people are lined up in a long row. Each person imagines dividing the row into four segments, each of which would be acceptable as his fair share of the items.

Each person inserts three distinguishable markers between the four segments that he/she imagined. The four people put their markers down at the same time so that no one is using the location of the other people's markers to improve the placement of his or her own markers.

To determine who gets which items, proceed as follows from left to right along the list of items:

1. The person with the earliest first marker gets all the items to the left of his/her first marker; all that person's markers are then removed.

2. The person with the earliest second marker gets all the items between his/her first and second markers; all that person's markers are then removed.

3. The person with the earliest third marker gets all the items between his/her second and third markers; all that person's markers are then removed.

4. The other person gets all the items to the right of his/her third marker.

Homework Problems for Fair Division Workshop

1. Three players want to divide a cake fairly using the lone divider method. The divider cuts the cake into three slices (slice 1, slice 2 and slice 3).

(a) Describe the possible fair division of the cake if the chooser declarations are:
Chooser 1: Slice 1
Chooser 2: Slice 3

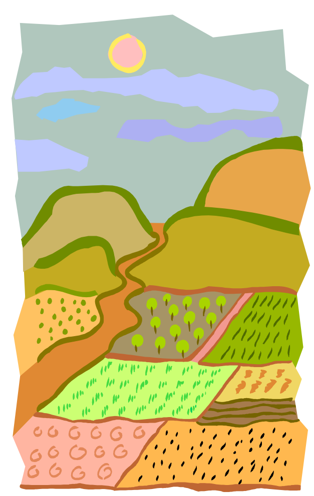
(b) Describe the possible fair division of the cake if the chooser declarations are:
Chooser 1: Slice 1, Slice 2, Slice 3
Chooser 2: Slice 3

(c) Describe the possible fair division of the cake if the chooser declarations are:
Chooser 1: Slice 2, Slice 3
Chooser 2: Slice 1, Slice 2

(d) Describe the possible fair division of the cake if the chooser declarations are:
Chooser 1: Slice 2
Chooser 2: Slice 2

2. Four partners (Adams, Benson, Cagle, Duncan) jointly own a piece of land which is subdivided into four parcels. The following table shows the percentage of the value of the land that each parcel represents to each partner.

	Parcel 1	Parcel 2	Parcel 3	Parcel 4
Adams	30%	24%	20%	26%
Benson	35%	25%	20%	20%
Cagle	25%	15%	40%	20%
Duncan	20%	20%	20%	40%



(a) Indicate which of the four parcels are fair shares to Adams.

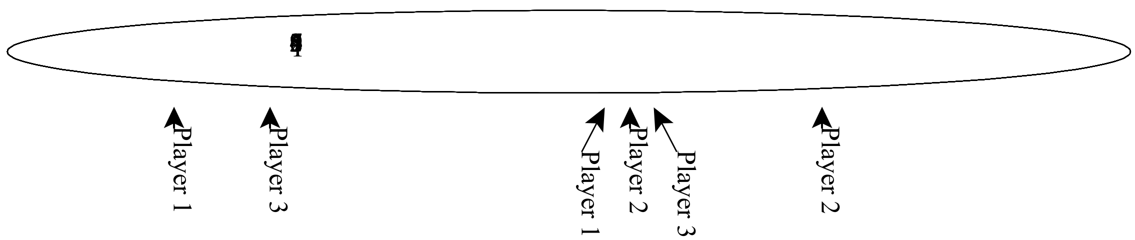
(b) Indicate which of the four parcels are fair shares to Benson.

(c) Indicate which of the four parcels are fair shares to Cagle.

(d) Indicate which of the four shares are fair shares to Duncan.

(e) Assuming that the four parcels cannot be changed or further sub-divided, describe a fair division of the land.

3. Three players agree to divide 12 items shown by lining them up in order and using the method of markers. The player's bids are as indicated.



(a) Describe the allocation of items to each player.

(b) Which items are left over?

(c) How might you divide the remaining items?

4. Every Friday night, Marty's Ice Cream Parlor sells "Kitchen Sink Sundaes" (KiSS) for $6.00 each. A KiSS consists of 12 mixed scoops of whatever flavors Marty wants to get rid of. The customer has no choice. Three friends (Abe, Babe, and Cassandra) decide to share one. Abe wants to eat half of it and pays $3.00 while Babe and Cassie pay $1.50 each. They decide to divide it by the lone-divider method. Abe spoons the sundae onto four plates (plate 1, plate 2, plate 3 and plate 4) and says that he will be satisfied with any two of them.

(a) If both Babe and Cassie find only plate 2 and plate 3 acceptable, discuss how to proceed.

(b) If Babe finds only plate 2 and plate 3 acceptable and Cassie finds only plate 1 and Plate 4 acceptable, discuss how to proceed.

(c) If Babe and Cassie both find only plate 3 acceptable, discuss how to proceed.

Fair Division

Resource Materials

Is this a good 2-player scheme?

- Alice and Bob have a cake to divide.
- Alice cuts.
- Then Alice chooses.

Is it Fair?

What, if anything, could go wrong?

Is this a good 2-player scheme?

- Alice and Bob have a cake to divide.
- Alice divides.
- Bob chooses.

Is it Fair?
What could go wrong?

This scheme is called

Divider-Chooser

3 Player Fair Division
Boyd's Method

Is this scheme fair?

1. Alice cuts a piece, (1/3), called A
2. Bob cuts the remainder into two pieces, B and C.
3. Carl picks one of the pieces, then Bob picks a piece, then Alice gets the last.

Is it Fair?

What could go wrong?

Lone Divider Method
Dewey's Method
Three people, Alice, Bob and Carl, wish to divide a cake between them.

1. Alice (the lone divider) cuts the cake into what she considers equal thirds.
2. Bob and Carl each secretly write down which of the three pieces(at least one) they would be happy with.
3. If this results in a satisfactory division, then they take their choices and Alice gets the remainder.
4. If both Bob and Carl will each only accept the same piece, say P, then they give one of the *other* pieces to Alice, put the remaining two pieces together, and use divider-chooser.

The Lone Chooser Method
Cheetham's Method

Three people, Alice, Bob and Carl, wish to divide a cake between them.

1. Alice and Bob use the divider-chooser method so that each gets what he or she believes is at least half.
2. Alice and Bob then each divide their halves into three pieces which they believe to be of equal sizes.
3. Carl then picks one of Alice's three pieces and one of Bob's three pieces, and that forms his share. Alice and Bob are left with the remaining two pieces from their halves.

3 Player Fair Division
Howe's Method

Is this scheme fair?

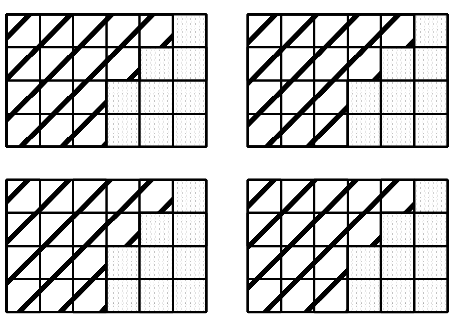
1. Alice cuts the cake into two pieces which she believes to be equal.
2. Bob then cuts each of these halves into three pieces which he believes to be equal.
3. Carl then picks one piece from each half.
4. Bob picks one piece from the two remaining pieces in each half, and Alice gets the last pieces in each half.

What could go wrong?

It is often better to be the **chooser** rather than the **divider**.

$$: Divider: Diagonal____Unshaded____
$$: Chooser: Diagonal____Unshaded____

Solution? Flip a coin first. Still, each player will get at least ½ (in their own estimation) But chooser may get more



Fair Division Scheme

Should be:

- **Fair**: If there are 8 players, for example, then each player can guarantee that he receives 1/8 of the whole.
- **Internal**: The scheme should not require the intervention of any outside judge or arbiter.
- **Decisive**: If the rules are followed, the scheme should always yield a fair division.

Handout #1—Fair Division

Here are four methods for dividing a cake among three people. Try them out in your groups. Analyze them to see which ones can fail to be fair and which ones will always yield a fair division. Remember that people may have their own ideas of fair which differ from yours, and so may do some strange things which seem strange to you but are considered fair by the person who does them.

Assume that Alice, Bob and Carl are the three players.

Boyd's Method:
1. Alice cuts a piece (called A) which she believes to be 1/3 of the pie.
2. Bob cuts the remainder into two pieces, B and C, which he believes to be equal.
3. Carl picks one of the three pieces, then Bob picks a piece from the remaining two, and Alice gets the last.

Dewey's Method:
1. Alice cuts the cake into what she believes to be equal thirds.
2. Bob and Carl each secretly write down which of the three pieces he believes to be at least a fair share.
3. If there are two different pieces, one that Bob liked and one that Carl liked, then these are given to them and Alice gets the remaining piece. Otherwise if Bob and Carl both liked *only* one piece, and it was the *same* piece, then Alice is given one of the other two pieces, which neither Bob nor Carl liked, and the remaining two pieces are put together. Bob and Carl then use the divider-chooser method to divide it up.

Cheetham's Method:
1. Alice and Bob us the divider-chooser method so that each one gets what he or she perceives to be at least half of the cake.
2. Alice and Bob then each divide their halves into three pieces which they believe to be of equal sizes.
3. Carl then picks one of Alice's three pieces and one of Bob's three pieces, and that forms his share. Alice and Bob are left with the remaining two pieces from their halves.

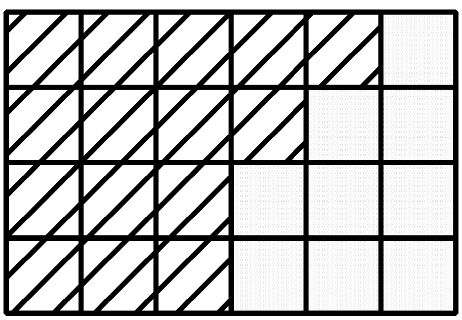
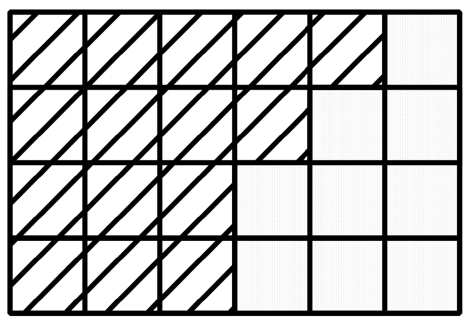
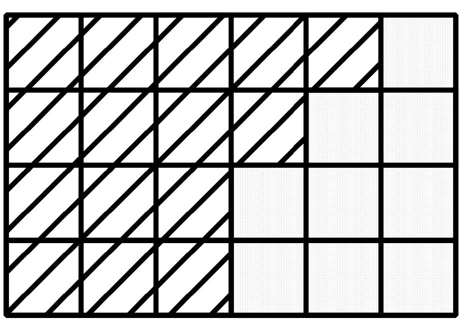
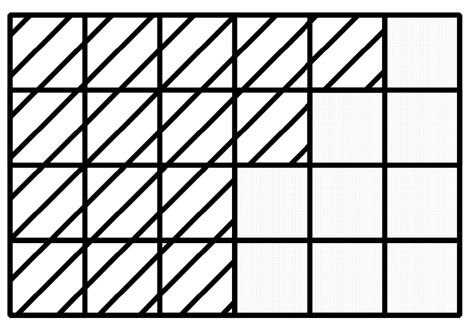
Howe's Method:
1. Alice cuts the cake in two pieces which she believes to be equal.
2. Bob then cuts each of these halves into three pieces which he believes to be equal.
3. Carl chooses one of the three pieces from each half.
4. Bob chooses one of the two remaining pieces from each half, and Alice gets the remainder.

Handout #4 — Fair Division.

You and a partner will use the Divider-Chooser method to fairly divide the plot of land below. (4 copies of it are shown.) This exercise will help to emphasize just what is so fair about fair division.

The figures below represent a plot of 24 acres of land consisting of woodland (diagonal lines) and grassland (unshaded).

1. You and your partner should each secretly write down how much you feel an acre of each type of land is worth (pick a whole number from 1 to 20 representing that many hundred dollars per acre.)
2. You and your partner should then decide the total value of the 24 acre plot, based on 1.
3. Use the divider-chooser method four times, each time the divider should choose a different way to divide up the land. You can decide who plays which role, on each of the four trials.
4. When this is done 4 times, compare what each person got to what each person expected to get. Is there a correlation between being the divider or chooser and how much land you got?

The Inspection Method
"The Last Diminisher"

1. **Ann cuts a part that she *feels* is exactly one third.**

2. **Bart inspects the piece. If he feels it is more than one third, he cuts enough from it so that he feels it is one third. The removed portion is returned.**

3. **Carl now inspects this piece with the same option.**

4. **The portion is given to the last person to cut from it.**

5. **The process continues in the same way.**

The Moving Knife Method

1. **A participant or a neutral party moves a knife across the object.**

2. **When any participant feels the knife has reached a fair third, he or she says "cut" and is given the resulting piece.**

3. **The process continues in the same way.**

There seems to be more coming after this, but I don't have a copy of last year's Resource Book, so I don't know what should be here.

Also, I need to add Hyperlinks to this document. Some of the referenced TSPs may be off.