Master Document

Putting Arrows on Graphs

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised November 30, 1997

Follow-up — Putting Arrows on Edges

Activity #1 — Euler Circuits in undirected graphs (15 min.)
(Allocated time = 15 minutes)

A. Put up the map of Discretia (TSP#2), distribute HO#1, and have the participants work in pairs to decide whether or not this map has an Euler circuit which starts and ends at the maintenance building.

Don't use the term "Euler circuit" at this point, since this would give it away. Rather, tell a story about a street cleaner or snowplow or something. Many participants will recall that a vertex of odd degree indicates that there is no such circuit, and this information will get around within each group pretty quickly. Regroup and discuss this with the class, having them point out the two vertices of odd degree, which the instructor may wish to label "A" and "B" for future reference and to ease discussion.

Now ask the participants to work within their groups to recall and discuss *the reason* that those vertices of odd degree preclude the existence of an Euler circuit. Give them a few minutes for this discussion, and when the groups seem to have each come up with some explanation, regroup and lead a discussion on the reason.

They should see that at A or B, you must enter, leave and then enter again, but then you can't leave because there are only 3 edges at that vertex. If they understand this reasoning, then the rule for directed graphs will seem very natural to them.

B. Put up the map of New Discretia (TSP#3), and ask the participants if this new layout will have an Euler circuit which starts and ends at the Maintenance building. According to the rule, it should. Distribute HO#2 and have them find such a circuit, labeling the edges in the order in which they are traversed.

It is good at this point to remind the participants that, not only does this new arrangement have an Euler Circuit, but it no longer has an Euler path. I.e., if you walk around the graph, crossing each edge exactly once, then you must end up where you start.

Activity #2 — Introduction to directed graphs and Eulerian digraphs
(Allocated time = 40 min.)

A. The mayor of Discretia has decided that traffic movement would be greatly facilitated if he made all the streets one-way. (He also realized this would help to increase revenues by giving more tickets to poor unsuspecting drivers.) Put up the map of New Discretia (TSP#4) with the one-way streets, distribute HO#3, and ask the participants: "Now what do you think? Is it possible for a snowplow (or something) to leave from the maintenance shed, traverse all streets exactly once in the proper direction, and end at the maintenance shed?"

In fact, you can't, and once again we are hoping that the participants will be able to discover the reason for this. The problems are that one vertex must be entered 3 times, but can only be exited once, and another vertex has the reverse problem. Give plenty of time for the groups to discover and verbalize these reasons. Encourage them to write down their reasons in their own words.

When they have come up with their reasons, regroup and put up the terminology slide (TSP#5), going over it with them, telling them that this is language that will facilitate the discussion of directed graphs. That done, ask them for their reasons that there is no circuit. Guide them until they start to talk in terms of "number of arrows into or out of a vertex", and then put up TSP#6 showing the definitions of indegree and outdegree.

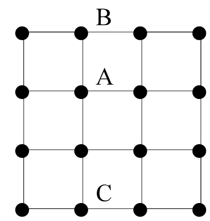
It is likely that at this point, some participant will suggest a way to change the orientations on two arrows in order to get an Eulerian digraph. (If not, then you can bring the issue up yourself.) You may wish to let participants volunteer ways to alter the graph to Eulerize it, and then lead them to notice that each way of reversing arcs to obtain an Eulerian digraph amounts to a path between the two offending vertices. This happened spontaneously once or twice in the past, and both times the discussion was worthwhile. But it might be a mistake to force the issue.

B. Distribute HO#4 (TSP#7) and ask the participants to determine which digraphs have an Euler circuit or Euler path. Let them try to develop criteria in their groups. Then put up the summary TSP#8 with criteria for both Euler circuits and paths. When this discussion is done, ask them to discover how the digraphs which did not have Euler circuits or paths could be modified so that they did have such circuits or paths.

There are two distinctions to make here: participants might add arcs or reverse arcs or try some combination of both. Since building a new road is much more expensive than replacing one-way signs, adding new arcs should be thought of as less attractive than reversing the direction of arcs. Thus it is reasonable to ask them to discover "the minimum number of arcs that need to be reversed so that the resulting digraph has an Euler path or circuit." Note: This discussion has the potential to bog down...be careful not to get sidetracked by too many tangents.

If the instructor has any need, TSP#9, showing several digraphs, can be used as further examples. Note that graphs C and D show digraphs which have Euler paths, but not Euler circuits.

Activity #3 — Digraphs model one-way street assignments
(Allocated time = 20 min)



A. On the floor, ahead of time, there should be taped a square grid looking something like the figure to the right. (The edges should be about 2 feet long each. There is no need to have pronounced vertices, such as by putting paper plates there.) Give a short arrow to each of 24 participants, so that there are as many arrows as edges. Explain to them that when you say "go," you want them to lay down their arrows so that each edge gets one arrow, but they can choose the directions however they wish. Then say "go."

When this is done, ask them to imagine that the resulting digraph models one-way street assignments in the a city. Ask them what properties this one-way street assignment has. (Since it has so-far been the topic du jour, the participants will probably point out that Eulerianness fails. Can this be fixed by reversing the directions on some of the arrows? No, since there are too many vertices of odd degree. So now make a big deal out of the fact that we are no longer interested in discussing the Eulerianness of a graph. Instead, we wish to discover another property that digraphs may or may not have---namely, strong-connectedness---the property that you can get from any vertex to any other vertex following the arrows.) Encourage suggestions and allow time for them to verbalize this property. It is likely that there will be at least one vertex with outdegree 0 or indegree 0, and this vertex will motivate the notion. After they have brought this up, ask them if the arrows can be rearranged so that "you can always get there from here."

The answer is yes. But how would you prove that a given arrangement of arrows has such a property? There are 16 possible starting vertices in this graph, and for each of those, there are 15 other vertices you could wish to visit...thus it looks like there are 240 pairs that need to be checked (this is a good review question to ask the participants.) But there is a much easier way to verify that you can always "get there from here," no matter where "here" and "there" are. Namely, you can show that you can start at some vertex, follow some arrows, visit every vertex (not necessarily exactly once), and end up where you started!

B. Now introduce the terminology "*strongly connected*." Recap the previous activity by asking "So, if I showed you this graph without the arrows" (*here pick up the arrows*) "could you plan ahead and then lay down the arrows so that the resulting digraph is strongly connected?" "Of course", should come back the answer. Now remove the tape corresponding to edge A in the figure above, and ask if the resulting graph still has the property that you can lay down arrows so that the resulting graph is strongly connected. Have the participants try

this...they should discover that you can. Okay, so pick up the arrows, remove the tape corresponding to edge C, and have the participants try again. Once more, they should discover that this is possible. Okay, pick up the arrows, remove the tape corresponding to edge B, and ask them to try again. This time, they should discover that it is not possible.

Allow some discussion, but since you'll be doing more discussion after the worksheet has been handed out, don't let it go too long. Rather, distribute HO#5 and put up TSP#10. Have them work it through, deciding which of these graphs have strongly connected orientations and which haven't, while keeping an eye out for what it is about a graph that prevents there being such an orientation. This done, lead the class in discussion, agreeing on which graphs have strongly connected orientations and which don't. For ones which did, let someone show the orientation; for those which did not, encourage them to verbalize why.

Gradually, the concept should develop of an "only-bridge" in a graph, i.e., an edge in a graph which makes a strongly-connected orientation impossible. Take a moment to point out that we are using "only-bridge" as a technical term. Also, "only-bridge" is not the word used in the literature. There the word "bridge" is used, but we thought that was confusing so we used "only-bridge" for clarification.

After the discussion, or during it if needed, put up TSP#11 showing the definitions of "strongly connected" and "only-bridge." Review the graphs on the previous TSP, finding only-bridges in those which didn't have a strongly-connected orientation. Finally, after they have discovered the theorem on their own, put up TSP#12 showing Robbins' Theorem.

One notion that has been of some interest to the participants in the past is how easy it is to state and prove theorems. Euler's theorem for directed graphs and Robbins' theorem are two examples of theorems which the participants were able to discover (and essentially prove) on their own, with the instructor merely asking some leading questions. So in real life, it is easy to get a theorem named after yourself...you just have to be in the right place at the right time — with a good idea, but not necessarily an incredibly deep idea.

Activity #4 — Tournaments
(Allocated time = 40 min)

A. First make sure that each table has either 4 or 5 participants, but try to make more tables of 4 than 5. Distribute the table-size arrows (5 long and 5 short for each table) and give the participants the following instructions, shown on TSP#13.

- **At each table, each pair of participants should play a game of Rock-Paper-Scissors. (If there is anyone who is unfamiliar with this game, you should take a minute to explain the rules, as shown on the slide.)**
- **After each game, an arrow should be directed *from* the winner *to* the loser.**

Walk around as they do this, reminding them as necessary that the arrow goes from

winner to loser. Give each table a blank TSP on which they should record the outcome of their tournament, by drawing the digraph which is produced on their table. Also, keep an eye out for tournaments which have interesting properties, such as a vertex with indegree 0 or outdegree 0, a Hamilton circuit or transitivity. You can draw on these interesting examples later when you answer some questions about tournaments. The definition of tournament is shown at the top of TSP#14.

After10 or 6 games have been played at each table, turn the class's interest to determining Rock-Paper-Scissors champions at each table. Display the TSP showing the tournament created at one of the tables, and explain that you will be finding champions. Explain that there may be more than one champion at a table. Some tables may have just one champion, and at some tables it might happen that everyone is a champion. "This is the way we will be using the word 'champion.'"
Tell them that you will use the following scheme to find champions: To determine the champion at a given table, you will look at the digraph that resulted from the tournament and attempt to find an ordering of the players A, B, C, D, E (at a table of 5, say) so that A beats B beats C beats D beats E, i.e., A⇨B⇨C⇨D⇨E. This is illustrated at the bottom of TSP#14. We will call such a sequence a "champion sequence." When such a sequence is found, we will call "A", the first person in the sequence, a champion. Now let them do this at their own tables to find champions.

> *As they work on this, walk around and continue to keep an eye out for tournaments with interesting properties. Later on, as you need examples of various things, you can ask a table to give you their slide. This will make your discussion much more tangible. It generally happens that some table will produce a transitive tournament*

When you regroup and begin class discussion of champions, ask them to volunteer interesting phenomena that they noticed as they looked for champions. Interesting phenomena might include:

- A group had more than one champion.
- Someone won all their games or lost all their games.
- Each group had at least one champion.
- A group's digraph had a Hamiltonian cycle, giving lots of champions.
- A group may have counted the number of champion sequences that their tournament had (not easy to count for non-transitive tournaments.)
- The tournament was transitive, which may be phrased as "A won 4 games, B won 3, C won 2, D won 1, and E won 0 games," or in some other such way.

As you gather this information from them, put up slides from tables that can exhibit the properties. Things that you should really make a point of, so that the questions below will be somewhat motivated, even if you have to motivate it yourself, are

- Vertices of indegree 0 are the only candidates for championship.
- Vertices of outdegree 0 cannot be champions.

- If a tournament has the property that the competitors in the tournament can be lined up in a row so that all the arcs are directed from left to right, then there is just a single champion sequence in that tournament. A good way to make this point might be the following: put up a group whose tournament was transitive, and ask "if I had given you only a thin strip of transparency film, like this" <*cover up all but a 1-inch wide strip*> "and asked you to draw your tournament, then you might have drawn it this way:" <*and draw letters representing the participants from left to right, so that all the arrows will go from left to right*.> Looking at the tournament they had already drawn in a circular fashion, ask them how the arcs would go on the linear drawing. Then you can discuss the properties of this tournament, showing how all the arcs go from left to right, and how there is just a single champion sequence.

Turn now to discussing some questions which might have arisen while polling them on their tournaments.

- Reveal the top part of TSP#15, asking if it is possible to have a tournament with just a single champion. They should say "yes," since you already discussed this together when vertices with indegree 0 came up. (If it would help, you can let them manipulate the arrows on their tables to construct such a tournament.) Point out that, even though in this case there may still be several ways to write down a champion sequence (like A⇨B⇨C⇨D⇨E), still A will always be the first term in any such sequence. You can mention the extension question, but save any explanation for later, if you have time.
- Reveal the question on the top part of TSP#16, asking if it is possible to have a tournament which has just a single champion *and* just a single champion sequence, showing that he is the champion. They should recall that the transitive tournament (here called "perfectly ordered" tournaments) seen earlier had this property. If it helps, again, they can manipulate the arrows on their tables to construct such a tournament. Show the extension problem. Again, if time permits, you can show a solution later.
- Reveal the third question (TSP#17), asking if it is possible to have a tournament which has no champion. Now encourage them to manipulate the arrows on their tables to try to construct such a tournament. Lead them to discover that there is no such tournament! Indeed it is true, that every tournament has an ordering of the players so that the first beats the second beats the third etc... You can let them try the extension problem if time permits.
- Finally, TSP#18 contains the very interesting Redei's theorem, but can be omitted if need be. The proof of this is involved, and won't even be attempted here. The interesting thing is that Redei's theorem implies that there will always be a champion sequence, and hence a champion.

Activity #5 — An algorithm for finding champion sequences
(Allocated time = any remaining min.)

Distribute HO#6 and put up TSP#19 showing a disguised transitive tournament. Have them work individually to find a champion sequence in this digraph. Since there's only one, unless they have a strategy for finding it, they will likely fail. The purpose of this exercise is to motivate the need for an algorithm for finding champion sequences in tournaments.

This graph is too easy, and needs to be changed!

Discuss the activity just completed and mention how good it would be to have an algorithm for finding a champion sequence in a tournament. Also, what we will be doing here is developing a proof that there will always be a champion sequence. Ask for 8 volunteers, and have them line up in the front of the room, on the left, holding the letters A, B, C, D, E, F, G, H written large on sheets of paper.

- Say to the class "pick a letter, any letter." Move that person (say A), with his letter, to the right side of the room, and announce that they are our first contribution to the champion sequence we are trying to build.
- Again ask "pick a letter, any letter." First try to put that person (say B) before A, which is only possible if B beats A. If not, then we can put that person after A, since then A beats B. This covers the first step as shown on TSP#20.
- Again ask "pick a letter, any letter." First try to put that person (say C) at the beginning of the champion sequence. If that fails, then try second, and if that fails, try last. This is step 2 on TSP#20. As you do this, the participants should be looking at the tournament (TSP#19) on the overhead projector or at their desks, and should be telling you where they can and can't be placed.
- Continue this until all 8 persons have been placed.
- Now lead the participants into noticing that this process can never fail, and so every tournament must have a champion sequence, and hence a champion.

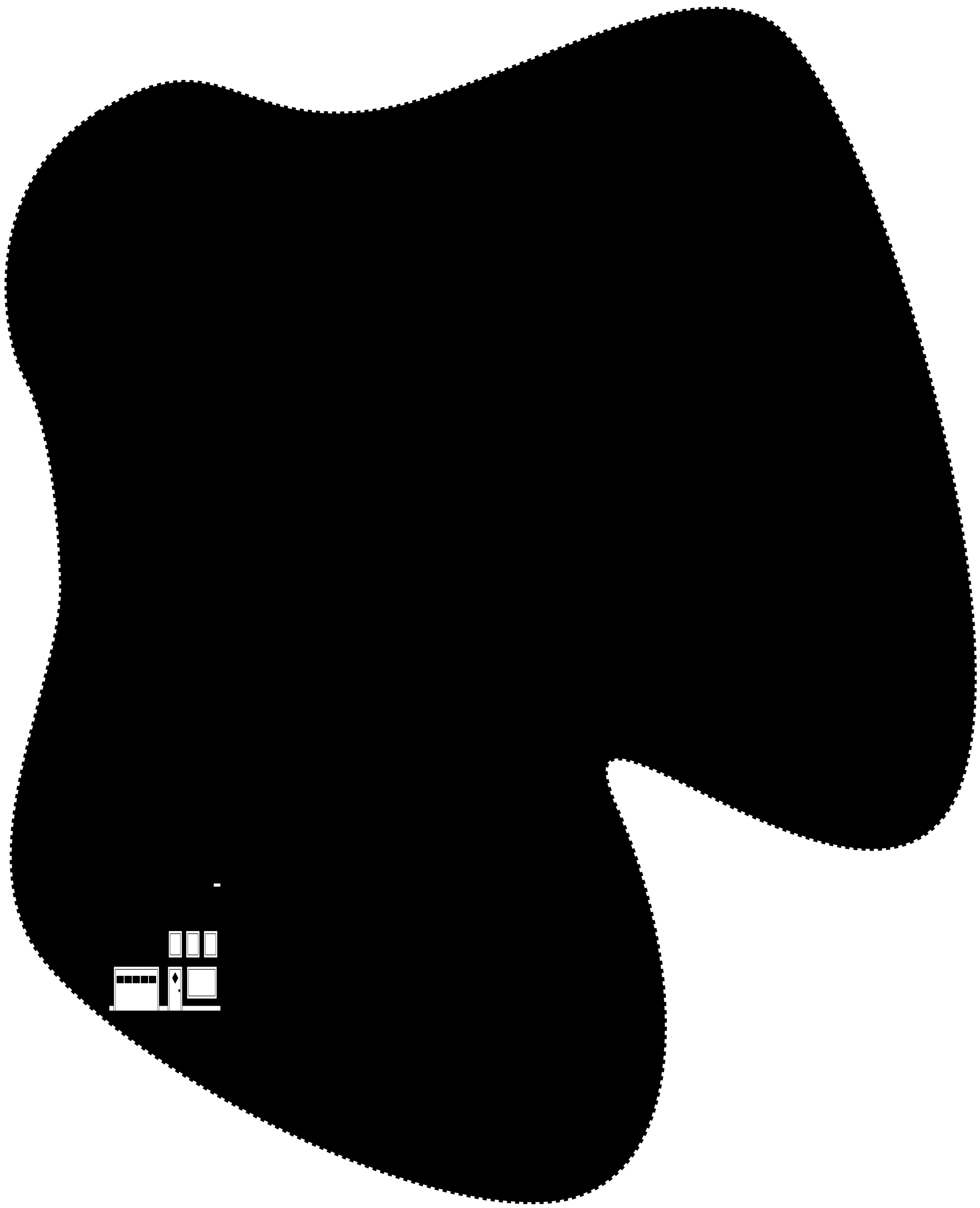
Put up TSP#20 showing a specific case of the algorithm (note, this is in their resource books), and then, if you think it will help, you can put up TSP#21 showing a more general algorithm (also in their resource books).

Putting Arrows on Graphs

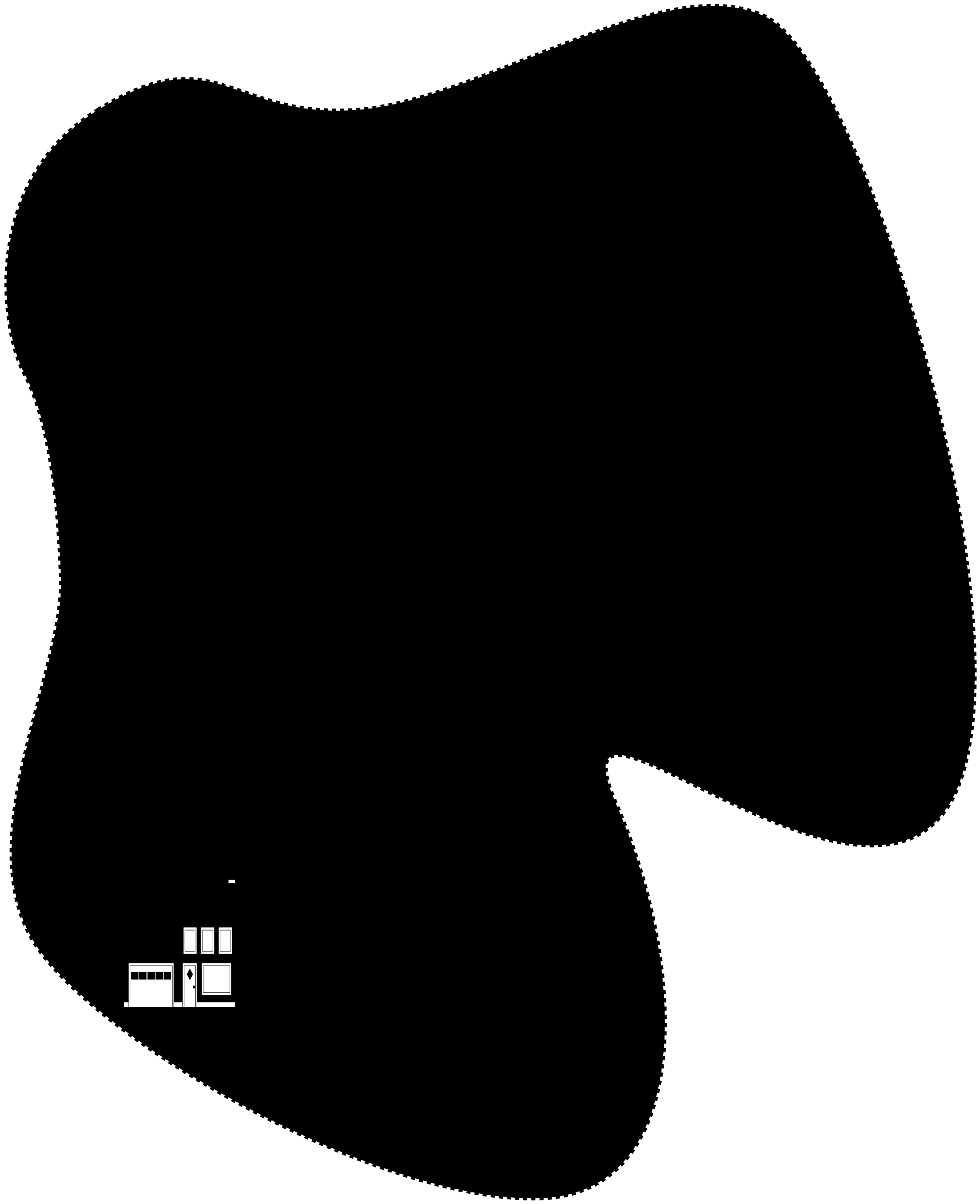
Follow-up session

December 1999

Discretia

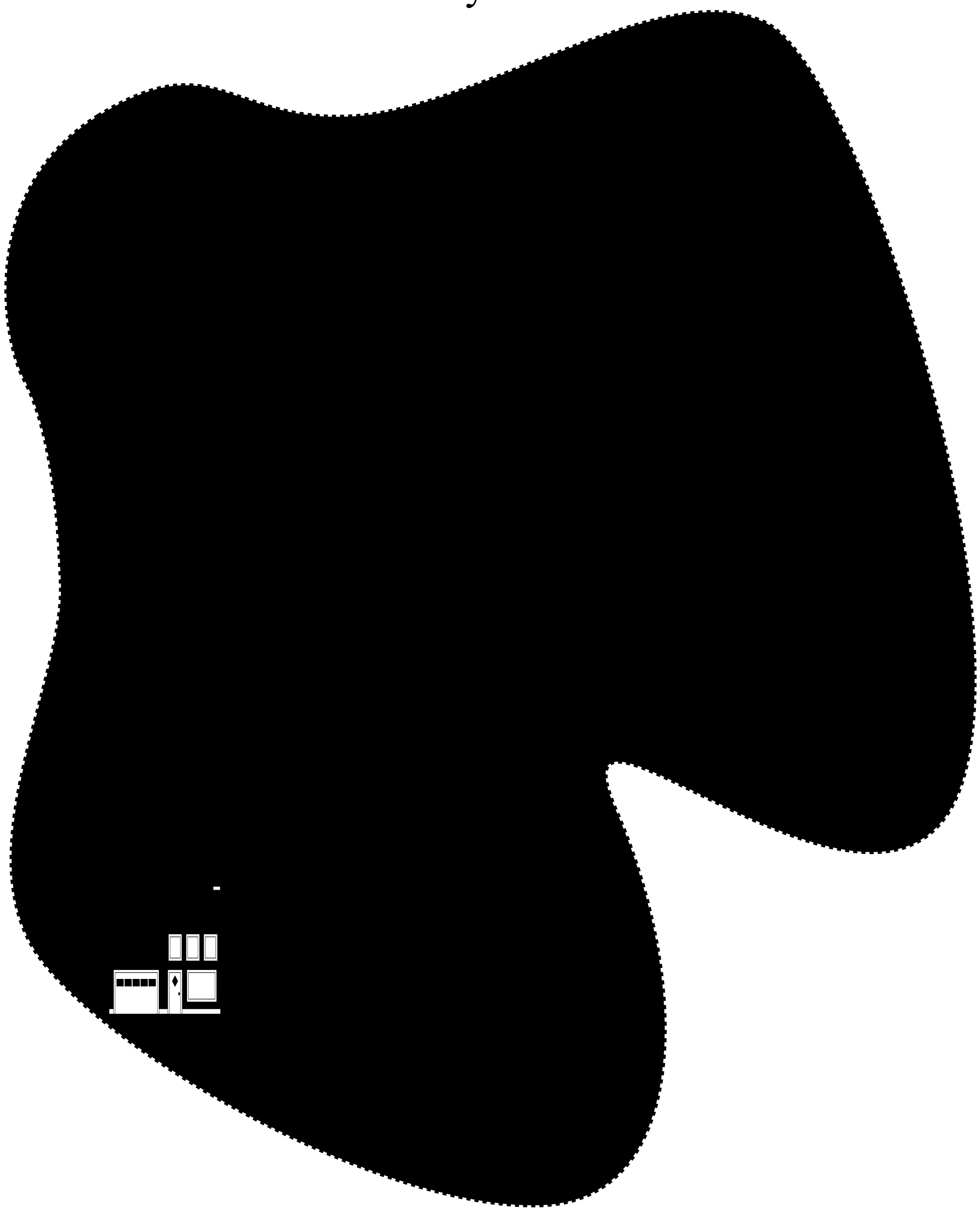


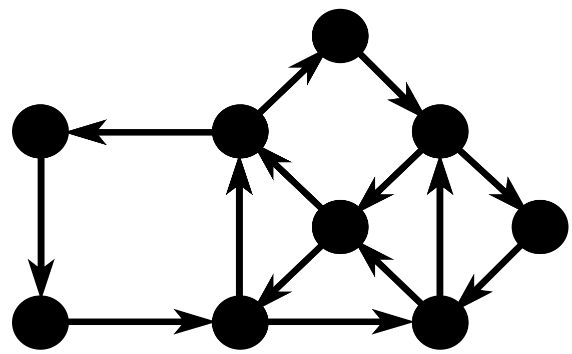
New Discretia



New Discretia

With one-way streets





Directed Edge: An edge with an arrow on it, usually restricting the direction of travel along that edge.

Arc: A directed edge.

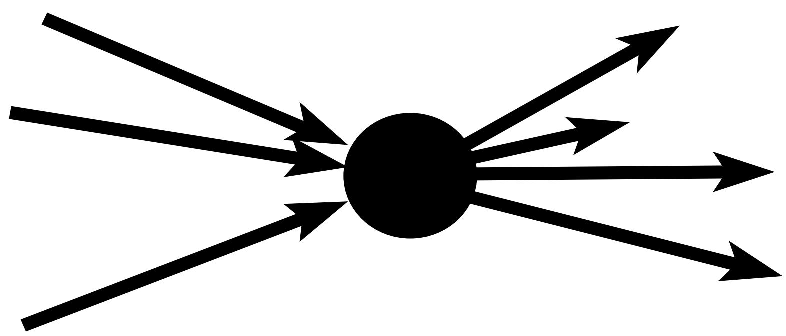
Directed Graph: A graph whose edges are all directed edges.

Digraph: A directed graph.

Directed Euler Circuit: A circuit in a digraph which follows the directions of the arrows and traverses each arc exactly once.

Indegree of a vertex: The number of arcs whose arrows point *into* that vertex.

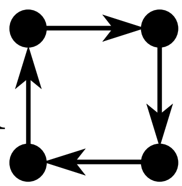
Outdegree of a vertex: The number of arcs whose arrows point *out of* that vertex.



Indegree = 3 Outdegree = 4

Let's try to discover when it is that a directed graph has an Euler Circuit or Path.

A



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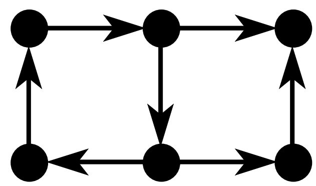

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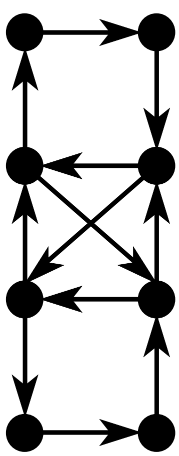

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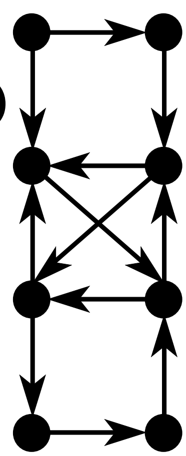

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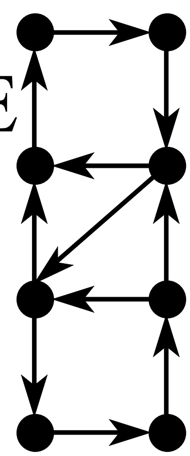

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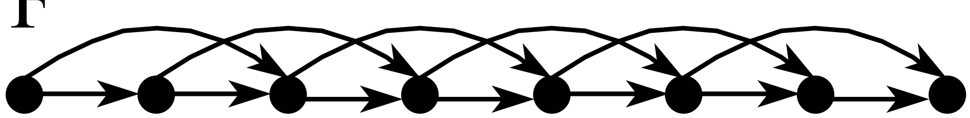


Euler's Theorems for directed graphs:

Circuits

A digraph has an Euler circuit only if indegree equals outdegree at every vertex.

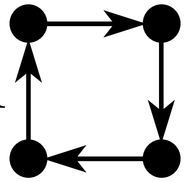
In fact, for any connected digraph, if indegree equals outdegree at every vertex, then there will exist an Euler circuit.

Paths

A digraph has an Euler path only if indegree equals outdegree at all but two vertices, and at those two vertices, indegree differs from outdegree by exactly one.

In fact, for any connected digraph, if indegree equals outdegree at all but two vertices, and at those two vertices, indegree differs from outdegree by exactly one, then there will exist an Euler path.

SOME EXAMPLES OF DIGRAPHS

A


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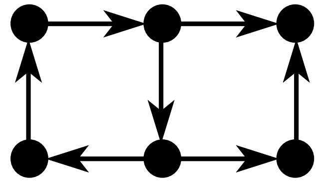

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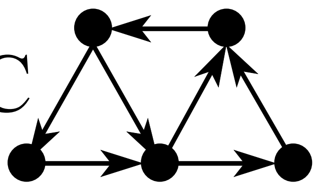

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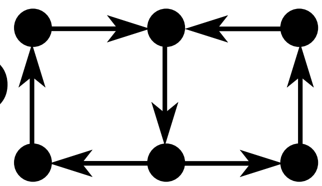

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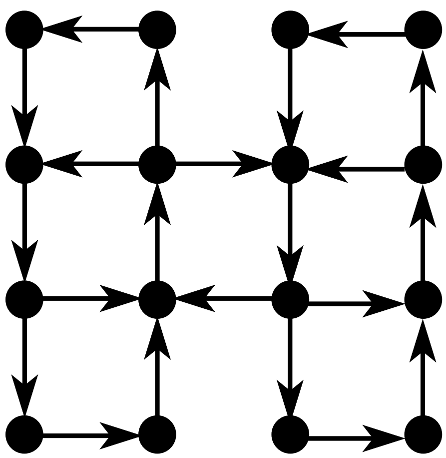

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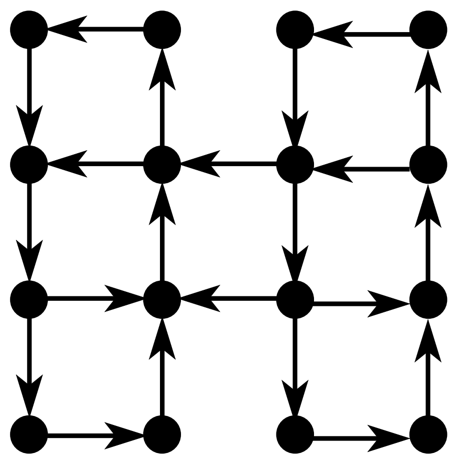

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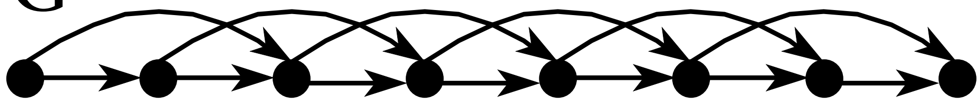
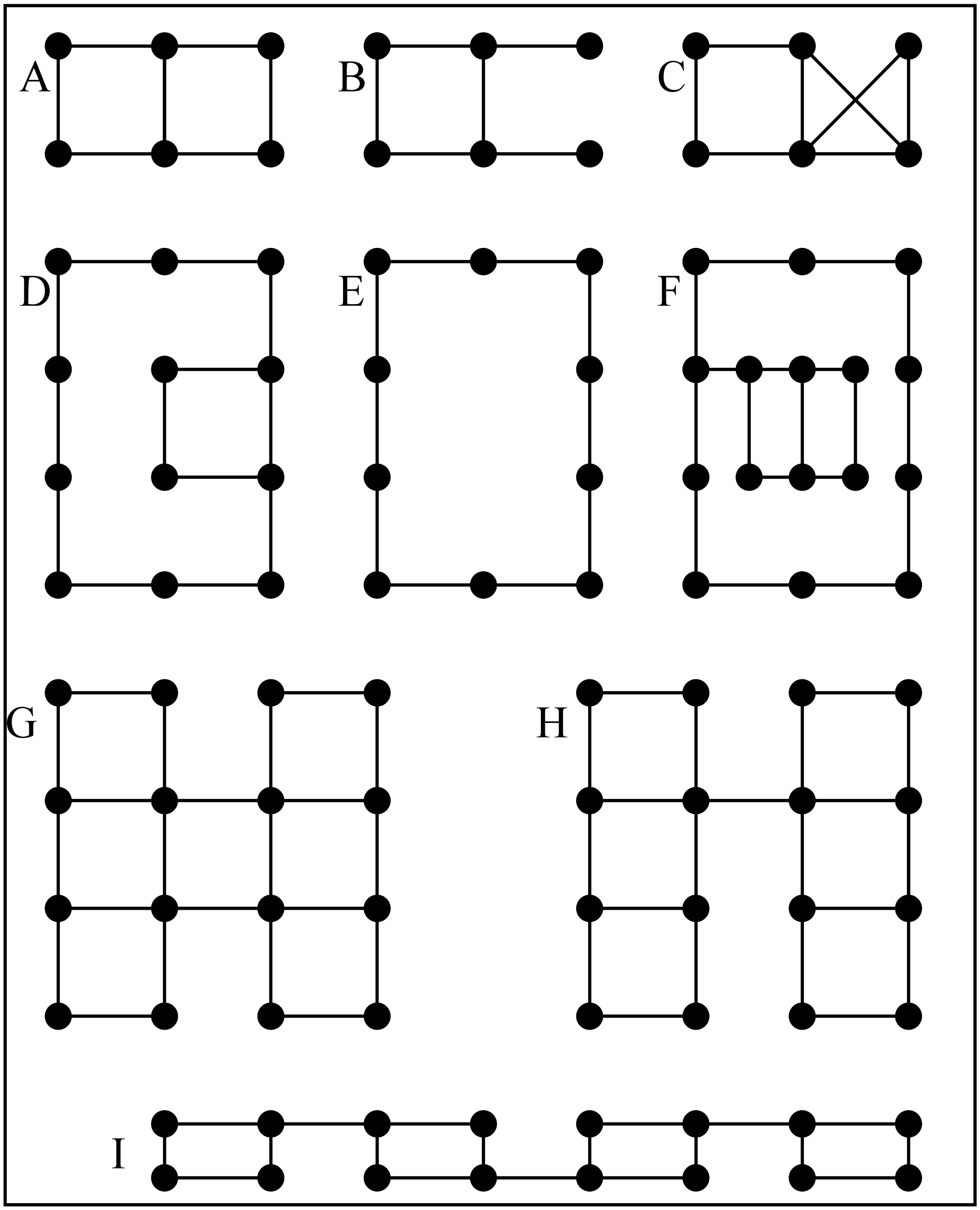
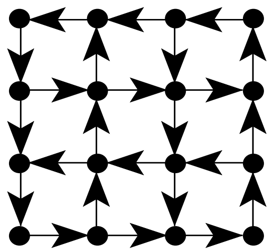


Which of these graphs can be oriented so that you can get from any vertex to any other vertex, while observing the directions on the edges?

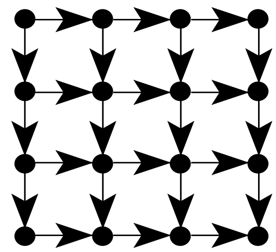


SOME TERMINOLOGY

Strongly Connected Digraph: A digraph is strongly connected if it is possible to get from every vertex to every other vertex by following a path in the direction of the arrows.

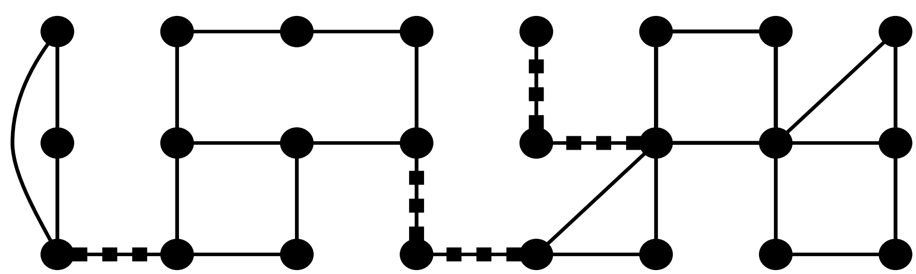


Strongly Connected



Not strongly connected

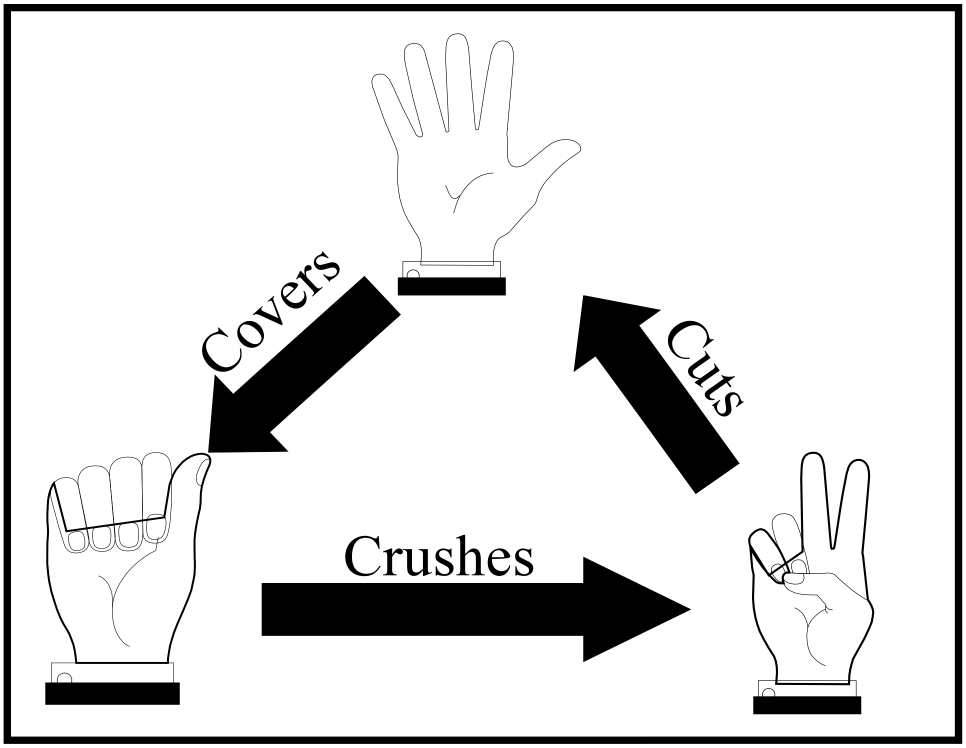
Only-bridge in a Graph: An edge in a connected graph whose removal results in a disconnected graph.



Robbins' Theorem

If the streets of a city form a connected graph which has no only-bridge, then the streets of that city can be made one-way so that any location can be reached from any other location by observing the directions on the streets

ROCK-PAPER-SCISSORS TOURNAMENT

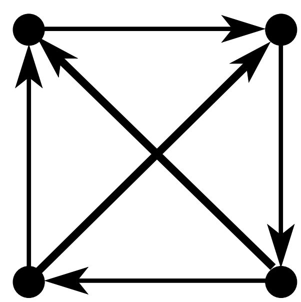


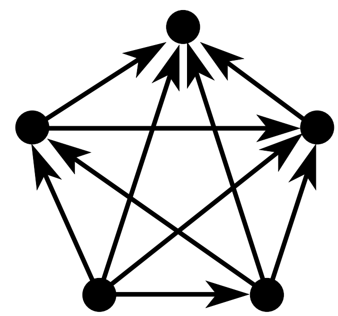
At each table, every pair of players should play a game of Rock-Paper-Scissors. Then lay down an arrow: Winner ⇨ Loser

TOURNAMENTS

Tournament: A complete graph in which every edge has been given a direction.

E.g.





Tournament Champions

Champion sequence: A sequence A, B, C, ... containing all the vertices, such that A→ B → C→ ···

Champion: The first vertex in a champion sequence.

Tournament Questions and Facts

Q1: Is it possible to have a tournament in which there is just a single champion?

A: Yes. If a tournament has a champion with indegree 0, then that vertex can be the only champion in the tournament.

Extension: Is there any *other* way for a tournament to have a just a single champion?

Tournament Questions and Facts

Q2: Is it possible to have a tournament in which there is just a single champion sequence?

A: Yes. If a tournament has an ordering of the vertices from left to right, so that every arrow goes from left to right, then this tournament will have just a single champion sequence. Such a tournament is called *perfectly ordered*.

Extension: Is there any *other* way for a tournament to have a just a single champion sequence?

Tournament Questions and Facts

Q3: Is it possible to have a tournament with no champion?

A: No. Every tournament has at least one champion.

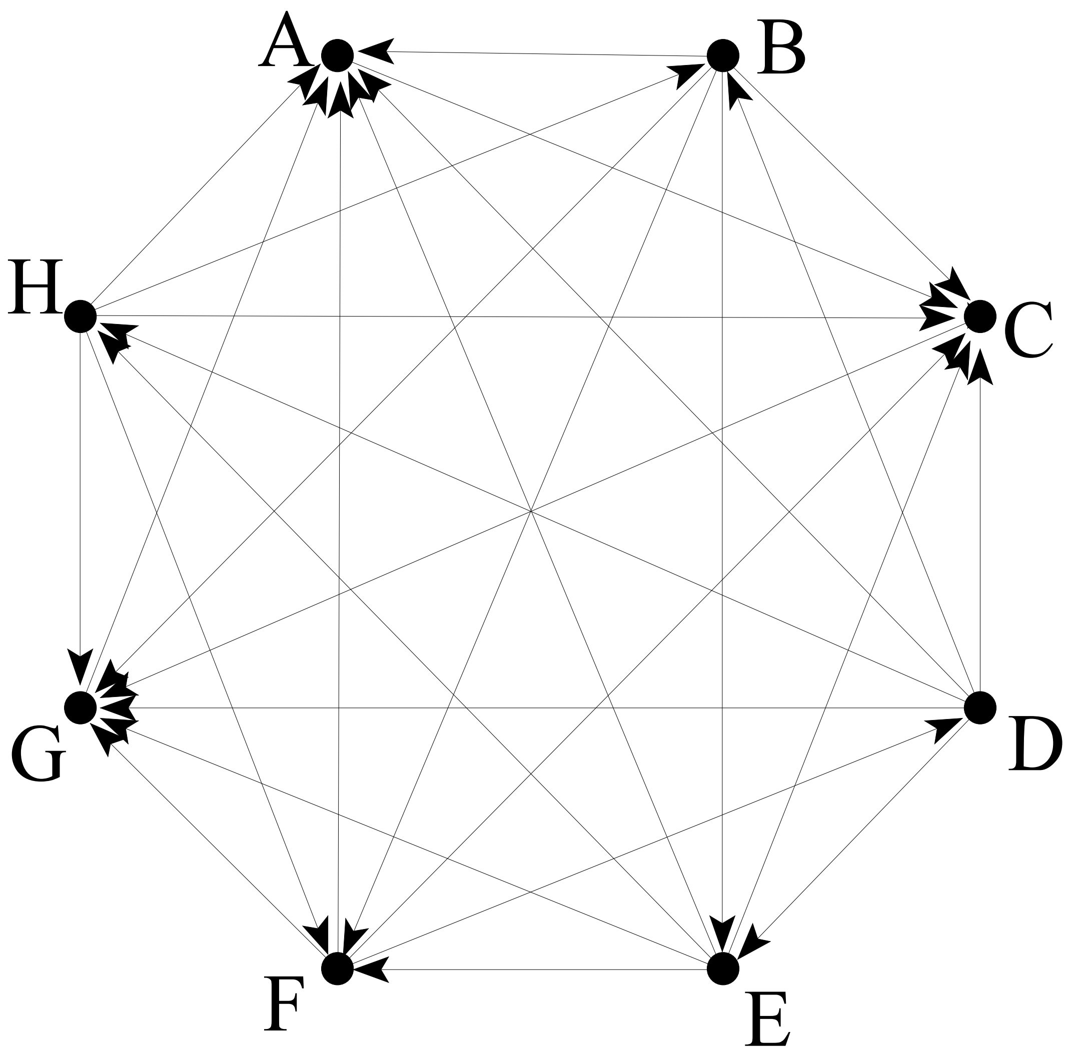
Extension: Is it possible to have a tournament with exactly 2 champions?

Tournament Questions and Facts

Q4: Is it possible to have a tournament with exactly 2 champion sequences?

A: No. Every tournament has an odd number of champion sequences. This is called Redei's Theorem.

Extension: Can you see why Redei's Theorem implies that every tournament has a champion?



Find a champion sequence in this tournament.

Algorithm for finding a champion sequence

1. Pick two vertices A and B. If A beats B, then write A→ B. Otherwise, write B→ A.

Let's assume B beats A, and consider the sequence B→ A. We will try to extend this sequence.

2. Pick a vertex "C" which is not on your sequence:
   a) If C beats B, then our new sequence becomes C→ B→ A.
   b) If C loses to B but beats A, then our new sequence becomes B→ C→ A.
   c) If C loses to B and A, then our new sequence becomes B→ A→ C.

Suppose that case b) happens, and we have B→ C→ A.

3. Pick a vertex "D" which is not on the sequence:
   a) If D beats B, then our new sequence becomes D→ B→ C→ A.
   b) If D loses to B but beats C, then our new sequence becomes B→ D→ C→ A.
   c) If D loses to B and C but beats A, then our new sequence becomes B→ C→ D→ A.
   d) If D loses to B, C and A, then our new sequence becomes B→ C→ A→ D.

Continue this until the sequence contains all vertices.

Algorithm for finding a champion sequence

1. Pick two vertices A and B. If A beats B, then write A→B. Otherwise, write B→ A. Call this your "champion-path-so-far".

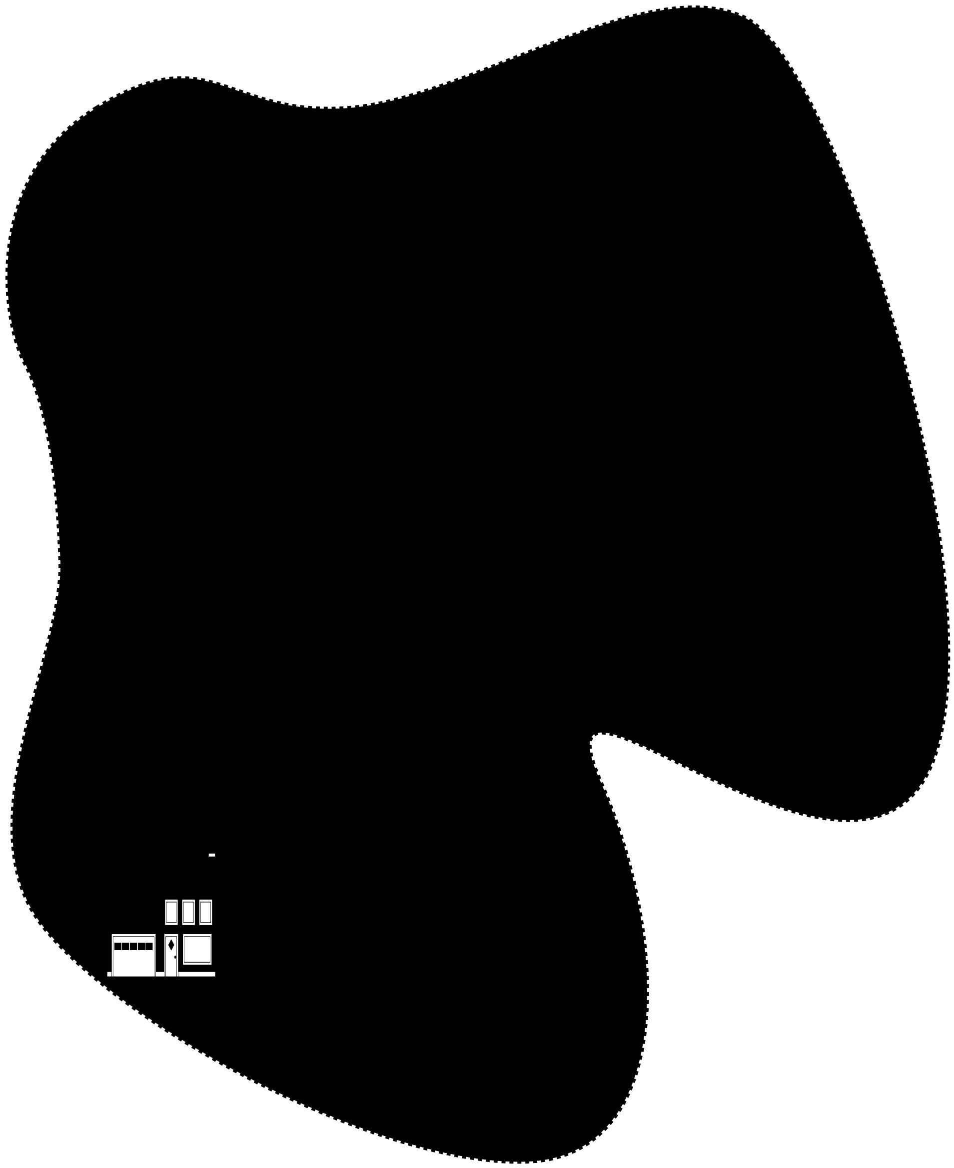
2. Pick a vertex "V" which is not on your "champion-path-so-far". If there is no such vertex, then we have found our sequence!

3. a) If V beats no vertex on your champion-path-so-far, then add it to the end, making the champion-path-so-far one vertex longer. Go to step 2.

b) If V beats the first vertex on your champion-path-so-far, then add it to the beginning, making the champion-path-so-far one vertex longer. Go to step 2.

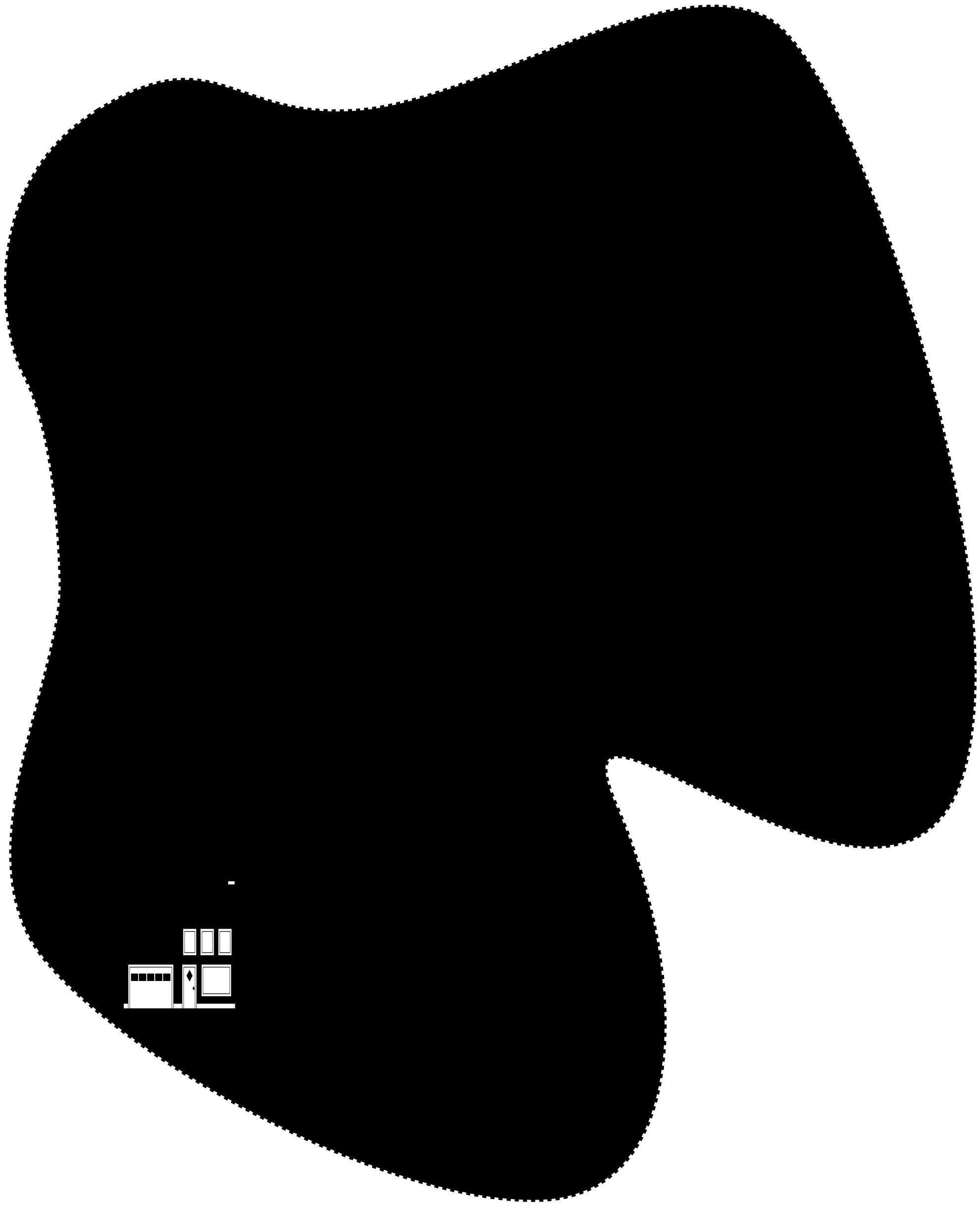
c) Otherwise, find the first vertex on the champion-path-so-far which V beats, and insert V into the champion-path-so-far directly before that vertex. Go to step 2.

Discretia



Is it possible to start at the maintenance building, drive along each road exactly once and end at the maintenance building?

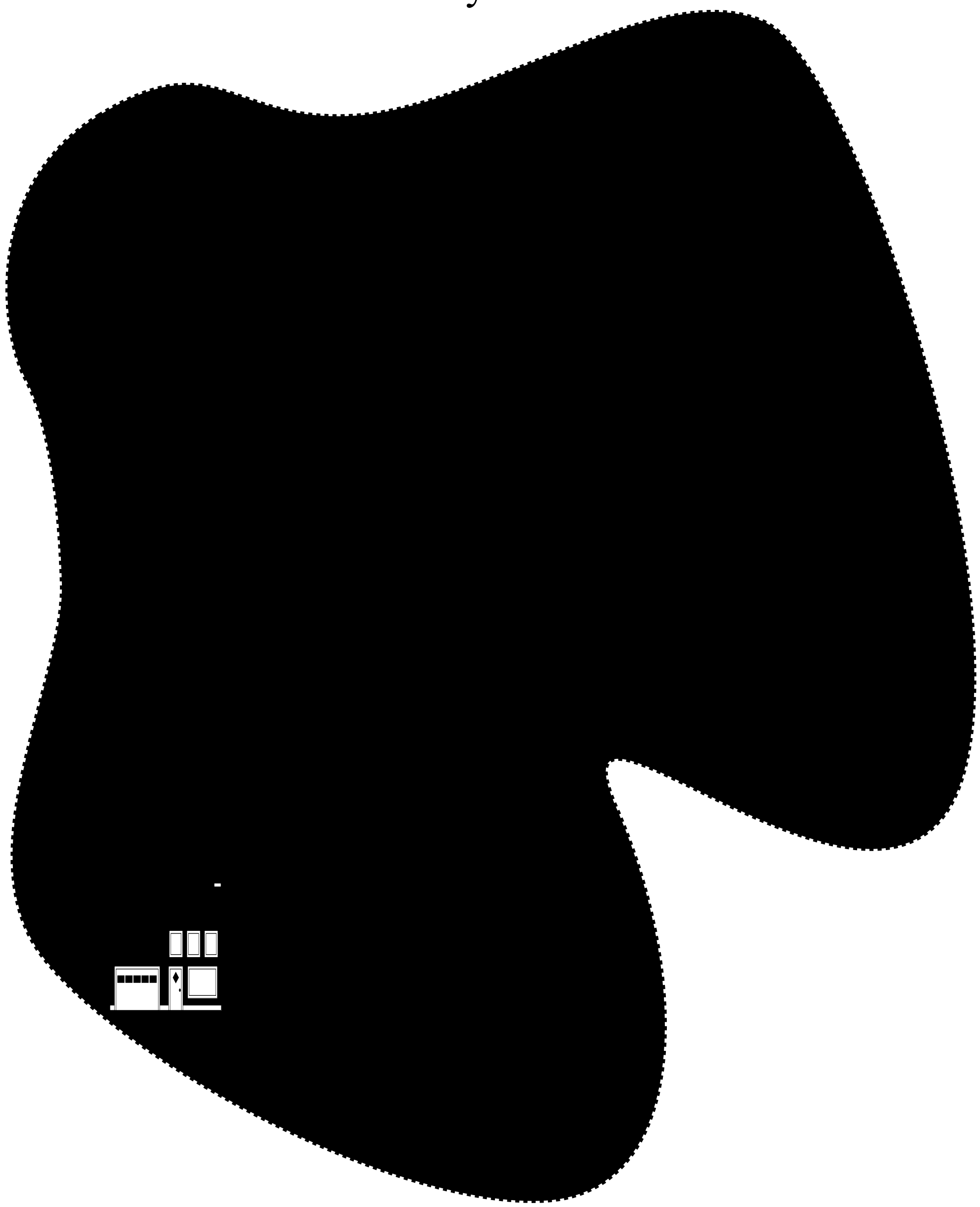
New Discretia



Now that we've added a road, is it possible to start at the maintenance building, drive along each road exactly once and end at the maintenance building?

New Discretia

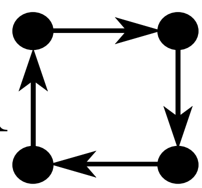
With one-way streets



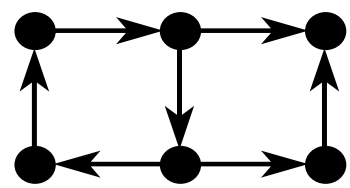
Is it possible to start at the maintenance building, drive along each road exactly once in the directions indicated by the arrows, and end at the maintenance building?

Let's try to discover when it is that a directed graph has an Euler Circuit or Path.

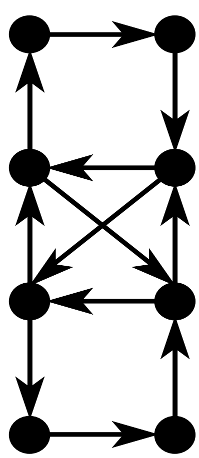
A



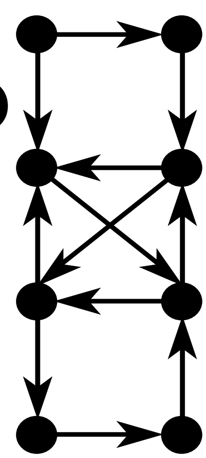
B



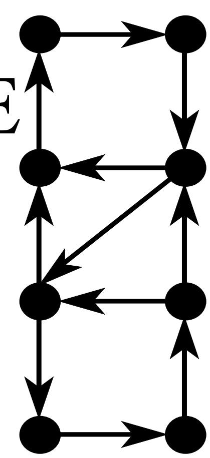
C



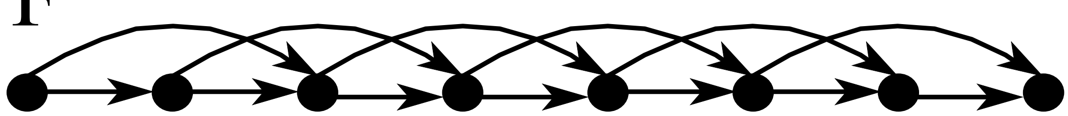
D



E

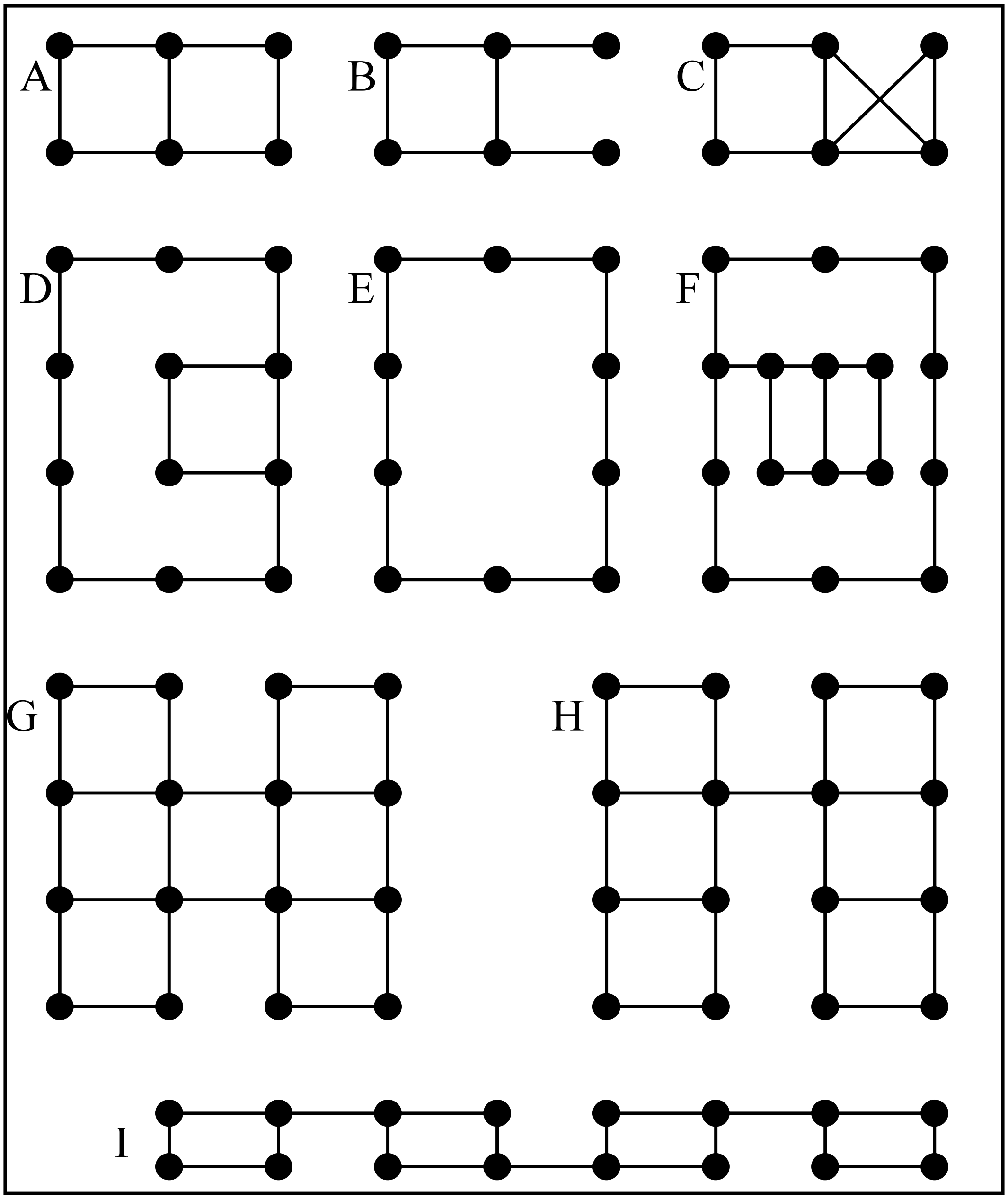


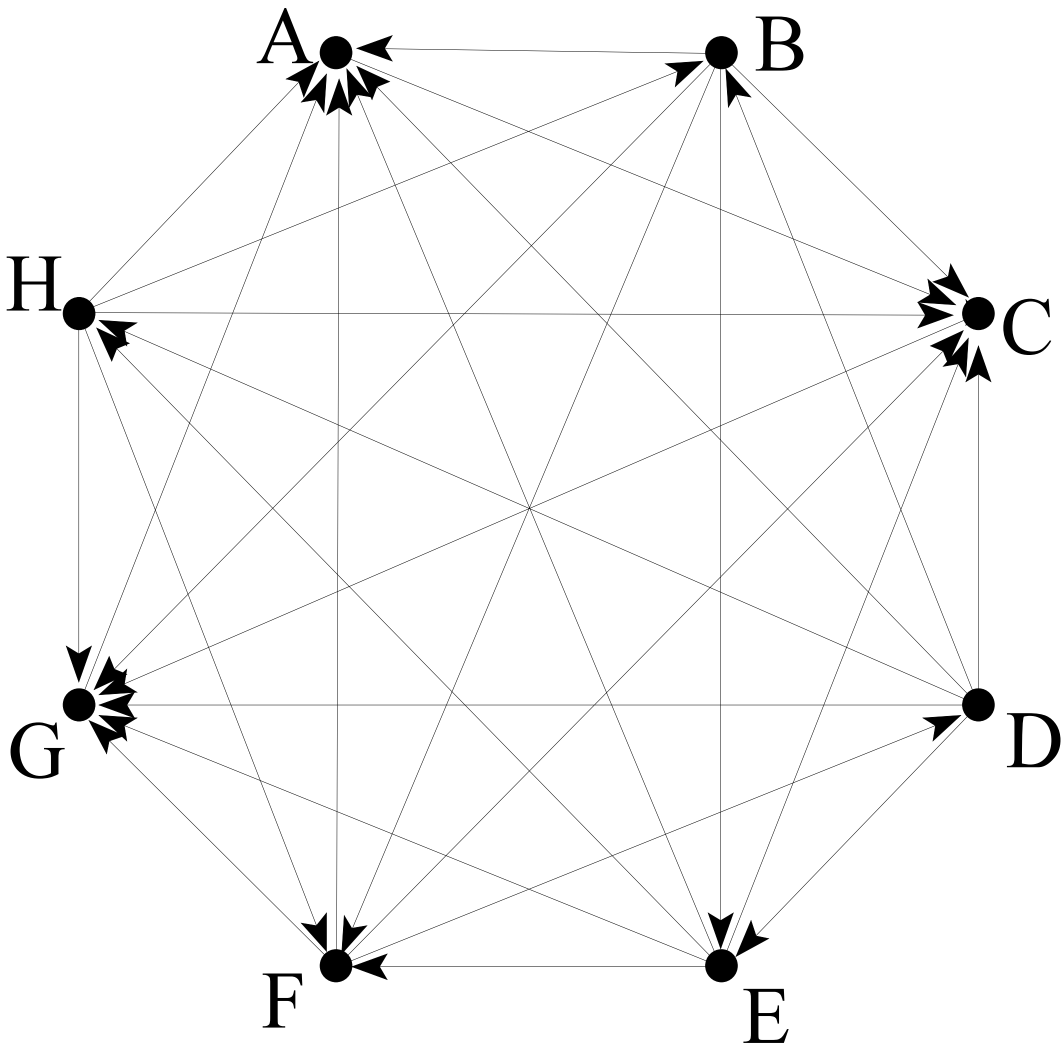
F



Which of the digraphs shown above have an Euler circuit or an Euler path?

Which of these graphs can be oriented so that you can get from any vertex to any other vertex, while observing the directions on the edges?





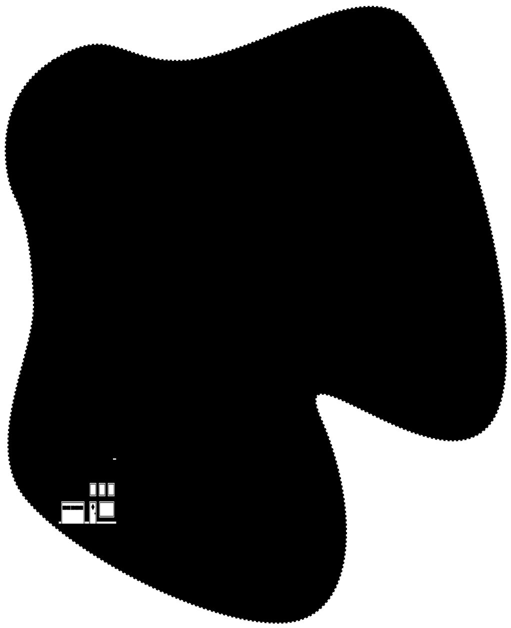
Find a champion sequence in this tournament.

Putting Arrows on Graphs

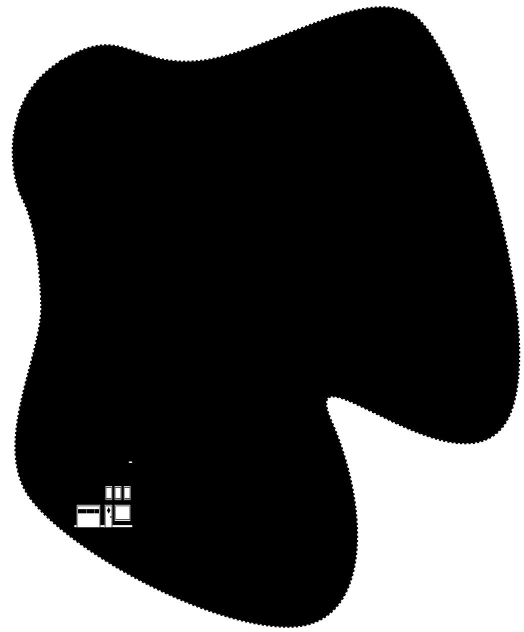
Follow-up session

December 1999

Discretia

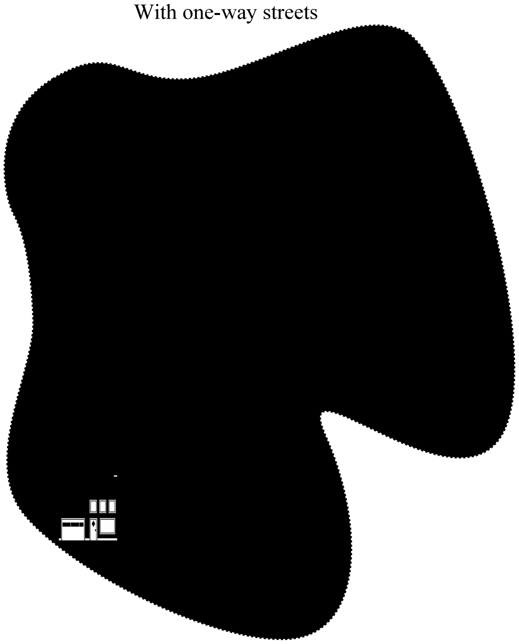


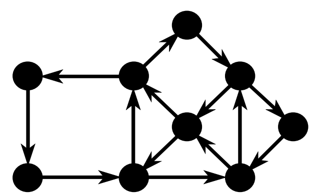
New Discretia



New Discretia

With one-way streets





Directed Edge: An edge with an arrow on it, usually restricting the direction of travel along that edge.

Arc: A directed edge.

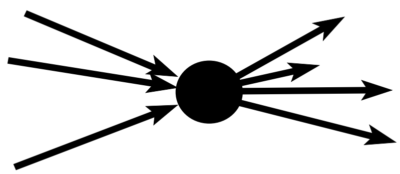
Directed Graph: A graph whose edges are all directed edges.

Digraph: A directed graph.

Directed Euler Circuit: A circuit in a digraph which follows the directions of the arrows and traverses each arc exactly once.

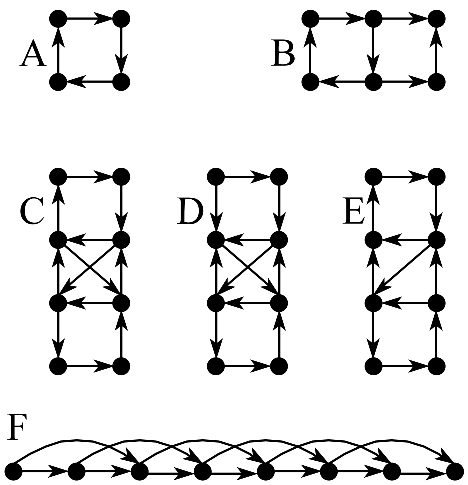
Indegree of a vertex: The number of arcs whose arrows point *into* that vertex.

Outdegree of a vertex: The number of arcs whose arrows point *out of* that vertex.



Indegree = 3 Outdegree = 4

Let's try to discover when it is that a directed graph has an Euler Circuit or Path.



Euler's Theorems for directed graphs:

Circuits

A digraph has an Euler circuit only if indegree equals outdegree at every vertex.

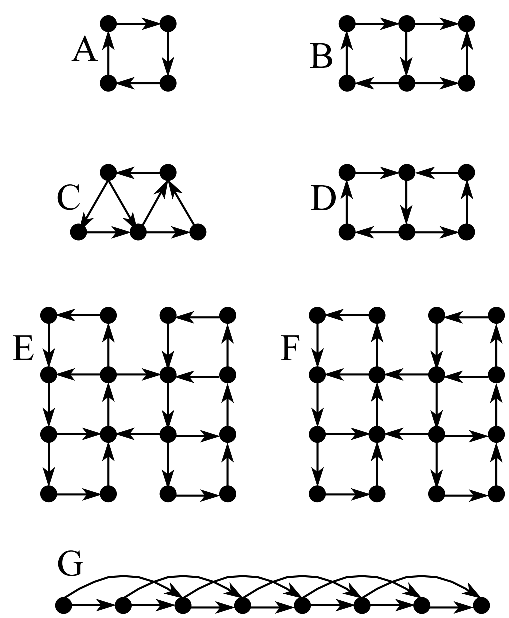
In fact, for any connected digraph, if indegree equals outdegree at every vertex, then there will exist an Euler circuit.

Paths

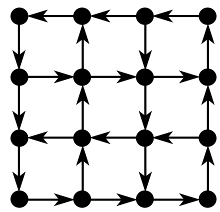
A digraph has an Euler path only if indegree equals outdegree at all but two vertices, and at those two vertices, indegree differs from outdegree by exactly one.

In fact, for any connected digraph, if indegree equals outdegree at all but two vertices, and at those two vertices, indegree differs from outdegree by exactly one, then there will exist an Euler path.

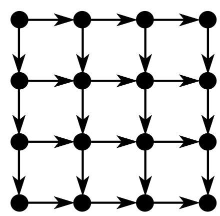
SOME EXAMPLES OF DIGRAPHS



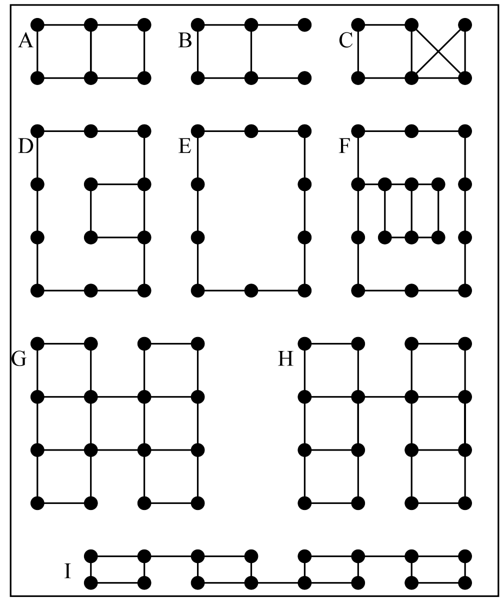
Robbinsville (plan 1)



Robbinsville (plan 2)

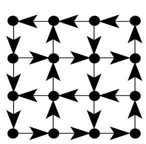


Which of these graphs can be oriented so that you can get from any vertex to any other vertex, while observing the directions on the edges?

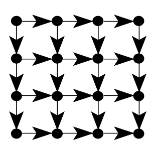


SOME TERMINOLOGY

Strongly Connected Digraph: A digraph is strongly connected if it is possible to get from every vertex to every other vertex by following a path in the direction of the arrows.

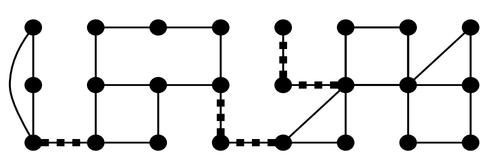


Strongly Connected



Not strongly connected

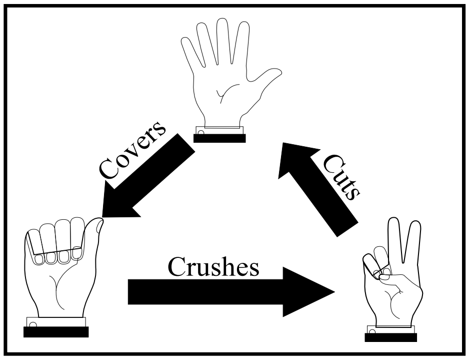
Only-bridge in a Graph: An edge in a connected graph whose removal results in a disconnected graph.



Robbins' Theorem

If the streets of a city form a connected graph which has no only-bridge, then the streets of that city can be made one-way so that any location can be reached from any other location by observing the directions on the streets

ROCK-PAPER-SCISSORS
TOURNAMENT

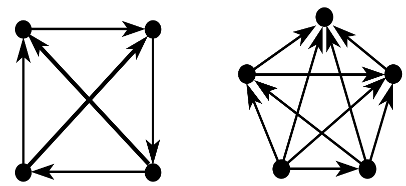


At each table, every pair of players should play a game of Rock-Paper-Scissors. Then lay down an arrow: Winner ⇨ Loser

TOURNAMENTS

Tournament: A complete graph in which every edge has been given a direction.

E.g.



Tournament Champions

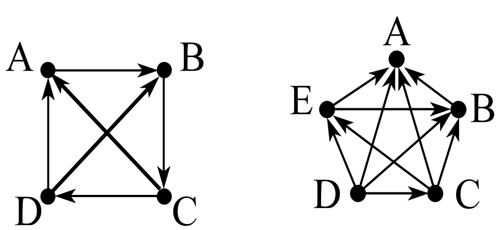
Champion sequence: A sequence A, B, C, ... containing all the vertices, such that A→ B → C→ ···

Champion: The first vertex in a champion sequence.

Tournament Champions

Champion sequence: A sequence A, B, C, ... containing all the vertices, such that A→ B → C→ ···

Champion: The first vertex in a champion sequence.



Tournament Questions and Facts

Q1: Is it possible to have a tournament in which there is just a single champion?

A: Yes. If a tournament has a champion with indegree 0, then that vertex can be the only champion in the tournament.

Extension: Is there any *other* way for a tournament to have a just a single champion?

Tournament Questions and Facts

Q2: Is it possible to have a tournament in which there is just a single champion sequence?

A: Yes. If a tournament has an ordering of the vertices from left to right, so that every arrow goes from left to right, then this tournament will have just a single champion sequence. Such a tournament is called *perfectly ordered*.

Extension: Is there any *other* way for a tournament to have a just a single champion sequence?

Tournament Questions and Facts

Q3: Is it possible to have a tournament with no champion?

A: No. Every tournament has at least one champion.

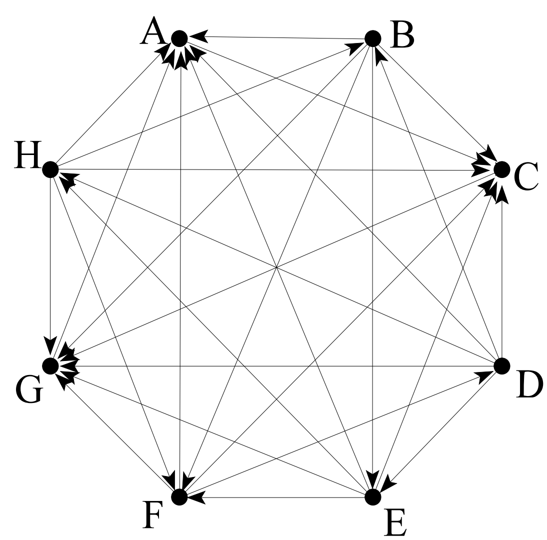
Extension: Is it possible to have a tournament with exactly 2 champions?

Tournament Questions and Facts

Q4: Is it possible to have a tournament with exactly 2 champion sequences?

A: No. Every tournament has an odd number of champion sequences. This is called Redei's Theorem.

Extension: Can you see why Redei's Theorem implies that every tournament has a champion?



Find a champion sequence in this tournament.

Algorithm for finding a champion sequence

1. Pick two vertices A and B. If A beats B then write A → B. Otherwise write B → A.
Let's assume B beats A, and consider the sequence B → A. We will try to extend this sequence.

2. Pick a vertex C which is not on your sequence.
   a) If C beats B then our new sequence becomes
   C → B → A.
   b) If C loses to B but beats A, then our new sequence becomes B → C → A.
   c) If C loses to B and A, then our new sequence becomes B → A → C.

Suppose that case b) happened and we have B → C → A.

3. Pick a vertex D which is not on the sequence.
   a) If D beats B then our new sequence becomes
   D → B → C → A.
   b) If D loses to B but beats C then our new sequence becomes B → D → C → A.
   c) If D loses to B and C but beats A, then our new sequence becomes B → C → D → A.
   d) If D loses to B, C and A, then our new sequence becomes B → C → A → D.

Continue this until the sequence contains all vertices.

Algorithm for finding a champion sequence

1. Pick two vertices A and B. If A beats B then write A → B. Otherwise write B → A. Call this your champion path:

2. Pick a vertex V which is not on your champion path. If there is no such vertex, then we have found our sequence!

3. a) If V beats no vertex on your champion path, then add it to the end, making the champion path 1 vertex longer. Go to step 2.

   b) If V beats the first vertex on your champion path, then add it to the beginning, making the champion path one vertex longer. Go to step 2.

   c) Otherwise, find the first vertex on the champion path that V beats, and insert V into the path just before that vertex. Go to step 2.