Master Document

Day 6 — **Systematic Listing and Counting**

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LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised August 2, 1999 (and most recently July 10, 2011)

Day 6 — Systematic Listing and Counting

Materials and Pre-Workshop Preparations	**Allocated Time**
Activity #1 — Addition Principle of Counting	**35 minutes**
• Triangular pattern blocks	
Activity #2 — Arrangements of 3 and 4 objects	**30 minutes**
• Communicators for each of the participants. (If they are not available, provide dnough slips of paper A, B, C or D written on them so that participants can each be assigned a letter, typically 8 of each.)	
Activity #3 — Multiplication Principle of Counting	**25 minutes**
Activity #4 — Practice with the Multiplication Principle	**35 minutes**
. . . . TOTAL WORKSHOP TIME:	**125* minutes**

* In addition, ten minutes are allocated for a break in this 2 ¼ hour workshop.

Word Wall: Systematic Listing, Systematic Counting, Addition Principle of Counting, Multiplication Principle of Counting, Slot Method

Activity #1 — Addition Rule for Counting
(Allocated time = 35 minutes)

A. Put up a picture of a triangulated Star of David (TSP #1) and say "*I am thinking of a problem related to this picture which I would like you to solve. But since mathematics involves asking questions as well as answering them, I am going to invite you to come up with some questions, prompted by this figure.*" Then take questions from the participants and write them on a blank transparency at the overhead projector.

Reassure them, if necessary, that they should not hesitate to ask anything. The question you are looking for is "how many triangles are there in this figure?," and they are likely to ask this question, but also questions about the number of rhombuses, parallelograms, and trapezoids, as well as questions about parallel lines, angles, areas, etc. Write down as many questions as they can come up with. They will probably see the relationship between this figure and the graphs they saw in the previous week. In case they don't do it themselves, encourage them to ask some graph theoretic questions. Some questions might be "does it have a Hamilton path/circuit?" or "does it have an Euler path/circuit?" or "what is its chromatic number? (They may also ask "how many colors are needed for the picture?," which has a different answer; in fact, it can be colored using only two colors.) If the questions they ask are relatively simple, you might encourage them to answer them as they come up. But be careful not to get sidetracked for too long.

B. When they've asked plenty of questions go back to the question "how many triangles are there in this figure?" (If no one asked it, then prompt for the question with some leading questions of your own.) Then distribute Hand-out #1 and triangular pattern blocks and challenge them to solve the problem on their own.

Occasionally a group finds counting triangles very easy, and then it is difficult for the instructor to build on the discussion. If this happens, a good alternate exercise is to have participants count rhombuses of any size, after giving a definition of "rhombus". In this case, the discussion below will be slightly different. Note that the corresponding problems for trapezoids or parallelograms take too long.

It is not a bad idea to act, for the entire exercise, as if you don't know the answer to this question. Let them work for a few minutes. You can walk around looking at their work and giving answers like "yes, of any size."*

C. After a few minutes, poll them for answers, and write all the answers on the board. Then pick a wrong answer, and ask them to show how they got that answer. Let the class find where the mistake is. Repeat this with a different wrong answer. Then pick a right answer, and ask them to show how they got that answer. Again, encourage the class to be critical. When that participant is done showing his or her solution, ask if anyone else got the same answer a different way, and let them show their method. Do this one or two more times.

The goal of systematic listing and counting is to make sure you "get them all, and get them once." This statement gets its own transparency (TSP #2), and should be used as often as needed during this workshop. One purpose of this activity is to show them *the value of organizing data and counting systematically.*

Wrong answers are as valuable to the instructor as correct answers, for they help illustrate what can go wrong. Typically, if participants fail to count in a systematic way, they will either miss some (and get a low answer) or count some more than once (and get a high answer). After completing this activity, it is worth reflecting on what was done and indicating to the participants the value of considering incorrect answers. This is, of course, hard to do if all of the participants get the correct answer.

D. Finally, contrast the methods that got correct answers with those that got incorrect answers, and show how important it is to organize data. When someone mentions the break-down into "small, medium and large" triangles, draw the six medium sized triangles and the two large triangles, as well as the twelve small triangles that are on the diagram. This provides a systematics listing of the triangles. Mention that counting them all is an example of the addition principle for counting. Show and discuss the addition principle on TSP #3. Note that the addition principle is applied when, for example, we count the total number of participants by counting the number at each table and then adding up the table numbers.

Note, however, that some ways of counting triangles might not involve the addition principle – e.g., count the number of triangles at each vertex, add the results, and divide by 3. Another method is to count the number of triangles inside each large triangle using the addition principle, then double the result (since there are two large triangles), and finally subtract the duplication.

Discuss how the addition principle could be used to help count the number of (small) diamonds (rhombuses) in TSP #1, without carrying out the calculation. (Note that small diamonds come in two forms — internal and external diamonds – so that there are three cases – large diamonds, small internal diamonds, and small external diamonds.)

As always, suggest some extension problems. A good one here is to have them count trapezoids (which are not also parallelograms). This is a good extension problem if they found the triangle problem easy; it is in the homework. It moves them beyond their comfort zone.

Ask them to suggest other techniques, besides "break into cases", that help counting. You can expect answers like "make a chart", "make a table", "count twice in two different ways" (that's a good one!), "listing systematically," "make a tree," etc. Write these on a transparency, and try to refer back to them during the coming days.

Activity #2 — The number of arrangements of 3 and 4 objects
(Allocated time = 30 minutes)

Before starting this activity, make a small speech to the participants discussing the following issue: The graph theory discussed in Week 1 was largely a level playing field for the participants, in the sense that they probably did not have widely varying levels of prior experience with the material, so that there weren't participants who could see all the answers right away while others took longer. This week, however, there are likely to be very different levels of prior experience, which could lead to frustration on the part of those who are new to the concepts. This is largely because counting techniques have been in some school curricula whereas graph theory hasn't. So a message to those of you who have seen some of this material before:

Please be considerate of your colleagues, and don't give the answers away if you have already dealt with this material; rather, let those who are seeing it for the first time experience the discoveries for themselves.

A. Break the participants into groups of 4. Within each group of 4, have the three participants write A, B, and C on their three communicators; the 4th person will serve as recorder. (If communicators are not available, give the participants slips of paper which have the letters A, B, and C on them; assigning letters in this way saves a lot of time.) Tell them that this is their letter for the entire activity. Then ask all the groups to arrange themselves alphabetically by first name and to determine the ordering of A,B,C that this yields; have the lead teachers demonstrate this first. When this is done, have them arrange themselves alphabetically by last name, and then write their orderings on the board again. Then repeat this entire activity where the 4 participants in each group write A, B, C, and D on their communicators. Have them line up alphabetically by first name, by last name, by middle initial, and by whatever interesting criteria you can think of, such as height, birth-month, birth-date, age (ahem), number of years teaching, number of children, grade taught, etc. After each line-up, have the lead teachers record the line-ups on the board.

Some people will undoubtedly mention "factorial" during this discussion. Try to parry them (repeating the cautionary note above) until after you discuss the multiplication principle.

B. Now you have on the board several orderings of A,B,C and A,B,C,D. Ask them (a) if all the possible orderings of A,B,C have appeared, (b) what orderings, if any, are missing, and (c) how they can justify that all orderings have appeared. [We will deal with orderings of A,B,C,D later.]

The goal is for them to make a systematic list of all possible orderings of A,B,C. There are different possible ways of creating a systematic list, and each person or group should be asked to explain the sense in which their list is systematic. Some may create a systematic list in which the possible orderings are listed alphabetically (or the reverse), whereas others may create a tree diagram and use that to generate their list. In any case, you should generate an alphabetical list of words and also a tree diagram of all words – after eliciting both ideas from the participants.

Note that the emphasis here is on systematic listing, and in particular the systematic completion of a list that has been generated unsystematically. This is not about the multiplication principle. Do not use the multiplication principle yet; we will introduce it after providing additional motivation.

This is a good time to introduce the notion of "words" as sequences of letters that may or may not make sense as actual words in English, so that the various sequences of A,B,C are referred to as "words."

C. Turn now to orders of A,B,C,D. Ask them (a) if all the possible orderings of A,B,C,D have appeared on the list generated earlier, and (b) what orderings, if any, are missing. Ask each table to take responsibility for the lists that begin with (a specified) one of the four letters. After they have added any missing words, ask how they can justify that all orderings have appeared.

During the discussion that follows, refer to and complete the tree diagram in TSP #4.

After some discussion, review TSP #5 and TSP #6. Referring to the tree diagram on TSP #4, create an awareness that the number of arrangements of A, B, C, D is four times the number of arrangements of A, B, C. Supplement this by covering up all but the last column (the D's column) of the second table on TSP #5, and noticing that there are 6 entries in that row, and that when you cover up the Ds in that column you see the 6 arrangements of A, B, C in the table right above.

Mention by way of extension that, in a similar fashion, the number of arrangements of A, B, C, D, E is five times the number of arrangements of A, B, C, D. Use the decorative tree diagram on TSP #7 to reinforce this fact.

[Time for a 5-10 minute break]

Activity #3. The multiplication principle of counting
(Allocated time = 25 minutes)

A. Indicate that there is another way of looking at what we have been doing and show them TSP #8. By asking "How many choices for what letter goes in the first

slot?" and then "How many choices for what goes in the second slot?" etc., and by referring to the 3 groups of 2 on the bottom of the TSP and the top quarter of the tree diagram created earlier on TSP #4, conclude that the total number of ways to order these 3 objects is 3×2×1=6, which agrees with our previous results.

We refer to this approach as the "Slot Method of Counting."

Use three objects that are handy (for example a pen, eraser and pattern block) and show that there are 6 ways to arrange these three objects. Then emphasize that it doesn't matter what the three objects are, only that there are 3 of them. Whether they are {a,b,c}, {1,2,3}, {president, vice president, secretary}, or any three objects, the number of ways to order them is 6.

B. Repeat this discussion for the number of orderings of A, B, C, D using TSP #9. In discussing why the total is 4×3×2×1, you should refer to the list at the bottom of the TSP, as well as the tree diagram constructed in TSP #4.

Always emphasize the method that leads to a formula rather than the formula itself.

C. Reinforce these new ideas by asking some other quick questions. For example, you can ask for the number of ways to make a 4-digit number from the digits 1, 2, 3, 4. Of course, they have already answered this, but in a different way. Then ask for the number of ways to make a 2-digit number from those digits. (In discussing this question it would be helpful to have four cards numbered 1, 2, 3, 4 that you can use to illustrate the answer to this question.) An explanation of the answer can be found on TSP #10. If more examples would be helpful, you can ask for the number of 2-letter words you can make from F, O, U, R, or the number of 3-letter words you can make from E, I, G, H, T. In either case, you can use both lists and tree diagrams to reinforce the new method of solving these problems.

D. Introduce TSP #11 with the multiplication principle for counting, relating the idea of successive cases to the example on TSP #10. Use the multiplication principle to find the number of ways of selecting and arranging 3 letters from a 7 letter word.

E. Reintroduce the factorial notation on TSP #12, pointing out that it is just shorthand for a quantity that appears quite frequently when counting. Then, referring back to the tree diagrams, note that 4! , the number of ways of ordering the letters A, B, C, D, is four times the number of ways of ordering the A, B, C, so that 4! = 4 x 3! Similarly, referring to the bottom of TSP # 6, note that 5! = 5 x 4!

Activity #4 — Practice with the Multiplication Principle
(Allocated time = 35 minutes)

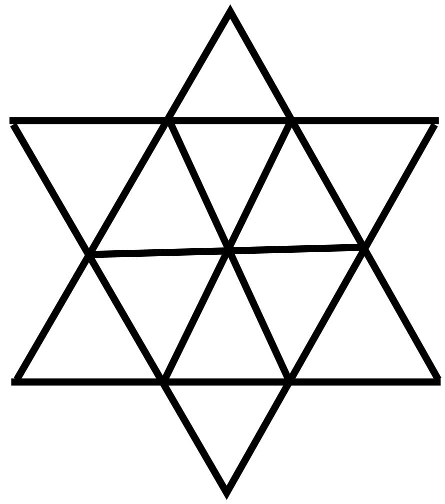
A. Distribute Hand-out #2 = TSP #13 with some practice problems involving the multiplication principle. Emphasize that the important question was "how" to do the problem, by envisioning a each situation as consisting of a sequence of parts and by envisioning the answer not as a number, but as a product. Ask them to do the first four problems, and review the answers to these problems with them. Then ask them to do the final two problems.

Note that even if they did not make a tree diagram to solve the first problem, nevertheless using a tree diagram (imaginary) to visualize the answer is important for many learners.

In the fifth problem, expect that someone will suggest that the answer is 20 and make a tree diagram to support this conclusion; if this happens then ask the following question "What are the instructions on the top of the test if this tree diagram lists all possible results?" and elicit the response that the instructions were "Answer one of the following T-F questions." Other answers that you might get to this problem are 10!, 10 x 10, and 10 + 10, and each should be discussed.

Problem #5 is of course similar to problem #4, and that should be elicited from the participants.

You may need to review exponents in discussing problem #5; TSP #14 can be used for this purpose. Take some time to review problem #6 — making words without repeating letters. Show how this problem is an example of the addition principle combined with the multiplication principle (see TSP #15). The addition principle is used to break the problem into cases, and then the multiplication principle is used to count the number of words in each case.



What are some questions you can think of to ask about this figure?

Systematic Listing and Counting

The Goal

count every possibility

• • •

count each possibility just once

The Addition Principle for Counting
(Separate Cases)

If you need to count the total number of possibilities in a certain situation, look to see if you can break the situation up into cases which don't overlap.

Then if you count the number of possibilities in each of the cases, and add them all up, that will give the total number of possibilities in the original situation.

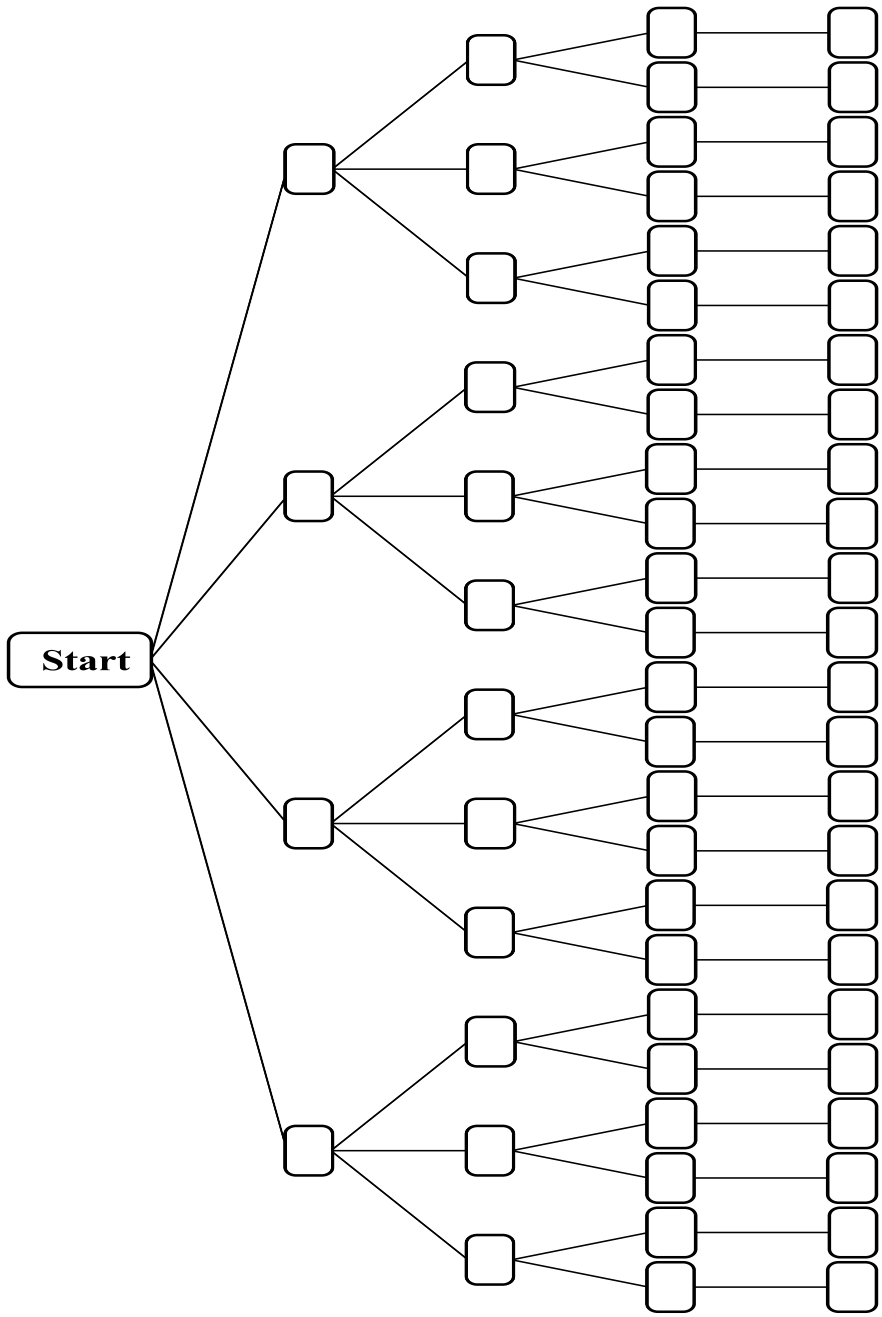
Example:

To count the number of triangles in the star pattern, break the situation up into three non-overlapping cases: small, medium, and large triangles.

The total number of triangles is the sum of the numbers of small triangles, medium triangles, and large triangles.

A Tree Diagram for Arranging 4 Letters

How would you place the letters A, B, C and D into these boxes to show all the ways to arrange these four letters?



Counting Arrangements of A, B, C

There are 6 ways to arrange the letters A, B, C:

A B C	B A C	C A B
A C B	B C A	C B A

If you start with A, there are 2 arrangements of B, C.
If you start with B, there are 2 arrangements of A, C.
If you start with C, there are 2 arrangements of A, B.

Counting Arrangements of A, B, C, D

There are 24 ways to arrange the letters A, B, C, D:

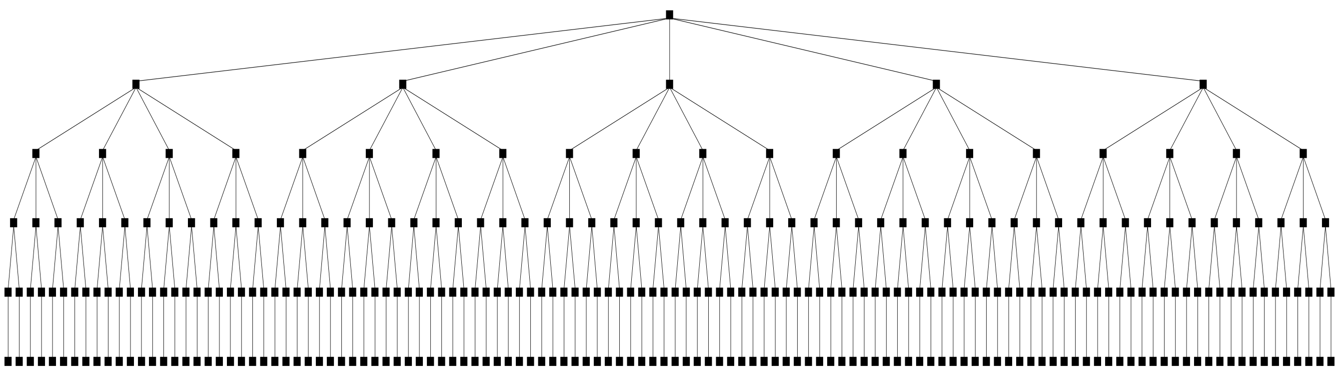
A B C D	B A C D	C A B D	D A B C
A B D C	B A D C	C A D B	D A C B
A C B D	B C A D	C B A D	D B A C
A C D B	B C D A	C B D A	D B C A
A D B C	B D A C	C D A B	D C A B
A D C B	B D C A	C D B A	D C B A

If you start with A, there are 6 arrangements of B, C, D.
If you start with B, there are 6 arrangements of A, C, D.
If you start with C, there are 6 arrangements of A, B, D.
If you start with D, there are 6 arrangements of A, B, C.

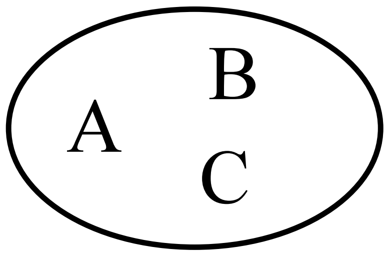
So the number of arrangements of four items is 4 times the number of arrangements of three items.
$4 \times 6 = 24$.

Counting Arrangements of A, B, C, D,E

There are 120 ways (5x24) of arranging the five letters A,B,C,D,E.



Number of Orderings of Three Objects



# of choices for item in first slot ____

# of choices for item in second slot ____

____ ____ ____

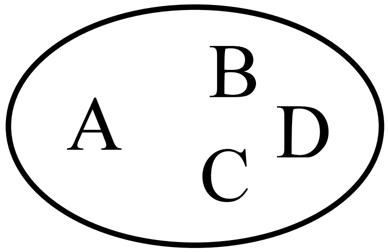
1st Slot   2nd Slot   3rd Slot

# of choices for item in third slot ____

So the total number of ways to order 3 objects is:

A B C	B A C	C A B
A C B	B C A	C B A

Number of Orderings of Four Objects



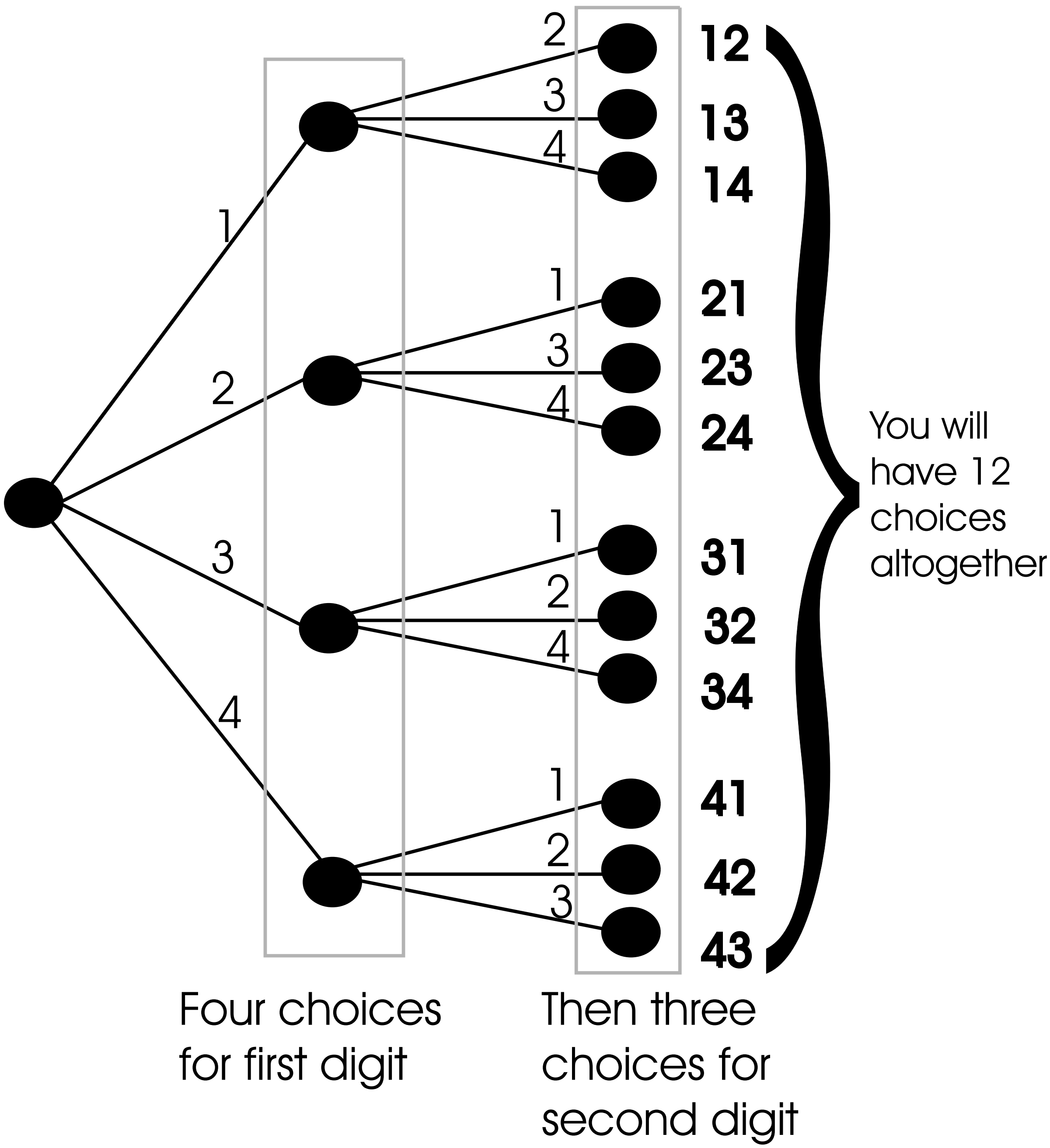
____ ____ ____ ____

1st Slot | 2nd Slot | 3rd Slot | 4th Slot

# of choices for item in first slot ______
# of choices for item in second slot ______
# of choices for item in third slot ______
# of choices for item in fourth slot ______
So the total number of ways to order 4 objects is:

A B C D	B A C D	C A B D	D A B C
A B D C	B A D C	C A D B	D A C B
A C B D	B C A D	C B A D	D B A C
A C D B	B C D A	C B D A	D B C A
A D B C	B D A C	C D A B	D C A B
A D C B	B D C A	C D B A	D C B A

How many 2-digit numbers using different digits can be made using 1, 2, 3, and 4?



THE NUMBER OF WAYS TO MAKE A 2 DIGIT NUMBER IS 4X3. YOU HAVE 4 CHOICES FOR YOUR FIRST DIGIT, AND <u>THEN</u> 3 CHOICES FOR YOUR SECOND DIGIT.

The Multiplication Principle for Counting
(Successive Parts)

If you need to count the total number of possibilities in a certain situation, look to see if you can break the situation up into a number of parts which follow one after the other.

Then if you count the number of possibilities in each part of the situation, and multiply them, that will give the total number of possibilities in the original situation.

Example:

If you want to know how many three-letter "words" can be made from the letters in the word "PRODUCT", decide how many possibilities there are for the first letter, then how many for the second letter, then how many for the third letter.

The total number of "words" is the product of these three numbers.

Caution: This only works if the number of possibilities in each part is independent of what is chosen in the previous parts.

Review of factorial notation

Any number "factorial" is the product of all the counting numbers up to that number.

For example,

$3! = 3\times2\times1$
$4! = 4\times3\times2\times1$
$5! = 5\times4\times3\times2\times1$

In general,

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 4 \times 3 \times 2 \times 1$$

Referring to the tree diagram, $4! = 4 \times 3!$

Similarly,

$5! = 5 \times 4!$ and $6! = 6 \times 5!$ and $7! = 7 \times 6!$ etc.

In general

$$n! = n \times (n-1)!$$

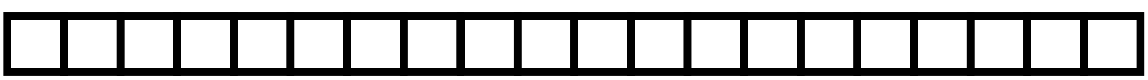
Practice with the Multiplication Principle

1. The secret password of the North California Cotton Club involves a letter followed by a single digit. How many possible code words are there?

2. In how many different ways can a pair of dice fall? (One die is red and the other is blue.)

3. The secret password of the South California Cotton Club involves a letter followed by a single digit and a letter. How many possible code words are there?

4. The strip of squares below is to be converted into a bar code by coloring each square either red or blue. In how many different ways can this be done?



5. A child is taking a true/false test that has 10 questions. How many different ways are there for him to answer the questions on this test, assuming that he isn't taking into account whether he will be correct or incorrect?

6. How many words of any length can be made from the letters in "TEAL," if you are not allowed to repeat a letter? (They do not have to be real English words.)

A child is taking a true/false test that has 10 questions. How many different ways are there for her to answer the questions on this test, assuming that she isn't taking into account whether she will be correct or incorrect?

— — — — — — — — — —

The first question can be answered T or F
The second question can be answered T or F
...
The tenth question can be answered T or F

By the multiplication principle of counting, the total number of possibilities is

2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2

or 2 multiplied by itself 10 times. We write this as 2^{10}

which is read

"2 to the tenth power"

or, more simply,

"2 to the tenth"

How many words of any length can be made from the letters in "TEAL," if you are not allowed to repeat a letter?

You can have 4-letter words, 3-letter words, 2-letter words, and 1-letter words.

By the addition principle of counting, once you find
•the number of 4-letter words,
•the number of 3-letter words,
•the number of 2-letter words, and
•the number of 1-letter words,
then the total number of words is the sum of these numbers.

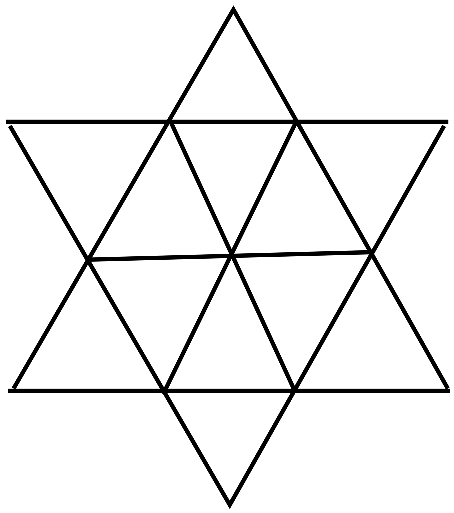
Using the multiplication principle of counting,
•the number of 4-letter words is 4x3x2x1 = 24,
•the number of 3-letter words is 4x3x2 = 24,
•the number of 2-letter words is 4x3 = 12, and
•the number of 1-letter words is 4.

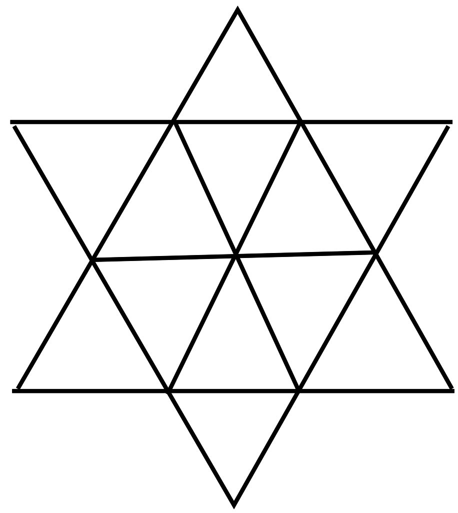
Conclusion: The number of words that can be made from the letters of "TEAL," without any repeats is

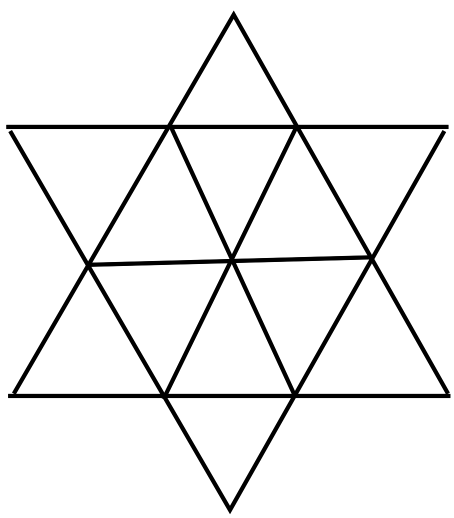
24+24+12+4 = 64.

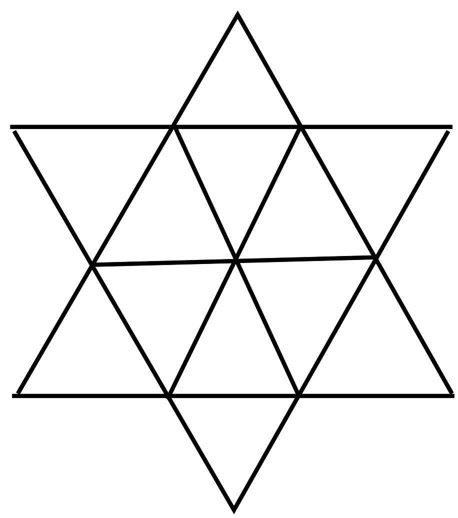
Hand-Out #1: Counting Triangles

How many triangles do you see in this figure? (Four copies are provided.)









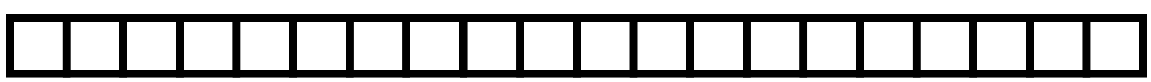
Handout #2 Practice with the Multiplication Principle

1. The secret password of the North California Cotton Club involves a letter followed by a single digit. How many possible code words are there?

2. In how many different ways can a pair of dice fall? (One die is red and the other is blue.)

3. The secret password of the South California Cotton Club involves a letter followed by a single digit followed by a letter. How many possible code words are there?

4. The strip of squares below is to be converted into a bar code by coloring each square either red or blue. In how many different ways can this be done?



5. A child is taking a true/false test that has 10 questions. How many different ways are there for him to answer the questions on this test, assuming that he isn't taking into account whether he will be correct or incorrect?

6. How many words of any length can be made from the letters in the word "TEAL," if you are not allowed to repeat a letter? Note, they do not have to be real English words.

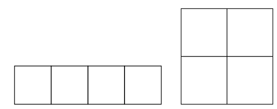
Workshop 6 — Systematic Listing and Counting — Exercises

Practice Problems:

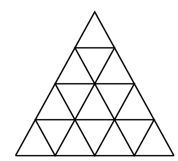
1. **Six different colored blocks are lined up in a row. How many color arrangements are possible?**

2. **A teacher who loves discrete math gives a workshop which causes 17 other teachers to love discrete math. Then each of those teachers teaches discrete math to 22 students, who gain such an appreciation of discrete math that they each go home and tell both of their parents an interesting discrete math fact. How many parents has the original teacher thereby influenced with her single workshop?**

3. **How many different colorings are possible if each small square in each of these arrangements is either shaded or left blank? Support your answers by listing systematically all of the possibilities. (Grids like these are provided on page EX 5.)**

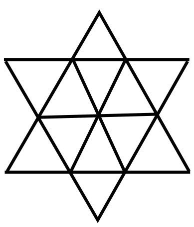


Problems for Study Groups:



4. **a. How many triangles of any size are there in the figure to the right? (More such triangles are on page 4.)**

   b. How many trapezoids of any size are there in the star figure at the left? (More such stars are on page 4.)



5. **Start with the letters in the word *graphs*.**
   - **a. How many ways can you arrange 2 of the letters? 1 letter? 1 or 2 letters? 3, 4, or 5 letters?**
   - **b. Explain the counting techniques used in terms of the multiplication and addition principles.**

6. **New Jersey license plates now read, left to right, one letter-two digits-three letters. They used to read three letters-two digits-one letter. And, for a while, they had three letters followed by four digits. How many possible license plates are there under each system?**

7. ***Agent 006* is sealed inside a room with no way to escape. With him in the room is a time-bomb set to explode in 18 minutes. It would take him just 30 seconds to defuse the bomb, except that it is locked inside a suitcase that has a combination lock on one of the clasps. To open the lock, he has to set three dials, each containing the digits 0-9, to the correct positions. He is able to check one of these combinations per second, thanks to his top-notch manual dexterity. Is he guaranteed to be able to open the suitcase with enough time left to defuse the bomb?**

8. **When Alice sends Bob a card, she always adds a string of 5 X's and/or O's after her signature, with one restriction: she never puts two O's next to each other, because Bob turns them into smiley faces, and she can't stand that. Thus, for example, "OXOXO" and "XOXXO" would be allowed, but "XOOXX" and "OOXOO" would not. Systematically list all the different ways for Alice to add X's and O's to the card. How many ways are there?**

9. a. How many 3-digit numbers are there where each digit is odd?

b. How many 3-digit numbers are there where each digit is odd and all three digits are different?

c. How many 3-digit numbers are there where each digit is odd and the three digits appear in increasing order? (For example, 157 is such a number.) Can you answer this question using the multiplication principle?

10. In this problem, we use the word "word" to mean any sequence involving the indicated letters. For example, SUSUSU is a six letter word using only S and U, as are SSSSSS and UUUSSS.

a. How many six letter words can you make using only S and U?

b. How many four letter words can you make using only R, E, D?

c. How many five letter words can you make using R, E, A, D?

d. If you wanted to make that many words (the answer to part c.) using only S and/or U, how many letters long would you have to make those words?

11. Make a tree diagram showing all arrangements of the letters in the word "REED". How many are there?

12. As I was going to St. Ives, I met a man with 7 wives. Each wife had 7 sacks, each sack had 7 cats, each cat had 7 kittens. Kittens, cats, sacks, wives ... how many were going to St. Ives?

Extension Problems:

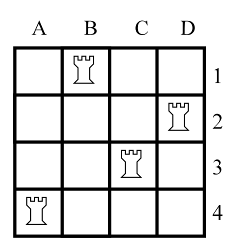
13. How many arrangements are there of the letters in the word "SEVEN"? In how many of these are the *E's* together? In how many are the *E's* separated?

14. Two players compete repeatedly until one wins two successive games or three games total. How many different sequences are possible? What if you change "three games total" to "four"?

15. A set of five regular octahedral dice have eight faces numbered 1, 2, 3, 4, 6, 8, 12, and 20. In how many different ways can these five dice fall? In how many ways will the five dice show a sum of six? of seven? What are the chances of rolling a sum of six or seven?

16. There were once five teams in the East and Central and four in the West Division of both the American and National Baseball Leagues. How many different standings were possible for the 28 teams when ranked within their divisions? within their leagues? all together? Assume that there are no ties. How many World Series team match-ups were possible?

17. The figure to the right shows 4 rooks placed on a 4x4 "chessboard" in such a way that no rook attacks any other rook. (In chess, a rook attacks pieces which lie in the same row or column as itself.)



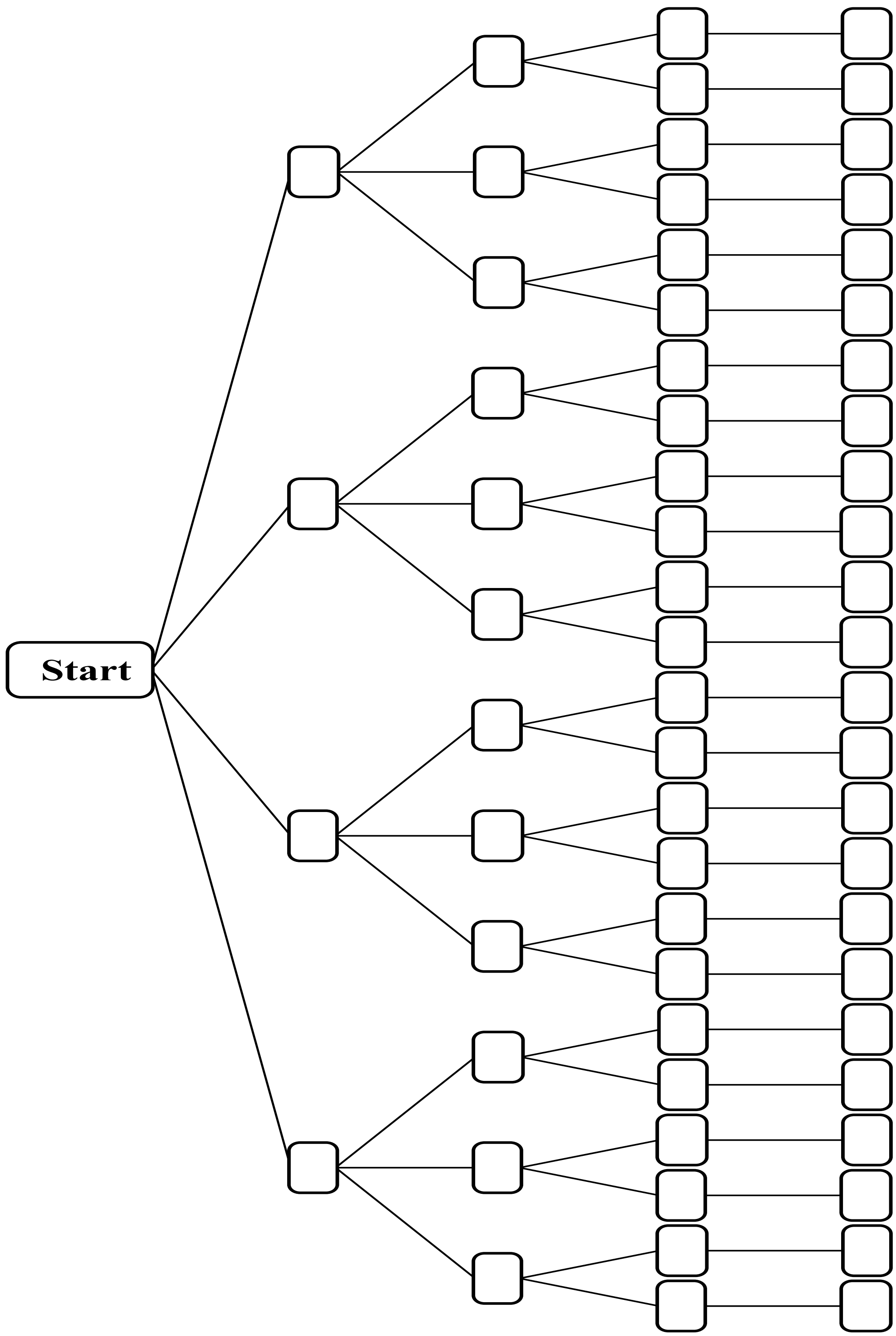
a. List systematically all the ways to place 4 non-attacking rooks on a 4x4 board. Use the grids on page EX 4. (In the diagram, the rooks are at squares A-4, B-1, C-3, and D-2.)

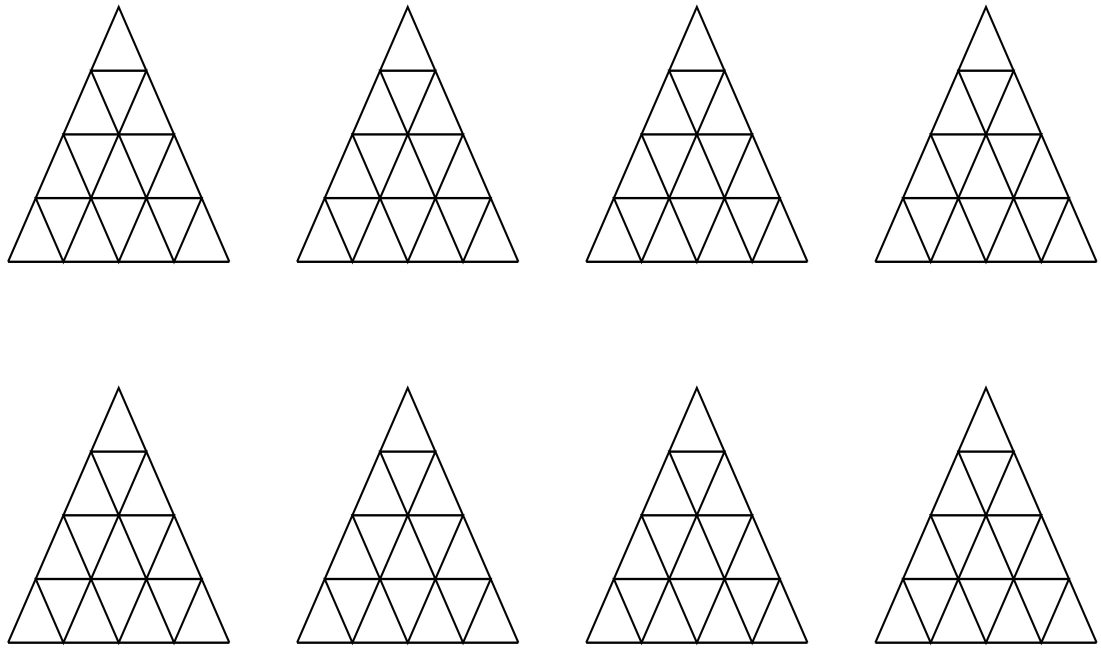
b. How many ways are there?

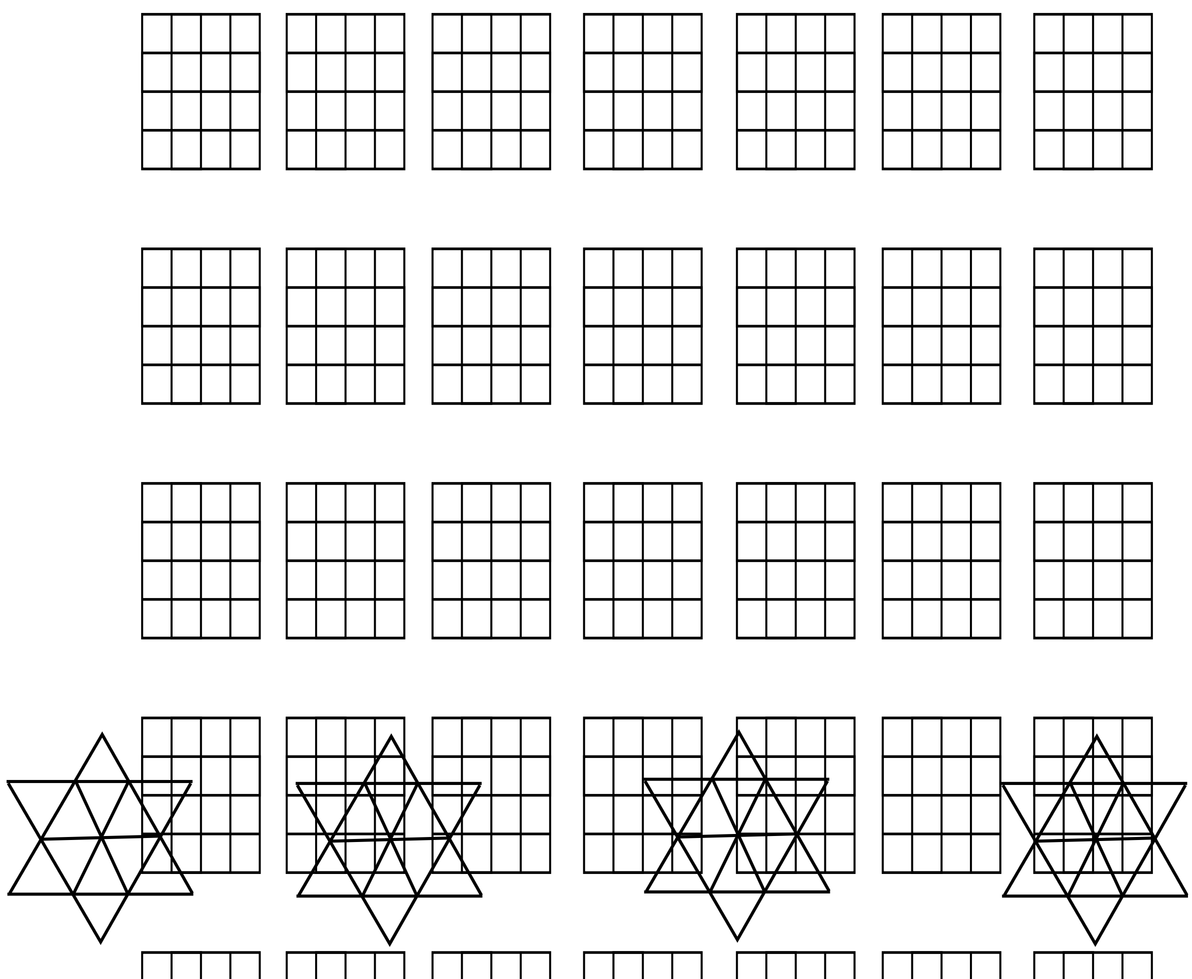
c. Is there a way to use the multiplication principle to find this answer?

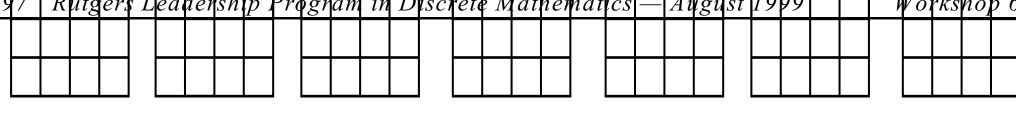
A Tree Diagram for Arranging 4 Letters

How would you place the letters A, B, C, and D into these boxes to show all the ways to arrange these four letters? How many ways are there altogether to arrange these four letters? Can you see how to get that number using the multiplication principle?

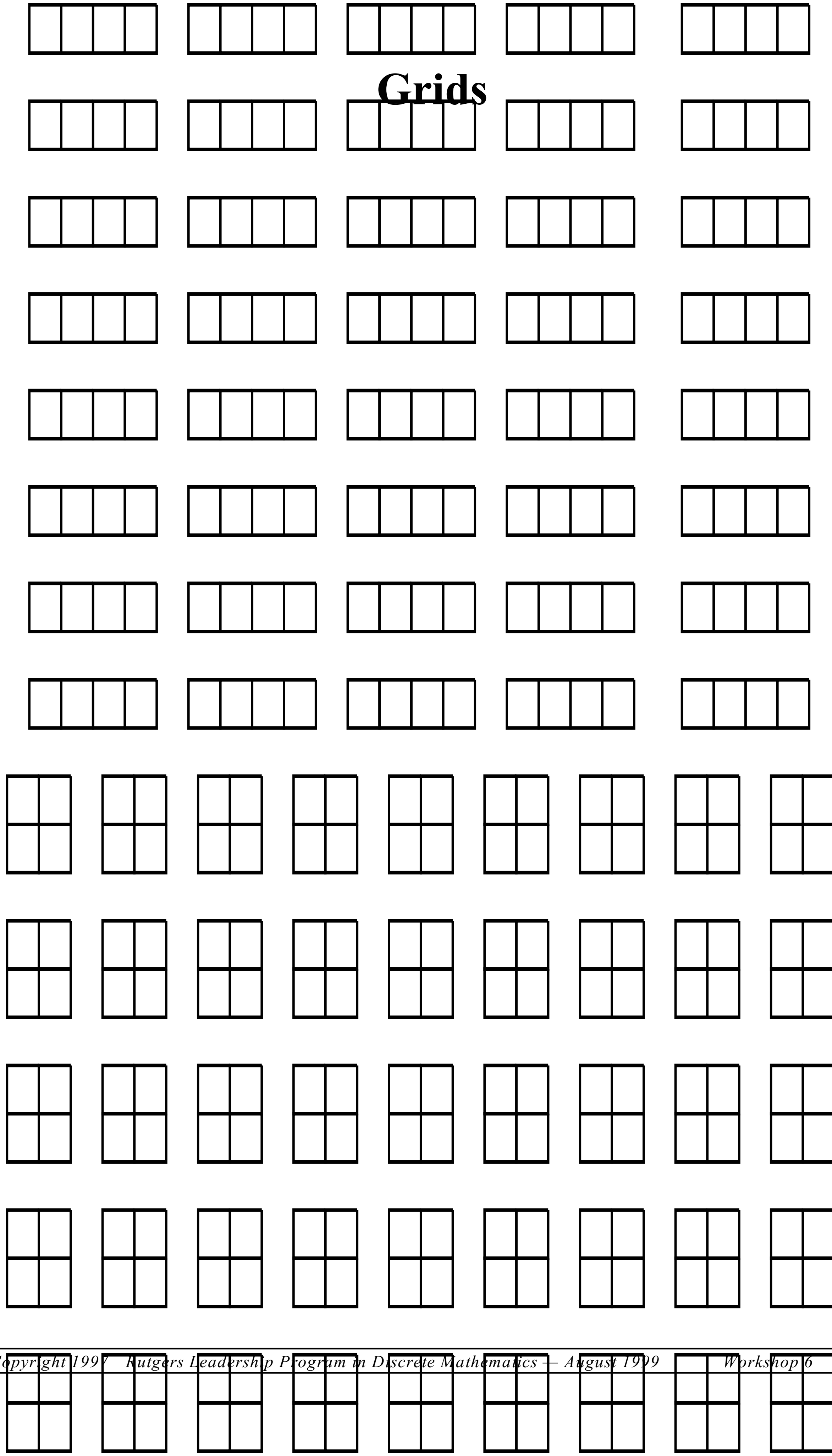








Grids



Resource Book

Workshop 6: Systematic Listing and Counting

Table of Contents

3. Mathematical Background.

4. Workshop Outline.

5. The "Counting Triangles" handout, which can also be used for counting diamonds or trapezoids.

6. A blank tree diagram to show the number of ways to arrange four letters

7. Five practice problems for the multiplication rule. The answers are 260, 36, 2^{20} = 1,048,576, 2^{20} again, and 4! + 4x3x2 + 4x3 + 4 = 24 + 24 + 12 + 4 = 64.

8.-9. Two transparencies which help to illustrate why there are 3!, 4!, or 5! ways to order 3, 4, or 5 objects.

10.-11. Two more transparencies illustrating how to use the multiplication rule to count orderings.

12. A transparency helping to explain factorial notation.

13. A worksheet for one of the homework problems — systematically list all 16 ways of shading the 4 squares.

14. This exercise, from Dale Seymour, will allow your students to get a very hands-on feeling for what 4! really equals. By creating the tree, the fact that the number of orderings of 4 objects should be 4×3×2×1 will become more tangible.

15. As is, this problem is not directly related to counting. But you can ask the question "If a student guesses on a problem, what are the chances that he will get it correct?" Since there are 3 numbers (2, 3 and 6) to place in the spaces, there are 3! = 6 ways to place them. If only one of them is correct, then the chances of guessing correctly for that problem is 1 in 6. For problem 1, there are 6 correct answers, so the chances of getting that once correct are 6 in 6, or certain. For problem 3, for example, there are 2 correct answers, making the chances of getting this problem correct be 2 in 6, or 1 in 3. This is an example of a straightforward problem yielding some more interesting questions if only the teacher looks for them.

16. Reading this problem, you see that there are 3 cow properties under consideration: color (black or white), spottiness (spotty or not) and hornage (has horns or doesn't). [Ed. note: hornage is probably not a real word.] This naturally leads a discrete mathematician to ask "Have all cow types been accounted for?" Use the multiplication rule to find out if the 8 cows

on top of the page cover it.

Resource Book

Workshop 6: Systematic Listing and Counting

17. **This problem asks for the number of ways to put together a face. As is, there are 48 ways ($2 \times 3 \times 2 \times 4$) to make a face...but for lower grades, where you would probably want the children to actually make all the faces, it might be better to use fewer variations. For example, you might affix the hair and nose beforehand, and give the students three choices for eyes and just two of the choices of mouth.**

18. **This problem asks children to discover 3!. You may wish to give the students a page with many copies of the vehicles and let them create all 6 permutations. Then you can ask the class to discuss a good way to systematically list all the ways.**

19. **This illustration may help students make the connection between counting and the multiplication rule. You can directly count the number of E's in the top row of the figure to get 8, but you can also see that 8 is equal to $2 \times 2 \times 2$.**

Resource Book

Workshop 6: Systematic Listing and Counting

Mathematical Background

- **Addition Rule for Counting: (Separate cases)**

  If you need to count the total number of possibilities in a certain situation, look to see if you can break the situation up into cases which don't overlap.

  Then if you count the number of possibilities in each of the cases, and add them all up, that will give the total number of possibilities in the original situation.

  For Example: To count the number of triangles in the star pattern, break the situation up into three non-overlapping cases: small, medium, and large triangles. The total number of triangles is the sum of the numbers of small triangles, medium triangles, and large triangles.

- **Multiplication Rule for Counting: (Successive parts)**

  If you need to count the total number of possibilities in a certain situation, look to see if you can break the situation up into a number of parts which follow one after the other.

  Then if you count the number of possibilities in each part of the situation, and multiply them, that will give the total number of possibilities in the original situation.

  For Example: If you want to know how many three-letter acronyms can be made from the letters in the word "PRODUCT", decide how many possibilities there are for the first letter, then how many for the second letter, then how many for the third letter. The total number of acronyms is the product of these three numbers.

- **Factorial:**

  For any counting number n, the quantity "n factorial" is given by

$$n! = n(n-1)(n-2) \cdots 3\times2\times1$$

  Note that $n!$ counts the number of ways to order n objects. For example, there are 6 ways to arrange the letters ABC, and $6 = 3! = 3\times2\times1$. Factorials are related by the simple rule $n! = n \times (n-1)!$ For example, $4! = 4 \times 3 \times 2 \times 1 = 4 \times 3! = 24$. To keep things consistent, $0!$ is defined to equal 1.

Resource Book

Workshop 6: Systematic Listing and Counting

Workshop Outline

- **Systematic Counting**
  - **We saw a figure of a triangulated six pointed star, and discussed some of the questions we could ask about it.**
  - **Then we answered one of those questions, namely that of how many triangles of any size there were in the figure:**
    - **We found that there were 20 of them;**
    - **And we found that it was easy to get the wrong answer unless you found a systematic way to list the various triangles.**
  - **It is important to count systematically to make sure that you don't miss any and that you don't count any twice.**
  - **There are often many ways to list systematically, and it is good to try more than one and see if they yield the same answers!**

- **Multiplication Rule for Counting**
  - **We broke into groups of 3 or 4 and ordered ourselves according to various criteria.**
  - **We wrote the orderings on the board, and decided whether or not all possible orderings of 3 or 4 letters appeared.**
  - **This required listing systematically, for example, according to first digit.**
  - **First we realized that the number of ways to order the letters A, B, C was 6, because there were three possible first letters, and then in each case, two ways to order the remaining two letters. The number of ways to order the letters A, B, C, D was 24 because there were 4 possible first letters, and then for each case, 6 ways to arrange the remaining three letters.**
  - **Then we saw another way to count the number of ways to arrange 3 or 4 letters. We discovered that the number of orderings of three letters was $3\times2\times1$, because there were 3 choices for which letter would be first, two choices for which letter would be second, and one choice for which letter would be third. By the multiplication rule, we multiply these together. Similarly, for 4 letters, there were $4\times3\times2\times1$ ways to order them.**
  - **We found that in general, there were $n\times(n-1)\times\ldots\times2\times1$ ways to order n objects. This quantity is denoted "n!" and pronounced "*n* factorial." We also saw that n! = n x (n-1)!**
  - **We used tree diagrams to help illustrate the multiplication rule.**

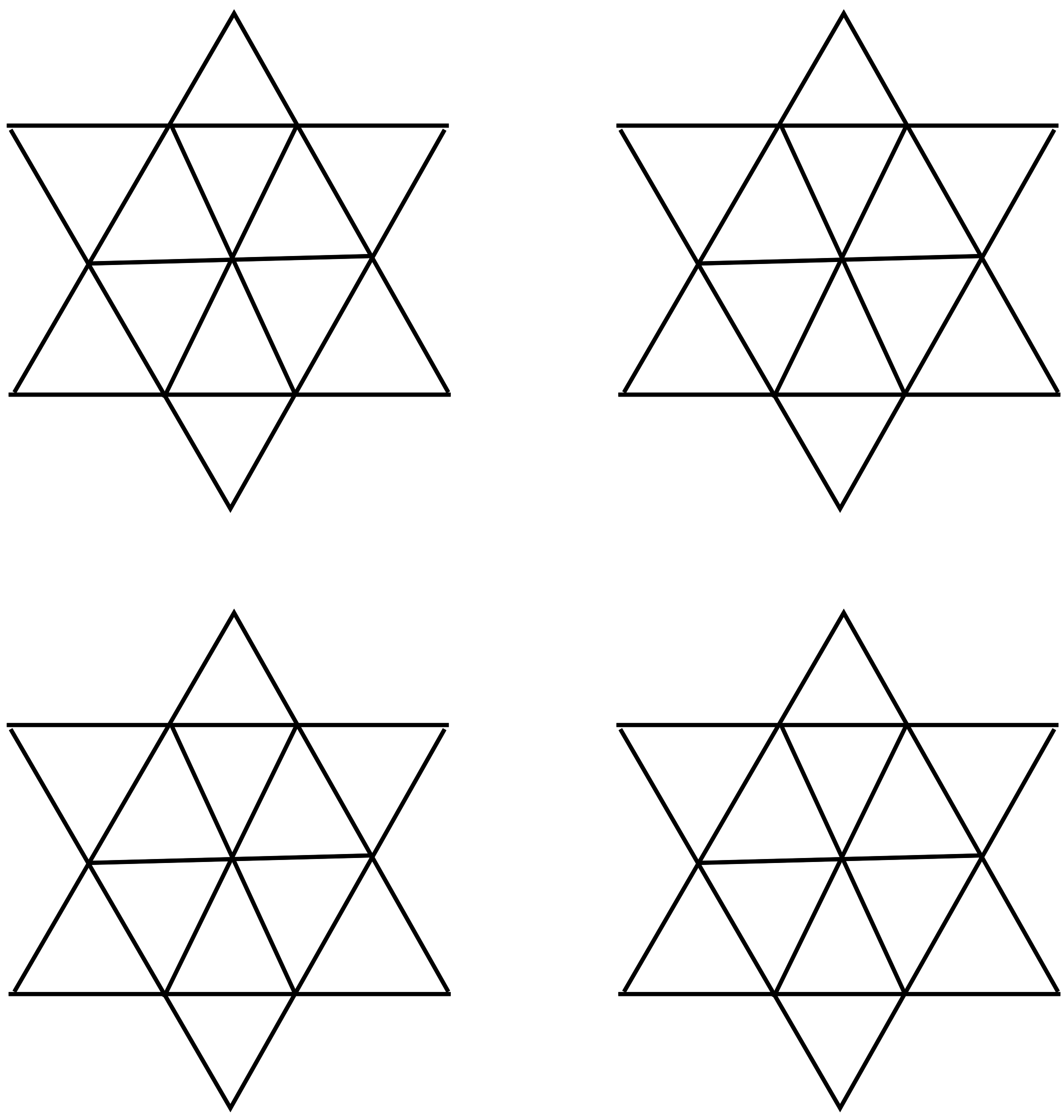
- **Practice with the multiplication rule**
  - **We did some problems illustrating the usefulness and simplicity of the multiplication rule.**
  - **Finally, we saw that the multiplication and addition rules sometimes need both to be used, as in the case of counting the number of words of any length which can be made from the letters of "TEAL" without repeating letters.**

Resource Book

Workshop <u>6</u>: Systematic Listing and Counting

Counting Triangles

How many triangles do you see in this figure? (Four copies are provided.)



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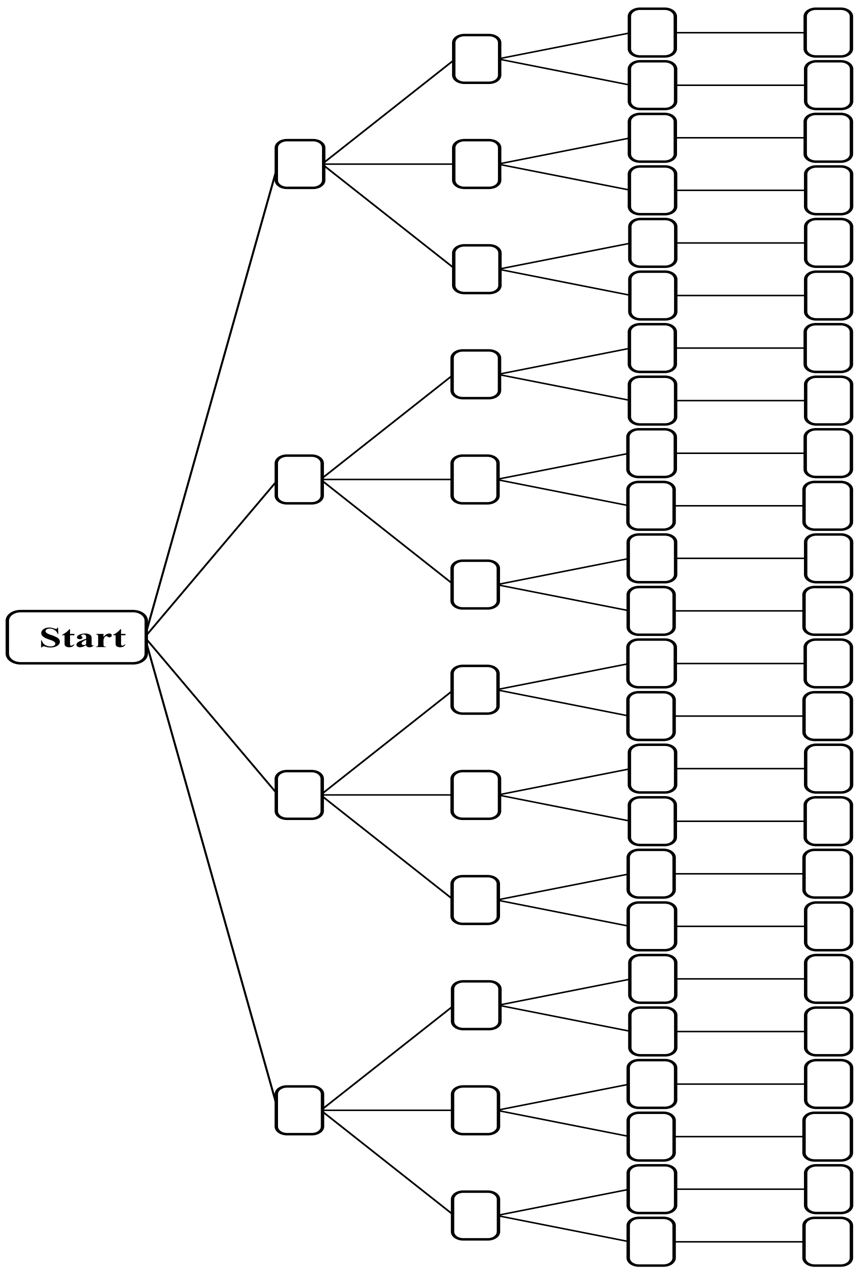
Workshop 6: Systematic Listing and Counting

A Tree Diagram for Arranging 4 Letters

How would you place the letters A, B, C and D into these boxes to show all the ways to arrange these four letters? How many ways are there altogether to arrange these four letters? Can you see how to get that number using the multiplication rule?



Resource Book

Workshop 6: Systematic Listing and Counting

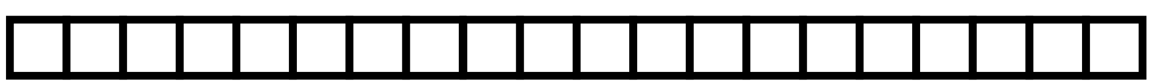
Practice with the Multiplication Rule

1. **The secret password of the California Cotton Club involves a letter followed by a single digit. How many possible code words are there?**

2. **In how many different ways can a pair of dice fall? (One die is red and the other is blue.)**

3. **A child is taking a true/false test that has 20 questions. How many different ways are there for him to answer the questions on this test, assuming that he isn't taking into account whether he will be correct or incorrect?**

4. **The strip of squares below is to be converted into a bar code by coloring each square either red or blue. In how many different ways can this be done?**



5. **How many words of any length can be made from the letters in the word "TEAL," if you are not allowed to repeat a letter? Note, they do not have to be real English words.**

Counting Arrangements of A, B, C and A, B, C, D

There are 6 ways to arrange the letters A, B, C:

A B C	B A C	C A B
A C B	B C A	C B A

If you start with A, there are 2 arrangements of B, C.
If you start with B, there are 2 arrangements of A, C.
If you start with C, there are 2 arrangements of A, B.

There are 24 ways to arrange the letters A, B, C, D:

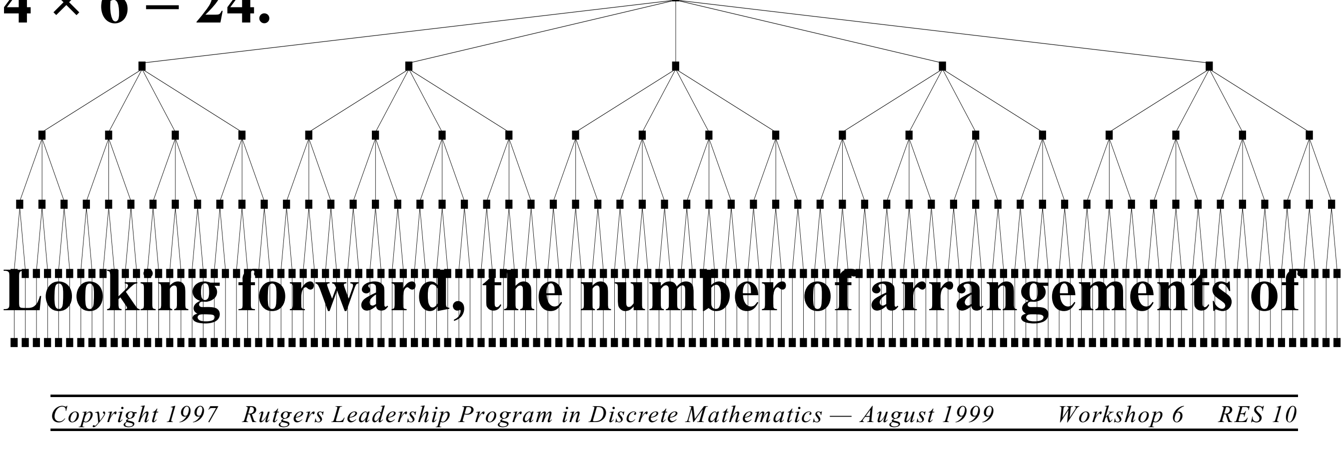
A B C D	B A C D	C A B D	D A B C
A B D C	B A D C	C A D B	D A C B
A C B D	B C A D	C B A D	D B A C

A C D B	B C D A	C B D A	D B C A
A D B C	B D A C	C D A B	D C A B
A D C B	B D C A	C D B A	D C B A

Counting Arrangements of A, B, C and A, B, C, D

If you start with A, there are 6 arrangements of B, C, D.
If you start with B, there are 6 arrangements of A, C, D.
If you start with C, there are 6 arrangements of A, B, D.
If you start with D, there are 6 arrangements of A, B, C.

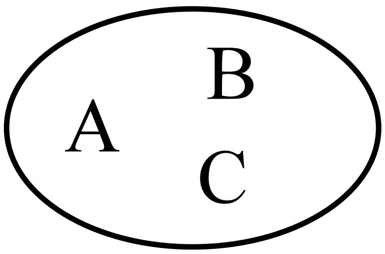
So the number of arrangements of four items is 4 times the number of arrangements of three items. $4 \times 6 = 24$.



Looking forward, the number of arrangements of

5 items will be 5 times the number of arrangements of 4 items. $5 \times 24 = 120$.

Number of Orderings of Three Objects



____ ____ ____

1st Slot 2nd Slot 3rd Slot

# of choices for item in first slot ____

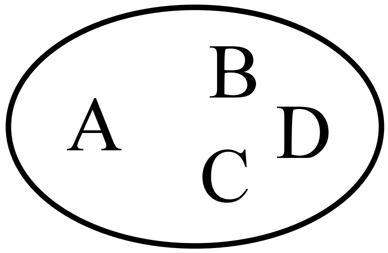
# of choices for item in second slot ____

# of choices for item in third slot ____

So the total number of ways to order 3 objects is:

A B C	B A C	C A B
A C B	B C A	C B A

Number of Orderings of Four Objects



____	____	____	____
1st Slot	2nd Slot	3rd Slot	4th Slot

# of choices for item in first slot ______
# of choices for item in second slot ______
# of choices for item in third slot ______
# of choices for item in fourth slot ______
So the total number of ways to order 4 objects is:

A B C D	B A C D	C A B D	D A B C
A B D C	B A D C	C A D B	D A C B
A C B D	B C A D	C B A D	D B A C
A C D B	B C D A	C B D A	D B C A
A D B C	B D A C	C D A B	D C A B
A D C B	B D C A	C D B A	D C B A

Factorial notation

Any number "factorial" is the product of all the counting numbers up to that number.

For example,

$3! = 3\times2\times1$

$4! = 4\times3\times2\times1$

$5! = 5\times4\times3\times2\times1$

In general,

$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 4 \times 3 \times 2 \times 1$

Referring to the tree diagram, $4! = 4 \times 3!$

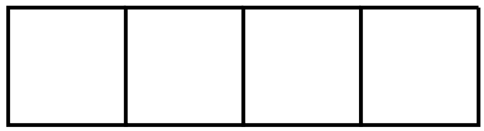
Similarly,

$5! = 5 \times 4!$ **and** $6! = 6 \times 5!$ **and** $7! = 7 \times 6!$ **etc.**

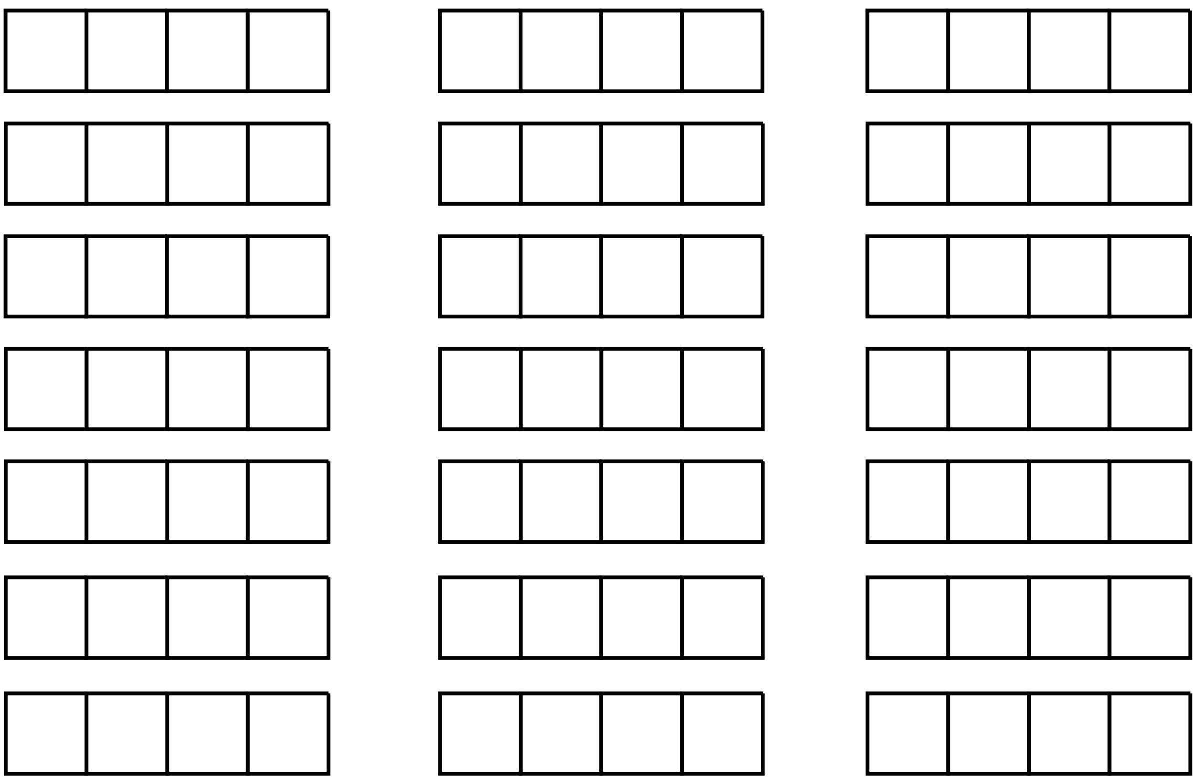
In general

$$n! = n \times (n-1)!$$

How many ways are there to shade this 1×4 figure?

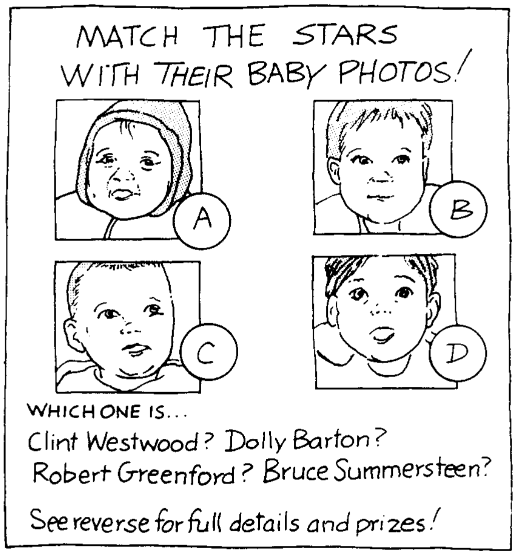


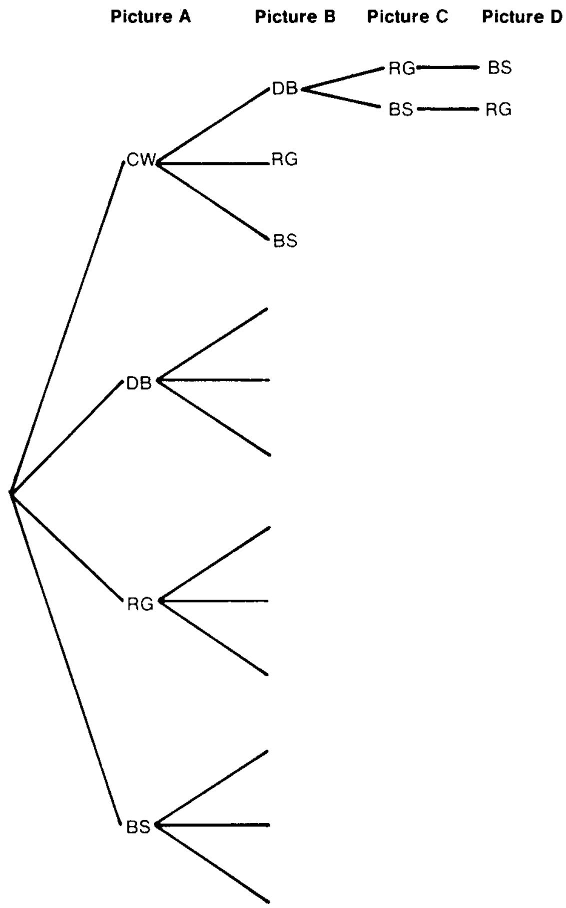
Examples



BY LISTING SYSTEMATICALLY, CAN WE FIND ALL THE WAYS TO SHADE THESE 4 SQUARES---WITHOUT MISSING ANY OR COUNTING ANY MORE THAN ONCE?

How Many Ways?





1 A popular magazine holds a contest in which entrants must match four film or recording stars to their baby photos. Travis decides to send in several entries—enough to ensure that one will be correct.

- **a** Complete the tree diagram.
- **b** How many different entries should Travis send?

2 Repeat problem **1** for a contest with three baby photos.

3 Repeat **1** for a contest with two baby photos.

4 Repeat **1** for a contest with five baby photos, but *without* using a tree diagram. (Hint: Set up a table and look for a pattern.)

Mathematical Investigations, Book 3
Dale Seymour Publications

NUMBER SHUFFLE

Make the number sentences true. Write 2, 3, or 6 in each circle, but use each number only once in a problem.

$\bigcirc + \bigcirc + \bigcirc = 11$

$\bigcirc + \bigcirc - \bigcirc = 7$

$\bigcirc + \bigcirc - \bigcirc = 5$

$\bigcirc \times \bigcirc + \bigcirc = 20$

$\bigcirc \times \bigcirc + \bigcirc = 15$

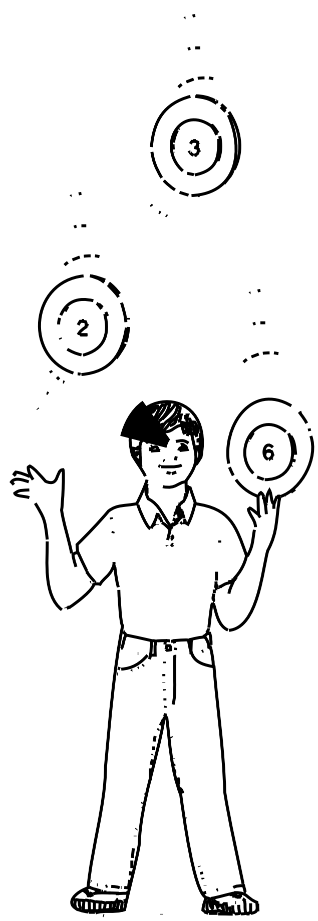
$\bigcirc \times \bigcirc - \bigcirc = 12$

$\bigcirc \times \bigcirc - \bigcirc = 16$

$\bigcirc \times \bigcirc - \bigcirc = 9$

$\bigcirc \times \bigcirc - \bigcirc = 0$

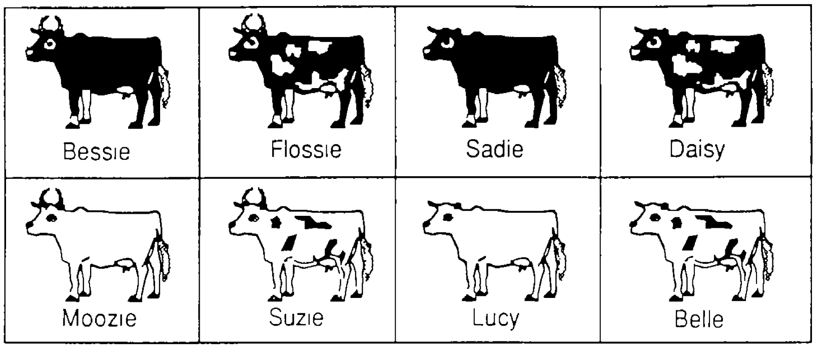
$\bigcirc \times \bigcirc \div \bigcirc = 9$



Resource Book

Workshop 6: Systematic Listing and Counting

OVER HERD **



Can you name each cow?

1. One cow was missing at feeding time. It was black and did not have spots or horns. Which cow was missing?

2. The first cow to get to the water trough was white with spots and horns. Which cow was first?

3. One sick cow was visited by the vet. Its white spotless coat seemed pink with fever. Even its horns were drooping. Which cow was ill?

4. The cow that was usually first to be milked was black, with spots and without horns. Which cow was it?

5. The cow that was usually last to be milked was black, with horns and without spots. Which cow was it?

6. One cow won a blue ribbon at the county fair. It was white and had spots but no horns. Which cow won the blue ribbon?

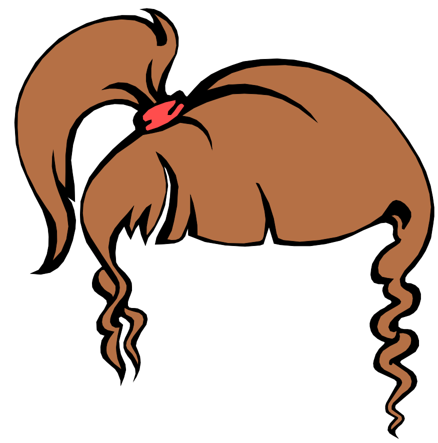
7. The white cow with spots and horns wears a bell around its neck. Which cow is it?

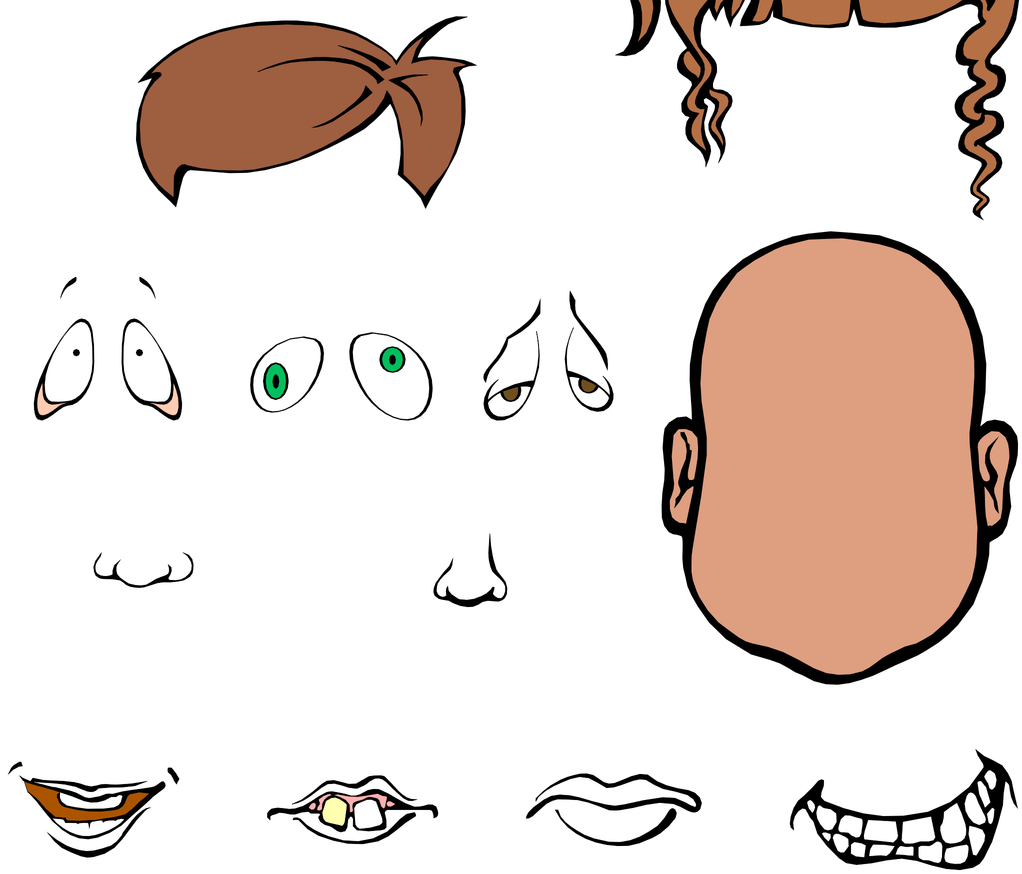
8. Two cows were strolling in the barn- yard. One was black, with spots and horns. The other was white, without spots or horns. What were their names?

9. Two cows won red ribbons at the county fair. One cow was black, with horns and without spots. The other cow was black, with spots and without horns. Which cows won red ribbons?

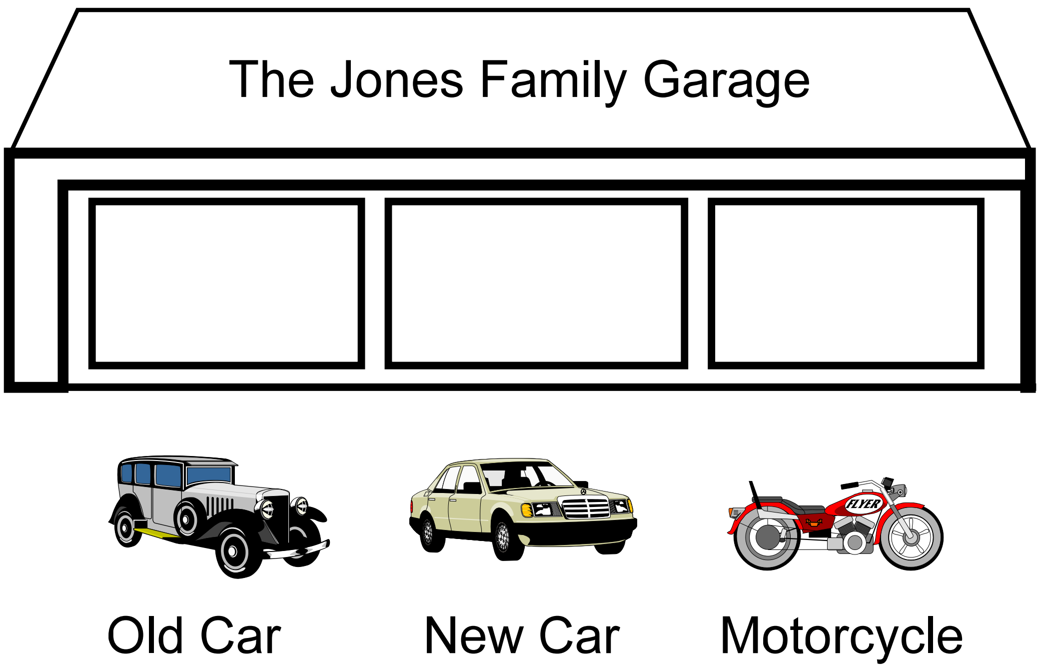
Workshop 6: Systematic Listing and Counting

How many ways are there to
give Silly Face
hair, eyes, nose and mouth?





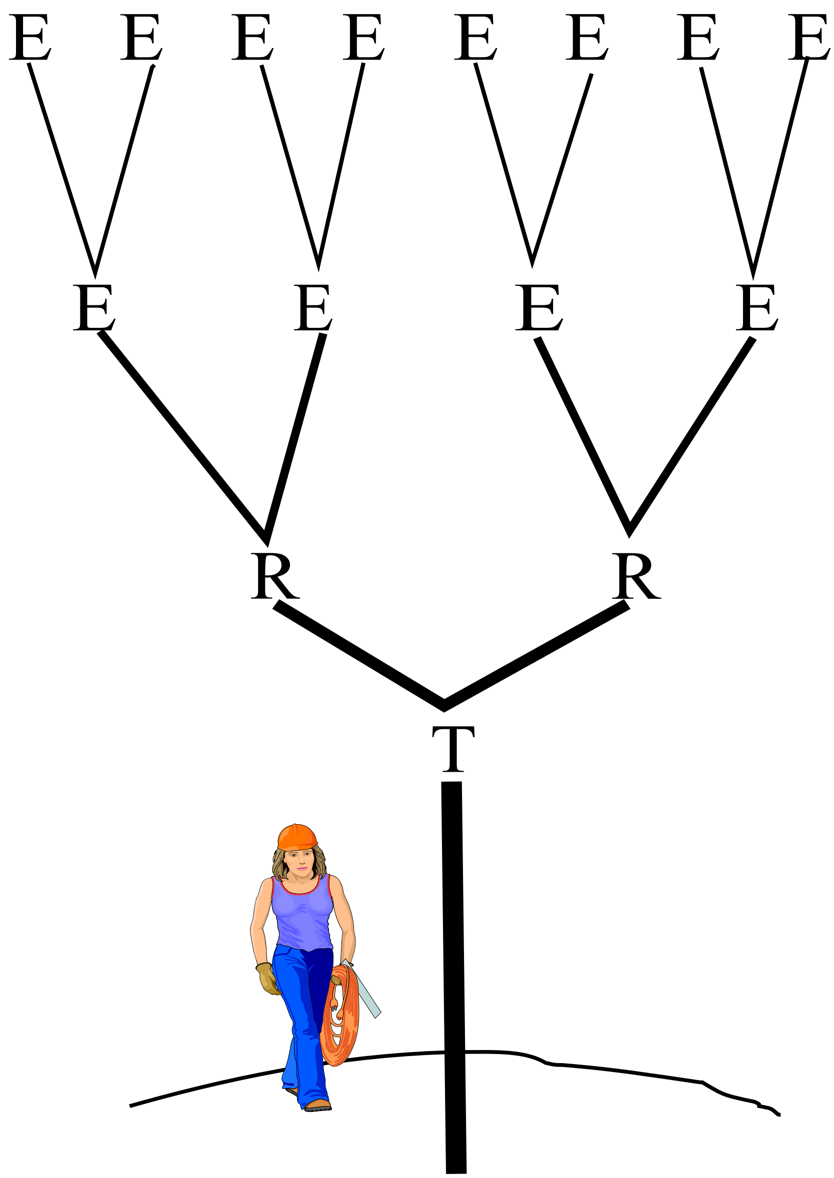
How many different ways are there for the Jones family to put their vehicles in the garage?



by Robert Hochberg. Copy as you wish

Resource Book

Workshop 6: Systematic Listing and Counting



How many ways can the tree trimmer climb to the top of this tree?

by Robert Hochberg. Copy as you wish