Master Document

Week 3, Day 2 — Matchings and Games

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised May, 2006

Week 3, Day 2 — Matching and Games

Materials Needed	**Allocated Time**
Activity #1 — Bipartite Graphs	20 minutes

- no materials needed

Activity #2 — Graph Coloring and Bipartite Graphs . . . . 25 minutes

- The large octagonal graph constructed the previous day using centers of paper plates and masking tape. (Each vertex is connected to five other vertices, excepting only the vertices which are two vertices away. Each exterior edge is about 3' long, so that opposite sides are about 7.2' apart.)
- Colored chips (circles with 3" diameter) to place on regions of the octagon — twenty-five chips of each of four colors.

Activity #3 — Matchings and Games . . . . 30 minutes

- no materials needed

Activity #4 — The Alabama Golden River Rock Game . . . . 50 minutes

- 25 brown "felt rocks"
- 1 gray rectangular "felt river bank"
- 9 yellow "felt golden rocks"
- 1 frog on a stick (or a frog mask).
- linker cubes (4 sets of 14 per table, if available)

. . . . TOTAL WORKSHOP TIME: 125* minutes

* In addition, 10 minutes are allocated for a break in this 2 ¼ hour workshop.
* Note that it is important that at least 45 minutes be spent on Activity #4.

Activity #1. Bipartite Graphs
(Allocated time = 20 minutes)

A. Recall the "School Dance" problem from the previous workshop's homework, and put up TSP #1 showing the graph of compatible dancers. Discuss briefly how this graph arises, with the vertices in two parts — the boys on one side and the girls on the other, naturally dividing the vertices into these two parts. Use TSP #1 to introduce the language of bipartite graphs, and to show that one very familiar class of graphs, namely the grid graphs, are really bipartite graphs. Use TSP #3 to explain why bipartite graphs are important.

B. Review TSP #4 on the terminology of matchings (a copy of yesterday's transparency TSP #14). Returning to TSP #1, ask why the dance problem had no perfect matching, with the expectation that some will respond that there were three boys who all wanted to dance only with two girls (or the complementary set). Ask what in general would exclude a perfect matching, with the expectation that someone will respond that one set could be larger than the set of all vertices to which its elements are connected. Then respond that in all other cases, there will be a perfect matching in a bipartite graph. Review TSP #5 and TSP #6, concluding with Hall's theorem on bipartite graphs.

C. Introduce the terminology of the "marriage problem" (TSP #7) and the translation of Hall's theorem for that situation. Compare the marriage and roommate problems (TSP #8), but do not actually try to solve the roommate problem presented there (that will be a homework problem). After they realize that the roommate problem is broader, review that using TSP #9. Note that the marriage problem and the roommate problem have different answers.

Activity #2. Graph Coloring and bipartite graphs.
(Allocated time = 25 minutes)

A. Coloring bipartite graphs. Referring back to TSP #1, ask participants how many colors you need for a bipartite graph, expecting that someone will suggest that two colors are sufficient. Ask whether anyone can think of other graphs that can be colored with two colors, and if so elicit the conclusion that those graphs are also bipartite. (Grid graphs should be mentioned here, even if no one suggests them, because they were discussed earlier as bipartite graphs.) In fact, if a graph can be colored with two colors, then it is bipartite — you can put the red vertices and the blue vertices into the two columns, since no edge connects two red vertices or two blue

vertices. So bipartite graphs are the same as graphs which can be colored with two colors. Moreover, a bipartite graph can only be colored in one way — once you pick the two colors and then color one vertex, the colors of all other vertices is determined. A common map that requires only two colors is a checkerboard. This information is summarized on TSP #10.

B. Two-colorable maps. Move to the octagonal graph on the floor. Ask how many colors are needed to color the map whose countries are the regions formed within the octagon on the floor. Have participants place colored markers on the map until they discover that it can be colored with two colors. In fact, it's easy to color with two colors, because once you put down one colored marker, everything follows from that. Recall that every map on the plane can be colored with four colors, but some can be colored with three colors, or even with two colors. Sometimes it is hard to tell whether or not a graph can be colored with three colors, but it is easy to tell whether or not a graph can be colored with two colors. You just use the greedy algorithm. If it works, the graph can be colored with two colors. If it doesn't work, the graph cannot be colored with two colors. After participants return to their seats, review all of this on TSP #11. Ask participants to draw a map which requires two colors. After a few minutes, construct a map by drawing several lines across the transparency and note that any such map can be colored with two colors.

C. Puzzles. In many problems and puzzles, the key to the solution is recognizing that the map or graph involved is bipartite. For example, ask participants to draw a 4x4 checkerboard (like that on TSP #10) and try to cover the checkerboard with 7 dominoes, leaving uncovered only two diagonally opposite squares. After they have had an opportunity to explore this situation, ask how many red squares and how many black squares would be covered by 7 dominoes. When appropriate, explain, using TSP #12 with the full 8x8 checkerboard, that this is impossible because each domino covers one red square and one black square, so 31 dominoes cover 31 red squares and 31 black squares; but diagonally opposite squares have the same color, so any covering of the remaining squares by 31 dominoes would have to cover 30 of one color and 32 of the other color.

Activity #3. Matching and Games
(Allocated time = 30 minutes)

A. Introduce Joe's Game on TSP #13, explain the rules, give participants a few minutes to digest them, and invite participants to play the game with you at the projector. They can choose whichever player they wish to be. They will likely correctly determine who wins in games A and B, but may choose incorrectly for C. (Don't go on to D, or you will surely run out of time for the later activities.) Analyze C completely; this is not too difficult since there are only two first moves for Player 1, and in each case essentially only one response for Player 2. If there is time, ask the participants to play the game on diagram D of TSP #13, and then discuss their conclusions about which player has the winning strategy; otherwise, D can be omitted. (For future reference: If the five crossings in D are added as vertices, Player I has a winning strategy.)

Many participants will have little concept of the idea of a strategy, so that they will in fact base their responses simply on several plays of the game, without further reflection; the goal is to get them to reflect on the games as well as play them.

What worked very well here was playing the game on an overlay TSP, with their moves in one color (where the move includes the edge to the new vertex selected) and the instructor's in another; the vertices should be numbered to indicate the sequence of moves. Save this TSP. Then later on, after they are shown the matching strategy for Joe's Game, you can review their winning plays, and show them that they were making all their moves along a matching in the graph.

TIME FOR A FIVE-MINUTE BREAK!
(Allocated time = 5-10 minutes)

B. Referring to TSP #13, ask participants what patterns they see. They will likely suggest that Player 1 has a winning strategy if there are an odd number of vertices and that Player 2 has a winning strategy if there are an even number of vertices. Try the game on the graph on TSP #14, and distribute HO #1. If there is time, have them play in pairs for a few minutes on this more challenging board, and then ask them who they think wins, and how this jives with their even-odd conjecture.

(Note: It is more important that participants have time to work on hand-outs #2 and #3, so that if you are at all behind schedule, please just do this activity at the overhead projector.)

After they realize that it is not evenness that makes for a winning strategy, play the

game several times with yourself as Player 1, following the strategy determined by a maximum matching that you have put on a blank transparency (but don't show the participants). Then reveal your strategy by showing your transparency and explaining how you used it. Then use TSP #15, where the secret of Joe's game is revealed, to discuss this strategy in general.

This example is a nice application of matchings, but also uses graph coloring to show that the graph doesn't have a perfect matching, and uses the "method of alternating chains" to show that Player 1 has the winning strategy. This method is not discussed during the workshop at all (we tried in the past, but it was hard to convince the participants of its usefulness) but can be "flashed" to individual participants subsequently in the following fashion: Imagine Player 1 following his winning strategy, by picking a matching, starting at an unmatched vertex, following matched edges, but blowing it and somehow losing. Then you can show that if all the edges along the path that was generated by game play were switched (into or out of the matching that Player 1 picked) then you have a larger matching, meaning that Player 1 blew the very first task of finding a maximum matching.

After revealing the secret, go back and review the plays as saved on the transparency generated in Activity 3A. Show how their winning moves corresponded to playing on the edges of a maximum matching.

Activity #4. The Alabama Golden River-Rock Game
(Allocated time = 50 minutes)

This activity will introduce the participants to games and strategies. The original version of the Alabama Golden River-Rock Game is an active version of the "Make-21 Game."

Note that we are now going to talk about games and strategies. Stress that what follows has nothing to do with bipartite graphs or matchings!! The common thread is games.

Note that the participants need time to play with these games, so make sure that the allocated time for this section is indeed available for playing the games.

A. Place the river bank and then 21 of the brown rocks in a long line. Select one participant to be the frog, and ask the frog to stand on the river bank. The frog has a burning desire to hop to the last rock!!

Now select two participants to play the game. Their job is to tell the frog how far it

should hop; but there are rules! At each turn, the frog is able to hop to either the next rock, or to the one after it (i.e. moving forward one or two rocks). The players take turns, and each turn they either tell the frog to hop to the next rock, or they tell the frog to hop over the next rock onto the following rock. The winner is the person who can legally tell the frog to hop on to the last rock.

It will eventually become apparent that the rocks could be numbered 1 to 21, and each step the frog moves to a rock that is 1 or 2 greater than where it was. However, we don't want to describe it this way at first.

The river bank is very important later on, as it acts as position 0; so when the frog takes its first step, it moves to rock number 1. It becomes quite confusing for the participants without the river bank.

Have the players play the game. Then select two new participants and a new frog to play it again. At some point, when the frog reaches the rock number 18 (don't call it 18; just point to it), one of the players will be aware that they have certainly won (obvious celebration beginning), or more likely, the player about to move the frog will realize they have lost (sagging shoulders, walking away, etc). At this point you must ask them what has happened. So you elicit that anytime a player moves the frog to this special rock they can win. Mark this by saying the last rock is special, because when you move the frog there you are the winner, so we will call it a golden rock. But also, we just observed that if ever you move the frog to "this" rock (i.e. rock #18), you can also ensure a win; so it too is a golden rock. It marks the way to the winning rock. So a golden rock is a position you move the frog to that has the property that NO MATTER WHAT THE OTHER PLAYER DOES, you can win by playing the right moves. You might as well just play to #18.

The important idea here is that we are setting up a visual understanding of a winning position; golden rocks become synonymous with winning positions. For the next two days, as we play all sorts of 2-player games, we will be asking participants "Can you see any golden rocks?" and "Where is the next golden rock?"

It may be worth mentioning now that even if you move the frog to rock #18, you still have to play the right moves if you are to win! For example, if the other player moves to #19, you can still lose by playing the bad move of moving the frog to #20.

B. Have three new participants play the game, looking for the signs of recognition that now rock #15 is another golden rock. Once that connection has been made, lay a golden rock on it.

Eventually the golden rocks will be on every third rock. Visually they now mark the way to win. Point out that when we started, we had no idea how to win this game. Gradually, working our way backwards, we found many winning positions, the golden rocks. So, as we came to understand the game, the fog of brown rocks everywhere in view cleared to reveal the golden rocks. Point out that over the next 2 days, we will play several 2-player games, and when we begin, all we will see is brown rocks all over the place. But there is gold in some of them, and that will be the challenge - find those golden rocks!!

Some participants will notice that the golden rocks are on every third rock. If this happens, point out that sometimes it is possible that you can find a rule that easily tells you where the golden rocks are. In this case, one such rule would be: golden rocks have names that are divisible by 3. Another such rule would be: whatever Player 1 makes the frog do (move ahead 1 rock or 2), Player 2 finds the next golden rock by making the frog move forward the other possibility (move ahead 2 rocks or 1 respectively).

Point out that now that we can see the golden rocks, we know that Player 2 will win every game (providing they always move the frog on to a golden rock whenever it is their turn). This is called a Player 2 Win Game. It doesn't mean that Player 2 will always win (probably you have examples when Player 1 won); but it does mean that Player 2 can always win, regardless of what Player 1 does, providing they follow the golden road!!

You may also want to point out that not all 2-Player games can always be won by the same person. See if they can think of an example - tic tac toe is the likely suggestion.

C. Have the participants return to their seats and d istribute Handout #1 (= TSP #15) which has on it the "Make-21 game". See below for a description of the game. Ask participants to work in groups of 2 — one person designated as Player 1 and the other designated as Player 2 — they will work cooperatively, not competitively, to figure out all of today's games. They should soon recognize that this is the same game as the Alabama Golden River-Rock Game.

Using TSP #17, introduce the notions of a winning position (like 18) (a Golden Rock), a winning move (moving to 18), a winning strategy (always moving to a multiple of 3), a player-1-win-game, and a player-2-win-game.

The "make 21" game.

THE SETUP: This is a two player game. On the board write a zero (the counter's initial value) and make two columns labeled "Player 1" and "Player 2". The rules are that a player can add 1 or 2 to the counter, and the first player to get to 21 wins.
THE PLAY: Have a player 1 play against a Player 2 while you write the sums on the board (they don't have to do this when they play amongst themselves, it's just for your record-keeping.) A sample game is shown to the right. Then, when they have the idea, have them play amongst themselves, again trying to figure out what is a good strategy.
THE REGROUP: This turns out to be a player 2-win-game, because, if he plays ideally, player 2 can guarantee a win no matter what player 1 does. His winning strategy is to do the following: if Player 1 adds 1, then he adds 2, and if Player 1 adds 2, then Player 2 adds 1. This keeps Player 2 on the multiples of 3, which are the **winning positions** for this game. If player 1 can somehow get onto a multiple of 3 (by Player 2's fault) then he can stay there and win the game. The multiples of 3 are really winning positions for whoever's on them.
THE GOALS: To reinforce the notions introduced in the first activity. Also to make connections with the curriculum, of course. This is a nice addition game, and can be adjusted, as in the homework, to adding more numbers.

Player 1	Player 2
1	3
4	6
8	9
10	12
14	15
17	18
19	21

D. Then have the participants work in groups of 2 to play the "Make-25 Game." You could tell them that they are playing the Alabama Golden River-Rock Game but with 25 brown rocks.

The participants may at first want the golden rocks to be the ones divisible by 3 but eventually they realize that rock #22 is now a golden rock.

E. Once the participants realize that rock #1 is a golden rock, return to the line of rocks. Move the river bank back and add 4 more brown rocks to the beginning of the rock path (doing it this way allows all the golden rocks to still be in correct positions, and for the same reason as was used before!). Ask the participants which of the new rocks are golden (#4 and #1).

Discuss with the group the "Make-25 game" and conclude that Player 1 has a winning strategy, which involves starting with 1, since the winning positions are 22, 19, 16, 13, 10, 7, 4, and 1. Note that Player 1 here has stolen Player 2's winning strategy, and has in effect become Player 2 in a game beginning with 1.

Summarize the activity, starting not knowing what to do, gradually coming to understand the problem and identifying the moves to winning positions (golden rocks).

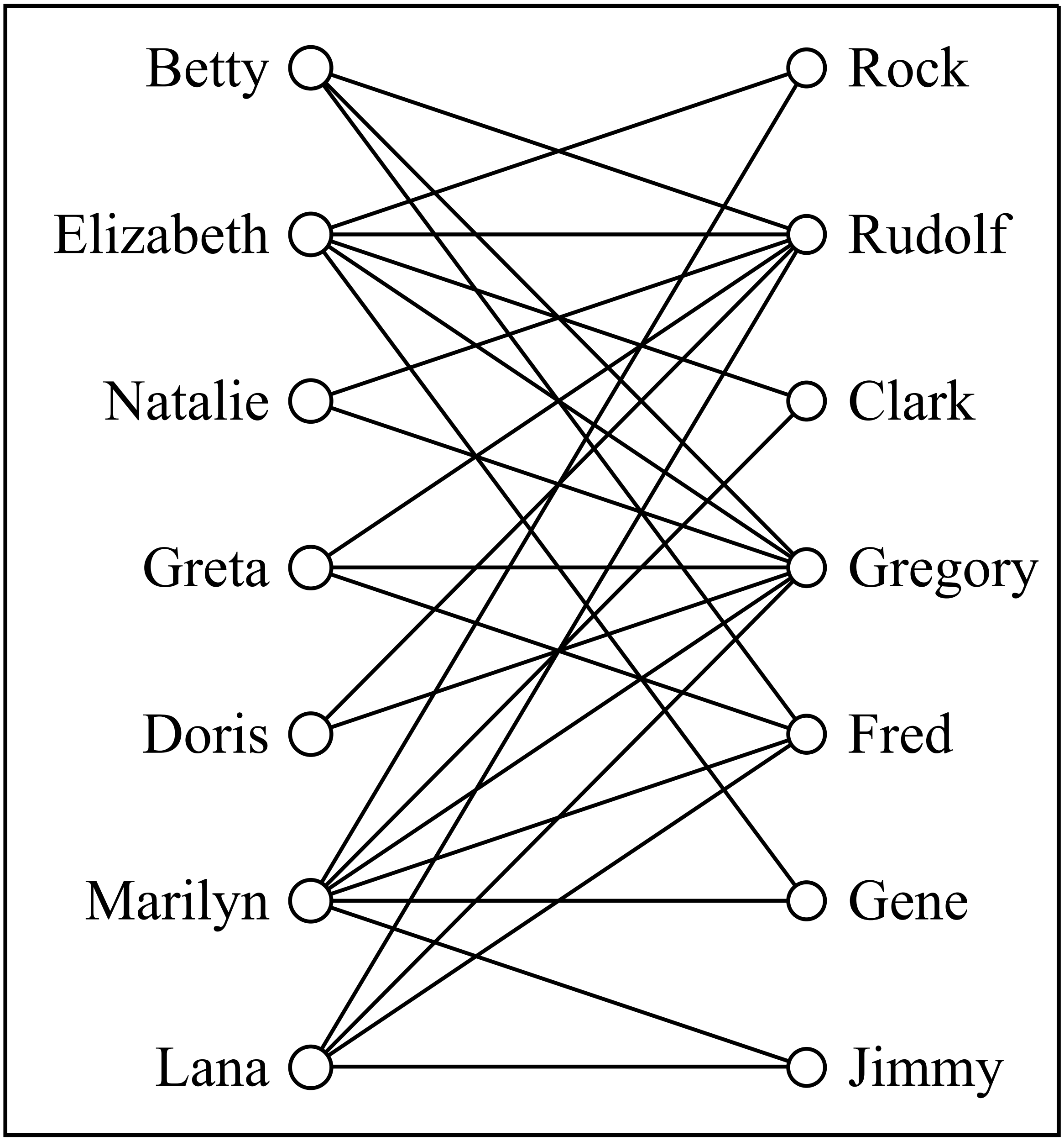
It will be invaluable over the next 2 days to constantly use the "golden rock" metaphor. Traditionally, understanding the idea of a winning position is one the participants find difficult, but when we constantly used this visual metaphor it was smooth sailing!

F. Have the participants again return to their seats. Distribute linker cubes (four sets of 14 per table, to represent the two piles) and handout #3 (= TSP #18) which has on it the "Two-pile Nim game". See below for a description of this game. Describe the game, play a sample game, have participants play the game for a few minutes, and have them describe who wins the game and how. Ask participants to work in groups of 2 — one person designated as Player 1 and the other designated as Player 2 — remind them that they should be working cooperatively, not competitively, to figure out all of today's games. Discuss the game in terms of the terminology used in TSP #17. Discuss the variation which occurs when the two piles are initially the same.

Two pile Nim.
THE SETUP: This game starts with two piles of any size. Players take turns removing any number of objects from a single pile, and the winner is the person who removes the last object. This can be played with coins, linker cubes, pattern blocks, paper clips, anything! THE PLAY: Again, have a player 1 play against a player 2 just to show the class how it goes. Then let them play amongst themselves for about 5 minutes. It is probably a good idea to standardize a starting configuration, say piles of size 6 and 8, for a while. Then you can consider generalizations during the regroup time.
THE REGROUP: This game is a first-player-win-game, and his winning move is to equalize the piles. Thereafter, his winning strategy is to keep the piles equal. Thus, the winning positions for this game are those that have piles of equal size.

Considering different starting positions, you can classify them into first-player-win-games and second-player-win-games. Of course, the games that have equal piles to start with are second-player-win-games, and the rest are first-player-win-games.

The School Dance compatibility graph

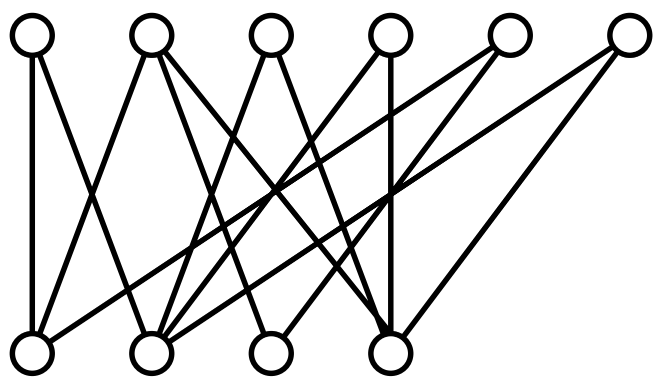


Can you find a perfect matching in this graph?

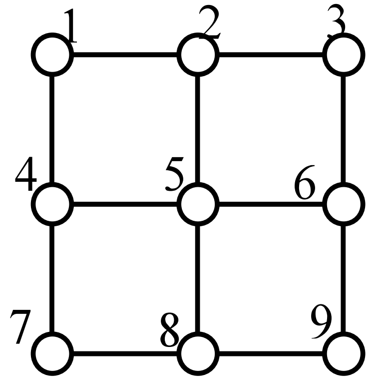
Bipartite graphs

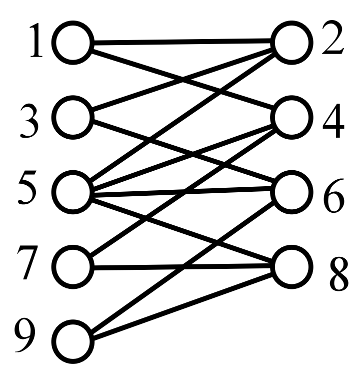
A graph is bipartite if its vertices can be separated into two groups so that each edge joins a vertex in one group with a vertex in the other group.

Example:



Sometimes, graphs which don't seem to be bipartite actually are. For example, every grid graph is bipartite.





What’s so Special about Bipartite Graphs?

Bipartite graphs are the graphs which are often used to model the applications of matchings where there are two types of vertices which are matched to each other — for example, tasks to days, or people to jobs.

We saw a few examples of this in the last workshop and on the homework:

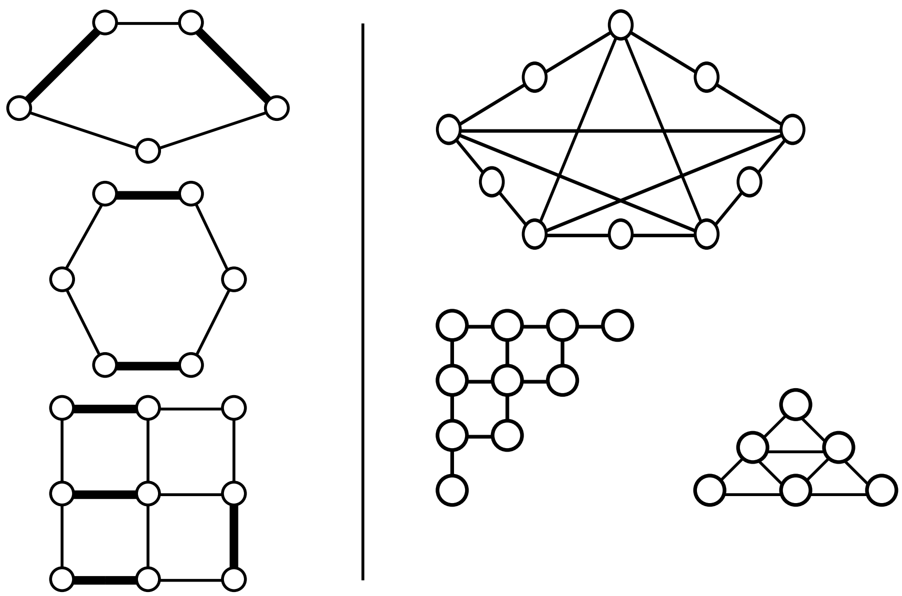
1. The Carpoolers Problem

2. Children and Pets

3. The School Dance

Matching in graphs

Matching. A matching M is a set of edges of the graph, no two of which share a vertex.



If one of the edges connects the vertex v to the vertex w, then we say that the matching M matches v to w.

Maximum matching. A maximum matching is one which matches as many vertices as possible.

Perfect matching. A perfect matching is one which matches every vertex; this can happen only if the number of vertices is even.

When does a bipartite graph have a perfect matching?

1. **If the two components have different sizes, then there can't be a perfect matching.**

Example: There is no perfect matching of the vertices of a 3x3 grid.

2. **If there is a set of vertices on one side which together are connected to a smaller number of vertices on the other side, then there can't be a perfect matching.**

Example: In the school dance problem, the four girls Betty, Natalie, Greta, and Doris together are connected to only three boys Rudolf, Gregory, and Fred.

The Good News

If neither of those situations apply, then the bipartite graph always has a perfect matching.

That is:

If the two components of a bipartite graph have the same size, and if each set of vertices on one side are together connected to at least that many vertices on the other side, then the graph has a perfect matching. (Philip Hall)

Moreover, there is a method (an algorithm) that will always find a perfect matching.

Another interpretation . . .

The question of whether a bipartite graph has a perfect matching is often referred to as . . .

. . . the marriage problem.

This is because . . . if you imagine the vertices on one side as men and the vertices on the other side as women, and an edge joining a man and woman if they are mutually acceptable . . . then a perfect matching would pair up all the men with all the women. (A perfect matching is called a solution to the marriage problem.)

Is there a solution to the marriage problem?

If the number of men and women are equal, and if for every group of people of one sex, the group of people of the other sex who are mutually acceptable to at least one of them is at least as large as the original group, then there is a perfect matching, and an algorithm for finding one. (Hall)

Two parallel problems.

Marriage Problem. Each person provides a list of mutually acceptable partners of the opposite sex. Is there a matching which pairs each person with an acceptable partner? If so, can you find it?

Roommate Problem. Each person provides a list of mutually acceptable roommates. Is there a matching which pairs each person with an acceptable roommate? If so, can you find it?

Are these two problems the same . . .

. . . or are they different?

Example:

Al — Bob, Cal, Fred	Fred — Al, Eric, Gil
Bob — Al, Dave, Hal	Gil — Dave, Fred, Hal
Cal — Al, Ira, Joe	Hal — Bob, Gil, Joe
Dave — Bob, Eric, Gil	Ira — Cal, Eric, Joe
Eric — Dave, Fred, Ira	Joe — Cal, Hal, Ira

Why are these problems different?

Which is more difficult to solve?

Marriage Problem: Does a bipartite graph have a perfect matching?

Roommate Problem: Does a graph have a perfect matching?

The Roommate Problem is broader than the Marriage Problem.

Solving the Marriage Problem: In the marriage problem (bipartite case), there is a way of telling whether there is a perfect matching, and a method of finding such a matching.

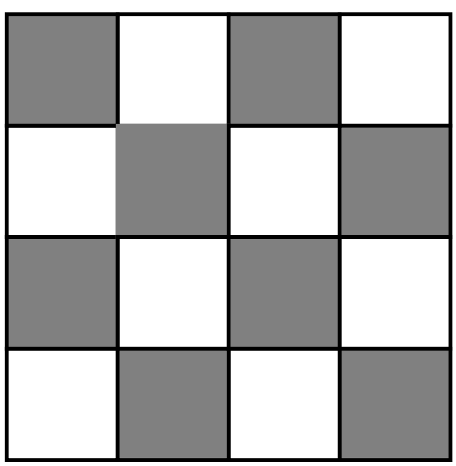
Solving the Roommate Problem: In the roommate problem (general case), there is no way of telling whether there is a perfect matching, and, if there is one, no efficient method that will always find it.

Coloring bipartite graphs

Every bipartite graph (such as grid graphs) can be colored with two colors. On the other hand, every graph which can be colored with two colors is bipartite.

Thus, the bipartite graphs coincide with the 2-colorable graphs.

Moreover, a connected bipartite graph can only be colored in one way — in the sense that — once you pick the two colors and then color one vertex, the colors of all other vertices are determined.



A Greedy Algorithm for 2-Coloring Bipartite Graphs

How can you tell whether a graph or map can be colored with two colors?

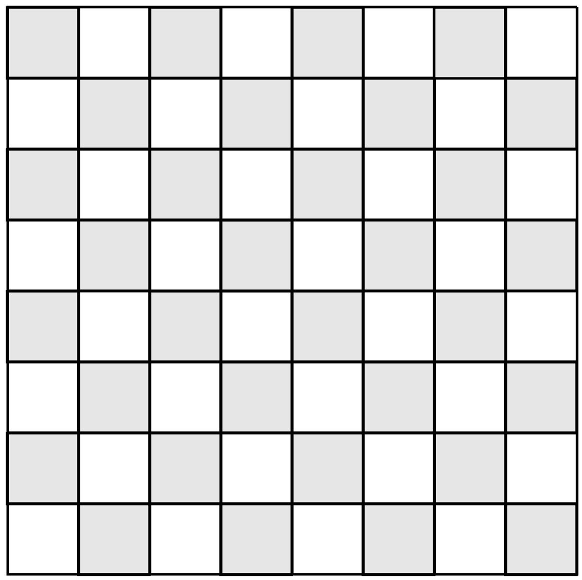
- Color one vertex blue, color all of its neighbors red, color all of their neighbors blue, color all of their neighbors red, etc.

- After each stage, stop to check whether any two adjacent vertices have been colored the same color; if they have, the graph was not bipartite, but if not, continue with the coloring.

- When you are finished, if no two adjacent vertices have been assigned the same color, then the graph is bipartite and a two-coloring has been found.

This algorithm (and a similar one for maps) will tell you whether a graph (or map) can be colored with two colors, and, if it can, will provide such a coloring.

Can you cover the checkerboard with 31 dominoes (each covering 2 squares) so that the top left and bottom right squares remain uncovered?



Any domino covers one shaded square and one unshaded square, so 31 dominoes would cover 31 shaded squares and 31 unshaded squares. The two squares remaining would have to include one shaded square and one unshaded square.

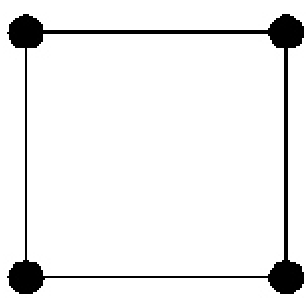
Joe's Game: This game can be played using any graph. Player 1 and Player 2 alternate picking vertices in the graph, subject to the following two rules:

1. Neither player can pick a vertex that has been picked earlier in the game.
2. Every vertex must be adjacent to the vertex picked just before.

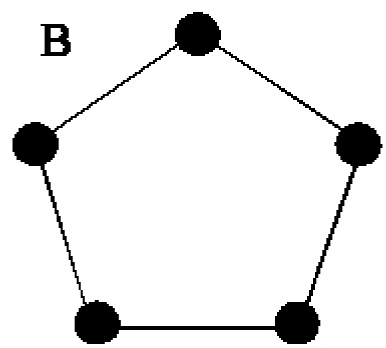
The winner is the last player who can choose a vertex.

Who wins if the game is played using each of the graphs below?

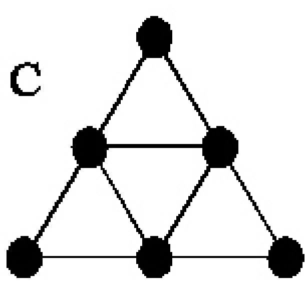
A



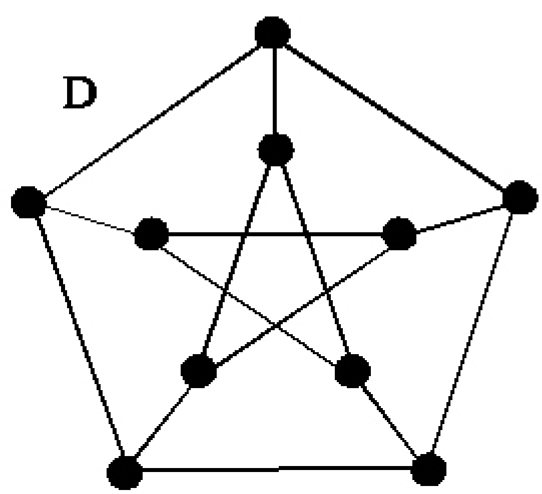
B



C



D

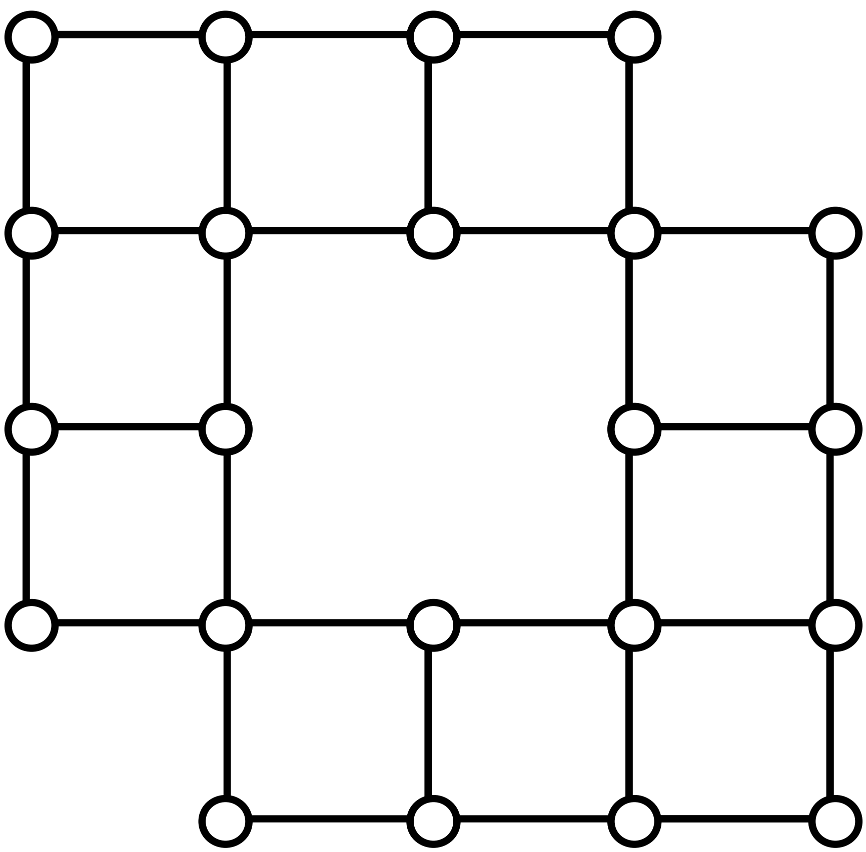


Joe's Game: This game can be played using any graph. Player 1 and Player 2 alternate picking vertices in the graph, subject to the following two rules:

1. Neither player can pick a vertex that has been picked earlier in the game.
2. Every vertex must be adjacent to the vertex picked just before.

The winner is the last player who can choose a vertex.

Who wins if the game is played using the graph below?



The Secret of Joe's Game:

Find a maximum matching in the graph.

If the maximum matching is perfect, arrange to be Player 2. Then whatever vertex Player 1 picks, always pick the matching vertex. This way you will always win, because you will always be picking a new vertex, and it will always be adjacent to Player 1's last choice.

If the maximum matching is not perfect, arrange to be Player 1. For your first turn, select a vertex which is not matched in the maximum matching. Then whatever Player 2 does, always pick the matching vertex. This way you will always be picking a new vertex which is adjacent to Player 2's last choice.

> But what, you may ask, if Player 2 picks a vertex which is not matched in the maximum matching. Then, dear soul, you blew it — what you started with was not a maximum matching.

This secret is not to be repeated ...
... and Joe gets 20% royalties of all winnings.

The “Make 21” Game.

Starting with a zero total, Player 1 and Player 2 alternate adding either 1 or 2 to the total. Whoever makes 21 wins. Which player has a winning strategy?

Work in pairs — one person designated as Player 1 and the other as Player 2. Your purpose, however, is not to vanquish your opponent, but to work cooperatively to figure out the game.

If you have figured out this game, try the next one:

The “Make 25” Game.

Starting with a zero total, Player 1 and Player 2 alternate adding either 1 or 2 to the total. Whoever makes 25 wins. Which player has a winning strategy?

Terminology

Winning position. A position in a game is called a winning position if, no matter what moves the next player makes, the other player can always win the game.

Winning move. A move is called a winning move if making it results in a winning position.

Winning strategy. A winning strategy is one which results in a victory for the player with the strategy. For example, a winning strategy might involve moving to a winning position, and then continuing to move to winning positions in response to the other player's moves.

Player-1-winning-game. A Player-1-winning-game is one where Player 1 has a winning strategy.

Player-2-winning-game. A Player-2-winning-game is one where Player 2 has a winning strategy.

“Two pile Nim”

Starting with two piles of six and eight cubes, Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last object wins. Which player has a winning strategy?

> Work in pairs — one person designated as Player 1 and the other as Player 2. Your purpose, however, is not to vanquish your opponent, but to work cooperatively to figure out the game.

If you have figured out this game, try the next one:

“Three pile Nim”

Starting with three piles of three, two, and one cubes, Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last object wins. Which player has a winning strategy?

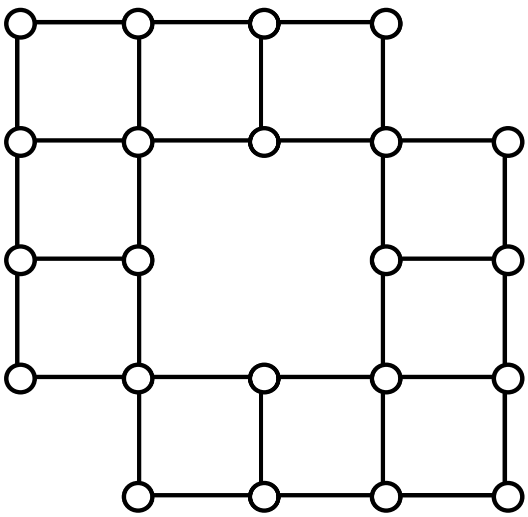
Handout #1 — Joe's Game

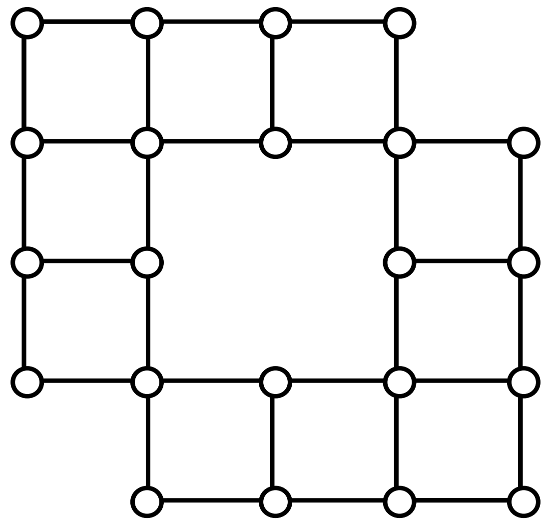
This game can be played using any graph. Player 1 and Player 2 alternate picking vertices in the graph, subject to the following two rules:

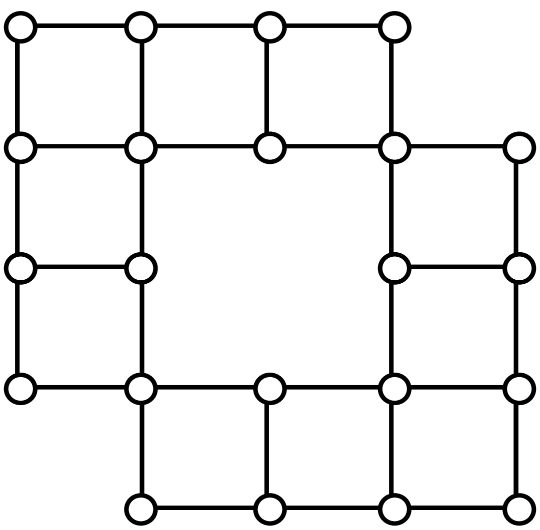
1. Neither player can pick a vertex that has been picked earlier in the game.
2. Every vertex must be adjacent to the vertex picked just before.

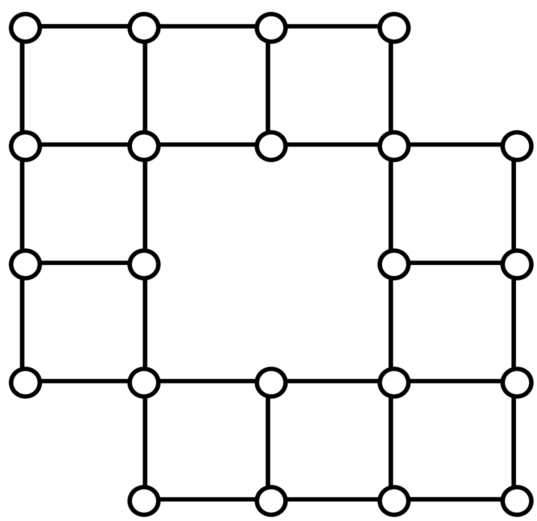
The winner is the last player who can choose a vertex.

Who wins if the game is played using the graph below? Four copies are provided.









Hand-out #2.

The “Make 21” Game.

Starting with a zero total, Player 1 and Player 2 alternate adding either 1 or 2 to the total. Whoever makes 21 wins. Which player has a winning strategy?

> Work in pairs — one person designated as Player 1 and the other as Player 2. Your purpose, however, is not to vanquish your opponent, but to work cooperatively to figure out the game.
>
> If you have figured out this game, try the next one:

The “Make 25” Game.

Starting with a zero total, Player 1 and Player 2 alternate adding either 1 or 2 to the total. Whoever makes 25 wins. Which player has a winning strategy?

Terminology

Winning position. A position in a game is called a winning position if, no matter what moves the next player makes, the other player can always win the game.

Winning move. A move is called a winning move if making it results in a winning position.

Winning strategy. A winning strategy is one which results in a victory for the player with the strategy. For example, a winning strategy might involve moving to a winning position, and then continuing to move to winning positions in response to the other player's moves.

Player-1-winning-game. A Player-1-winning-game is one where Player 1 has a winning strategy.

Player-2-winning-game. A Player-2-winning-game is one where Player 2 has a winning strategy.

Handout #3.

"Two pile Nim"

Starting with two piles of six and eight cubes, Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last object wins. Which player has a winning strategy?

> Work in pairs — one person designated as Player 1 and the other as Player 2. Your purpose, however, is not to vanquish your opponent, but to work cooperatively to figure out the game.
>
> If you have figured out this game, try the next one:

"Three pile Nim"

Starting with three piles of three, two, and one cubes, Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last object wins. Which player has a winning strategy?

Week 3, Session 2: Matchings and Games — Exercises

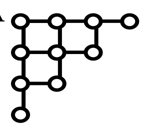
Practice Problems:

1. A stack of 21 pennies is on the table, and two players take turns removing 1 or 2 pennies from the stack. The winner is the player who takes the last penny.
   a. Which player should win this game if both players play as well as they can, and what is the winning strategy for that player?
   b. What if the stack starts with 20 pennies? Who wins and by what strategy?
   What if it starts with 19 pennies?

2. There are two stacks of 21 pennies on the table, and two players take turns removing 1 or 2 pennies from either stack; note that on a given turn, if two are removed from the stack, the player must remove both pennies from the same stack. The winner is the player who takes the last penny. Which player should win this game, and by what strategy?

Study Group Problems:

3. Who wins Joe's Game if the following two graphs are used? In each case, indicate the specific strategy that the player with the winning strategy should follow.

A

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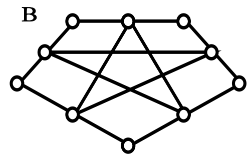
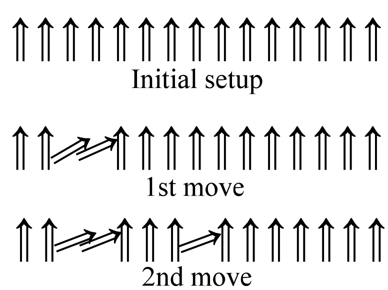


4. In the game of "Kaylls", 15 pins are set up in a row and players take turns knocking them down with a bowling ball. On each turn, a player can knock down either 1 or 2 pins — but if he knocks down 2 pins, they must be adjacent (since the ball is only so wide). The winner is the last person to knock down a pin. The first two moves of a sample game is shown to the right. Note: These do not have to be the first two moves of every game.



   a. Who should win the game of Kaylls, and by what strategy? (Hint: Do you see any connection with Nim?)

   b. Who should win the game, and by what strategy, if there are 16 pins? if there are *n* pins?

5. "Daisy Chain Game" — Thirteen beads are arrayed around a circle. Each player may remove either one or two adjacent beads. (The beads removed have to be adjacent to each other, not to the beads just removed. Also, the beads must have been adjacent in the original configuration.) Whoever removes the last bead wins the game. Who should win this game, and using what strategy? (Hint: Do you see any connection with Nim?)

6. "Northcott's Game" — Here the players are called "X" and "O" instead of "Player 1" and "Player 2", and each has two chips. Starting with the chips in the positions below, X and O alternate turns; at each turn, a player moves one of his/her tokens to another position on the same row (moving backwards is permissible) without jumping over the other player's piece. The last move in a game leaves the other player with no valid moves. Whoever makes the last move wins the game. Which player has a winning strategy?

X									O
X									O

7. a. Construct a bipartite graph with at least six vertices which has an Euler circuit and the same number of vertices in each of its two parts.
   b. Construct a bipartite graph with at least six vertices which has a Hamilton circuit and the same number of vertices in each of its two parts.
   c. Construct a bipartite graph with at least six vertices which has an Euler circuit, but a different number of vertices in each of its two parts.
   d. Construct a bipartite graph with at least six vertices which has a Hamilton circuit, but a different number of vertices in each of its two parts.

   Note: If, in any of the cases above, it is *impossible* to construct that type of graph, give the reason why!

8. a. Can twenty five students in a 5x5 array of seats each move to an adjacent seat and still have one person per seat?
   b. Is there a Hamilton circuit in a 5x5 grid?

9. Can you find a solution to the roommate problem involving ten students each of whom is willing to room with the indicated people:

   Al — Bob, Cal, Fred
   Bob — Al, Dave, Hal
   Cal — Al, Ira, Joe
   Dave — Bob, Eric, Gil
   Eric — Dave, Fred, Ira
   Fred — Al, Eric, Gil
   Gil — Dave, Fred, Hal
   Hal — Bob, Gil, Joe
   Ira — Cal, Eric, Joe
   Joe — Cal, Hal, Ira

10. Given any red square and any black square on a 4x4 checkerboard, can you always place 7 dominoes so that exactly those two squares remain uncovered? What about an 8x8 checkerboard and 31 dominoes? (Use the grids on page EX 3.)

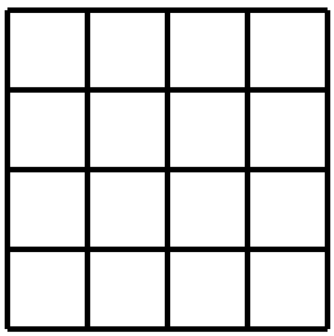
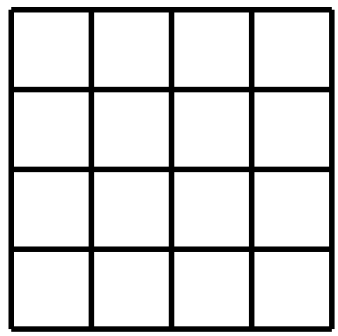
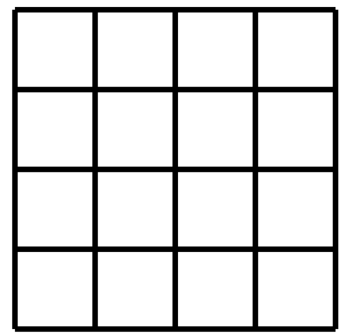
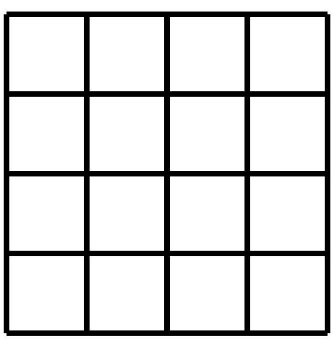
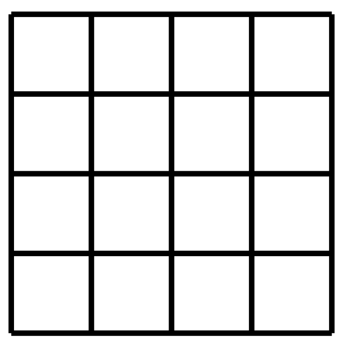
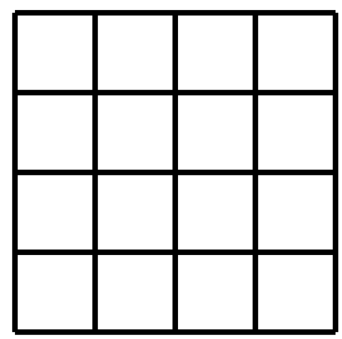
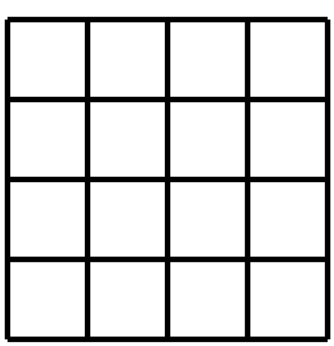
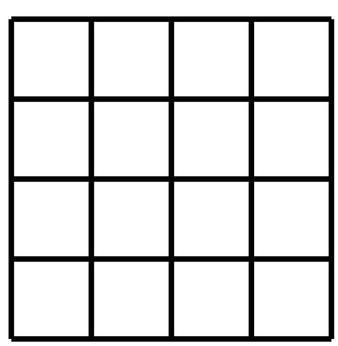
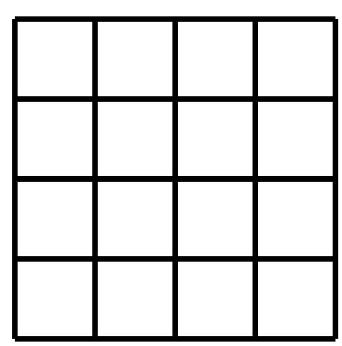
Extension problems:

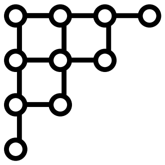
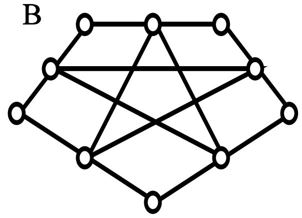
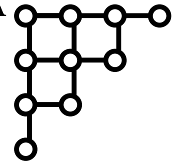
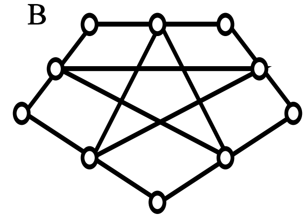
11. Each of two players alternate placing a domino on a 4x4 checkerboard. The winner is the person who places the last domino. Which player has a winning strategy, and what is that strategy?

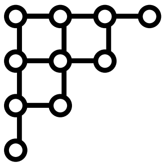
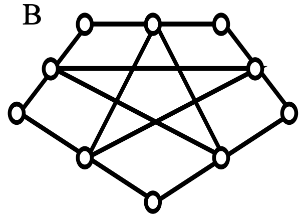
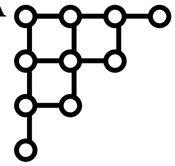
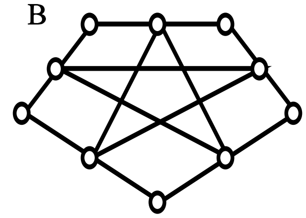
12. Play "dots and boxes" on the diagrams at the top of page EX 5. Player 1 and Player 2 alternate connecting dots. When a player completes a box, he puts 1 or 2 in the box and goes again. Whoever has their number in the larger number of boxes wins the game. What is Player 1's best starting move?

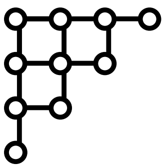
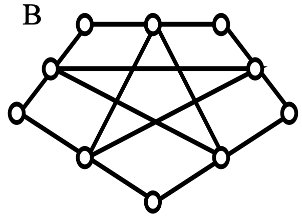
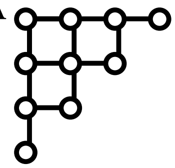
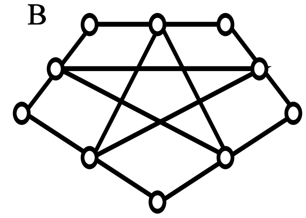
13. Play some games of "dots and triangles" on the diagrams on page EX 5. Player 1 and Player 2 alternate connecting dots. When a player completes a triangle, he puts 1 or 2 in the triangle and goes again. Whoever has their number in the larger number of triangles wins the game. Which player has a winning strategy?

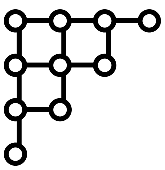
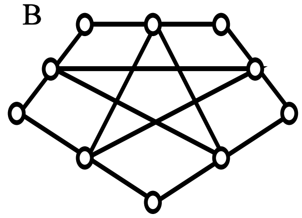
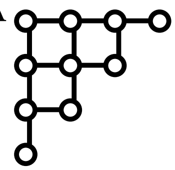
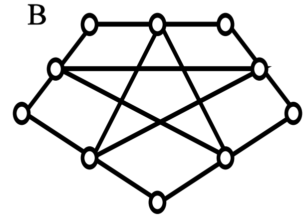
14. Play some games of "3x3 Hex" on the Hex grids on page EX 6. Players alternately place a marker of his/her color into an empty hexagon; Player 1 tries to complete a path between the top and bottom rows of hexagons, while Player 2 tries to complete a path between the left and right rows. The winner is the first player to complete such a path. (The first grid contains a sample game which Player 2 has won.) Can you find a winning strategy for either player?

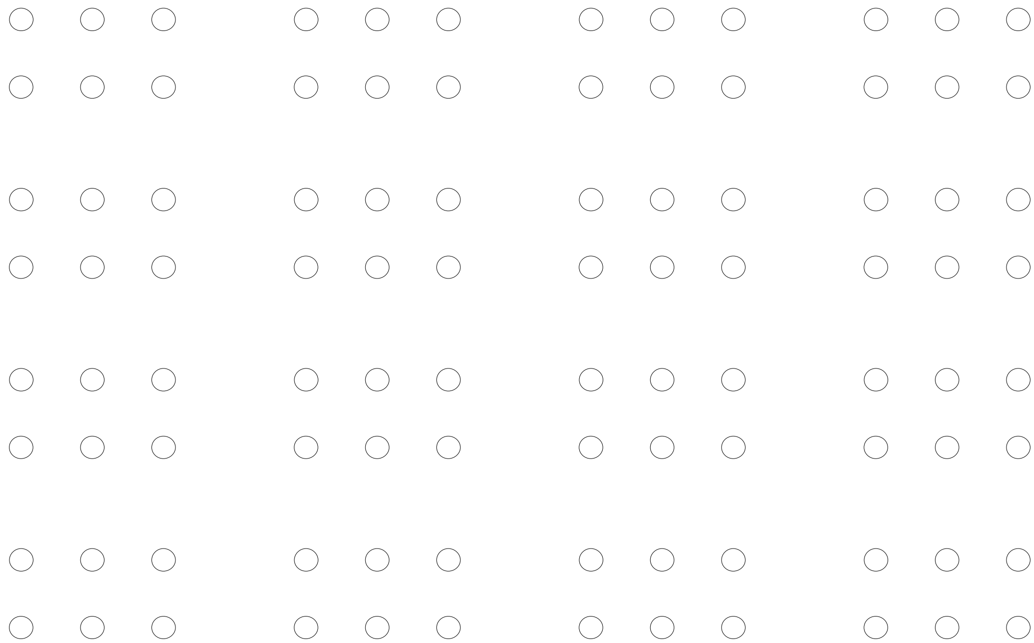
A  B  A  B 

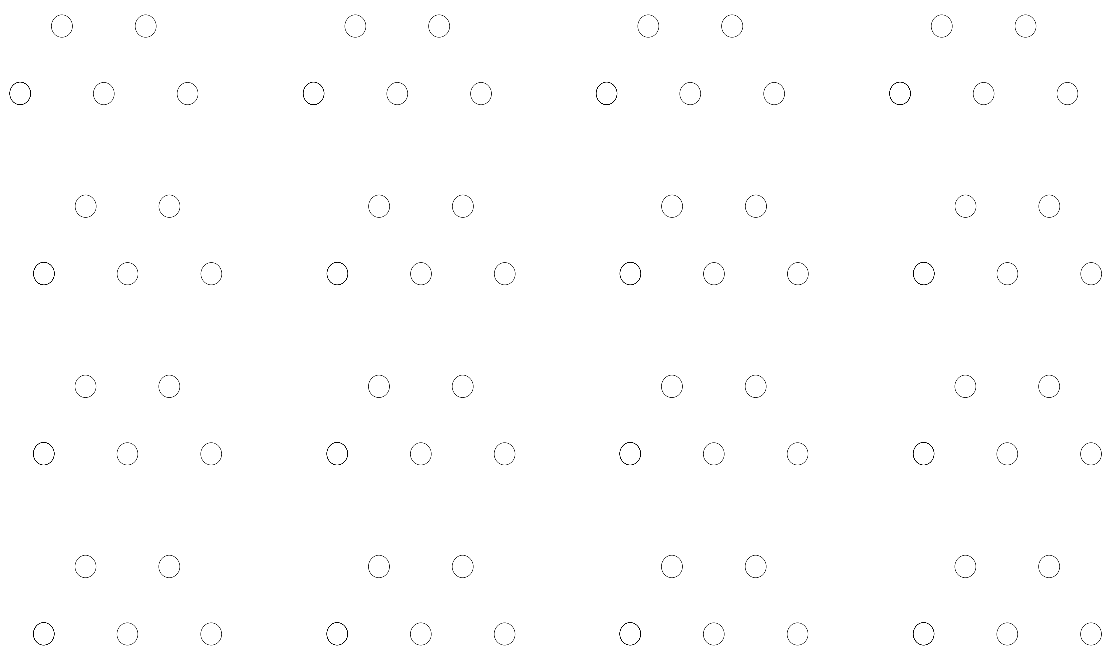
A  B  A  B 

A  B  A  B 

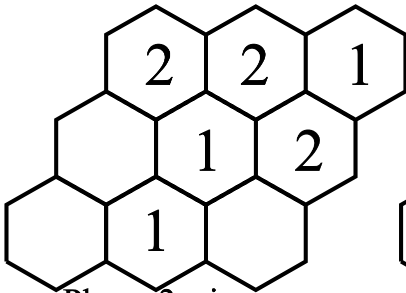
A  B  A  B 

Some grids for square and triangular "dots and boxes" games

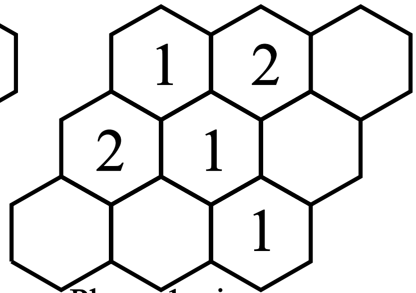




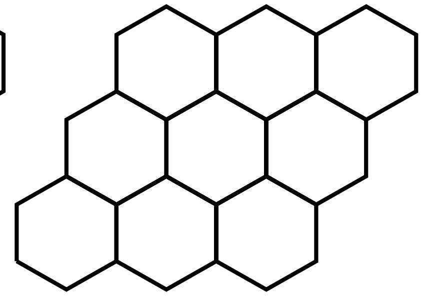
Some Hex Grids

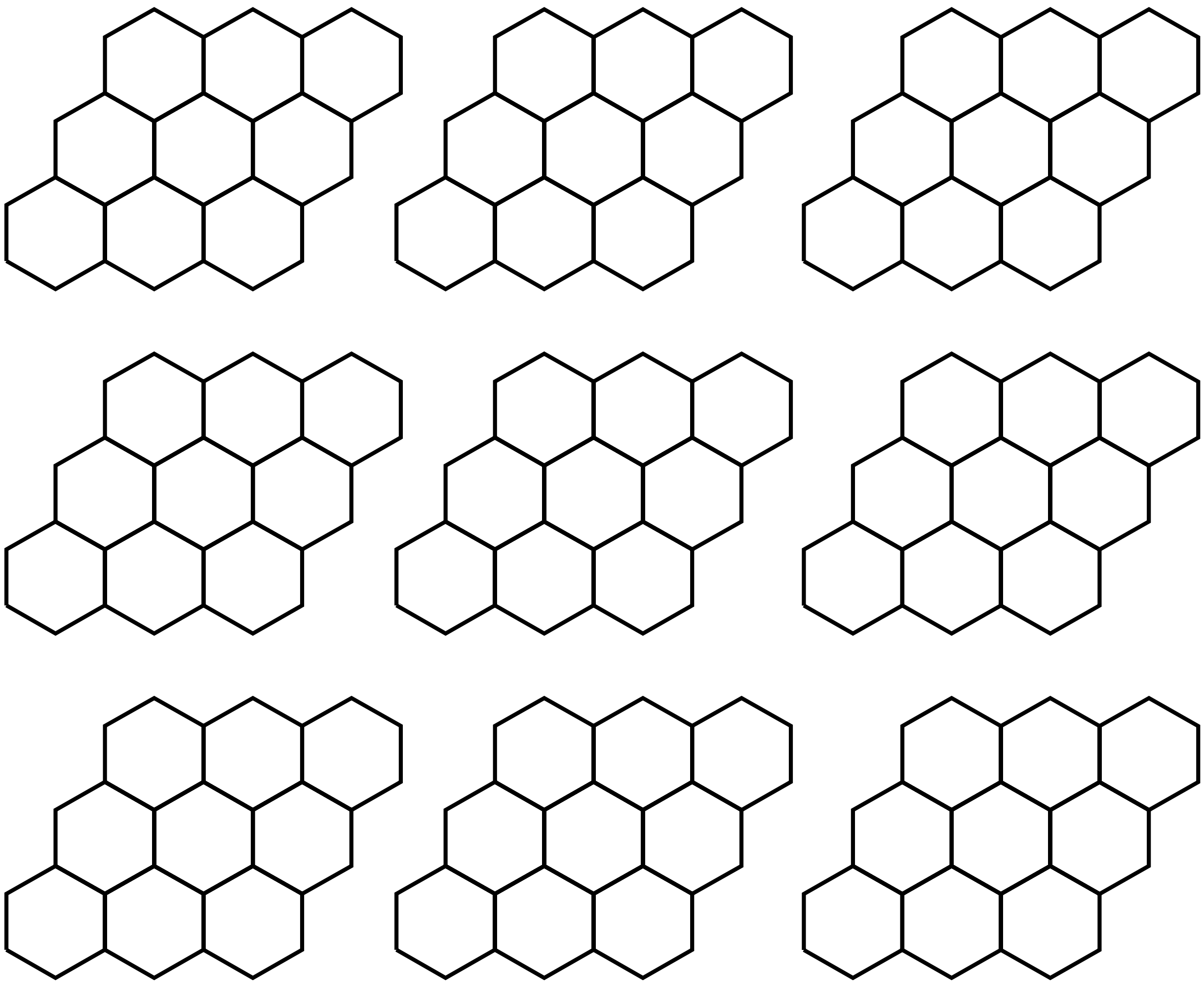


Player 2 wins



Player 1 wins





This is a 2 player game in which players take turns alternately placing a marker of his or her color into an empty hexagon. Player 1 tries to complete a path between the top and bottom rows of hexagons, while player 2 tries to complete a path between the left and right rows. The winner is the first player to complete such a path. This game can be played on a board of any size.

Resource Book

Week 3, Section 2: Matchings and Games

Mathematical Background

Matching. A matching M is a set of edges of the graph no two of which share a vertex. If one of the edges connects the vertex v to the vertex w, then we say that the matching M matches v to w.

Maximum matching. A maximum matching is one which matches as many vertices as possible.

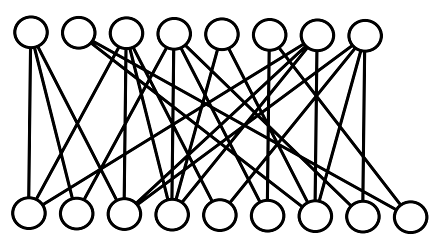
Perfect matching. A perfect matching is one which matches every vertex; this can happen only if the number of vertices is even.

Two parallel problems.

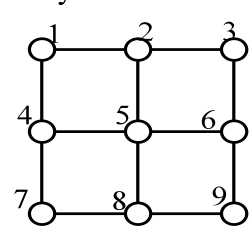
> **Marriage Problem.** Each person provides a list of mutually acceptable partners of the opposite sex. Is there a matching which pairs each person with an acceptable partner? If so, can you find it?
>
> **Roommate Problem.** Each person provides a list of mutually acceptable roommates. Is there a matching which pairs each person with an acceptable roommate? If so, can you find it?

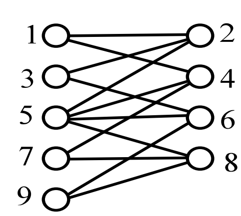
The Roommate Problem is broader than the Marriage Problem since the Roommate Problem asks whether an arbitrary graph has a perfect matching, whereas the Marriage Problem only relates to whether bipartite graphs have perfect matchings.

Bipartite graphs. A graph is bipartite if its vertices can be separated into two groups so that each edge joins a vertex in one group with a vertex in the other group. The graph at the right, for example, is bipartite.



Sometimes, graphs which don't seem to be bipartite actually are. For example, every grid graph is bipartite.





Bipartite graphs are the graphs which are often used to model the applications of matchings where there are two types of vertices which are matched to each other — for example, tasks to days, or people to jobs.

Resource Book

Week 3, Section 2: Matchings and Games

When does a bipartite graph have a perfect matching?

1. If the two components have different sizes, then there can't be a perfect matching. For example, there is no perfect matching of the vertices of a 3x3 grid.

2. If there is a set of vertices on one side which together are connected to a smaller number of vertices on the other side, then there can't be a perfect matching. For example, in the school dance problem, the four girls Betty, Natalie, Greta, and Doris together are connected to only three boys Rudolf, Gregory, and Fred.

If neither of those situations apply, then there always is a solution to the marriage problem. That is: If the number of men and women are equal, and if for every group of people of one sex, the group of people of the other sex who are mutually acceptable to at least one of them is at least as large as the original group, then there is a perfect matching. (Philip Hall)

In the marriage problem (bipartite case), there is a way of telling whether there is a perfect matching, and a way of finding such a matching.

In the roommate problem (general case), there is no way of telling whether there is a perfect matching, and, if there is one, no efficient way that will always find it.

Coloring bipartite graphs.

Every bipartite graph (such as grid graphs) can be colored with two colors. On the other hand, every graph which can be colored with two colors is bipartite. Thus, the bipartite graphs coincide with the 2-colorable graphs.

Moreover, a connected bipartite graph can only be colored in one way — once you pick the two colors and then color one vertex, the colors of all other vertices are determined.

There is a simple greedy algorithm which will tell whether any graph or map can be colored with two colors, and, if so, will provide such a two-coloring:

(1) Color one vertex blue, color all of its neighbors red, color all of their neighbors blue, color all of their neighbors red, etc.

(2) After each stage, stop to check whether any two adjacent vertices have been colored the same color; if they have, the graph was not bipartite, but if not, continue with the coloring.

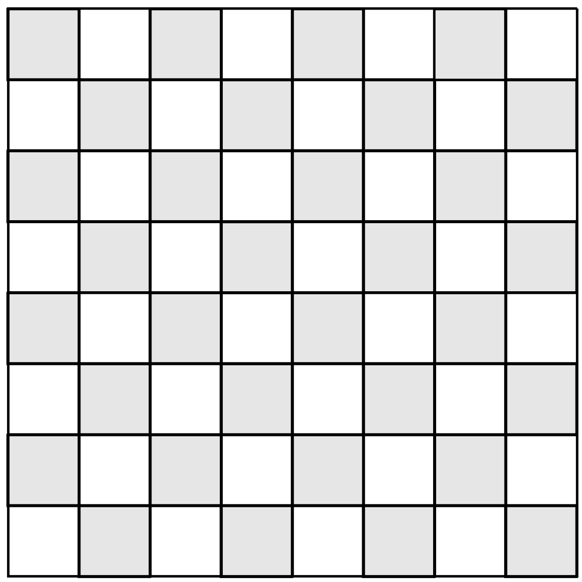
(3) When you are finished, if no two adjacent vertices have been assigned the same color, then the graph is bipartite and a two-coloring has been found.

Resource Book

Week 3, Section 2: Matchings and Games

In many problems and puzzles, the key to the solution is recognizing that the graph or map involved is bipartite.

Example: Can you cover a checkerboard with 31 dominoes so that the only squares uncovered are diagonally opposite corners of the board?



Terminology of Games

Winning position. A position in a game is called a winning position if, no matter what moves the next player makes, the other player can always win the game.

Winning move. A move is called a winning move if making it results in a winning position.

Winning strategy. A winning strategy is one which results in a victory for the player with the strategy. For example, a winning strategy might involve moving to a winning position, and then continuing to move to winning positions in response to the other player's moves.

Player-1-winning-game. A Player-1-winning-game is one where Player 1 has a winning strategy.

Player-2-winning-game. A Player-2-winning-game is one where Player 2 has a winning strategy.

Resource Book

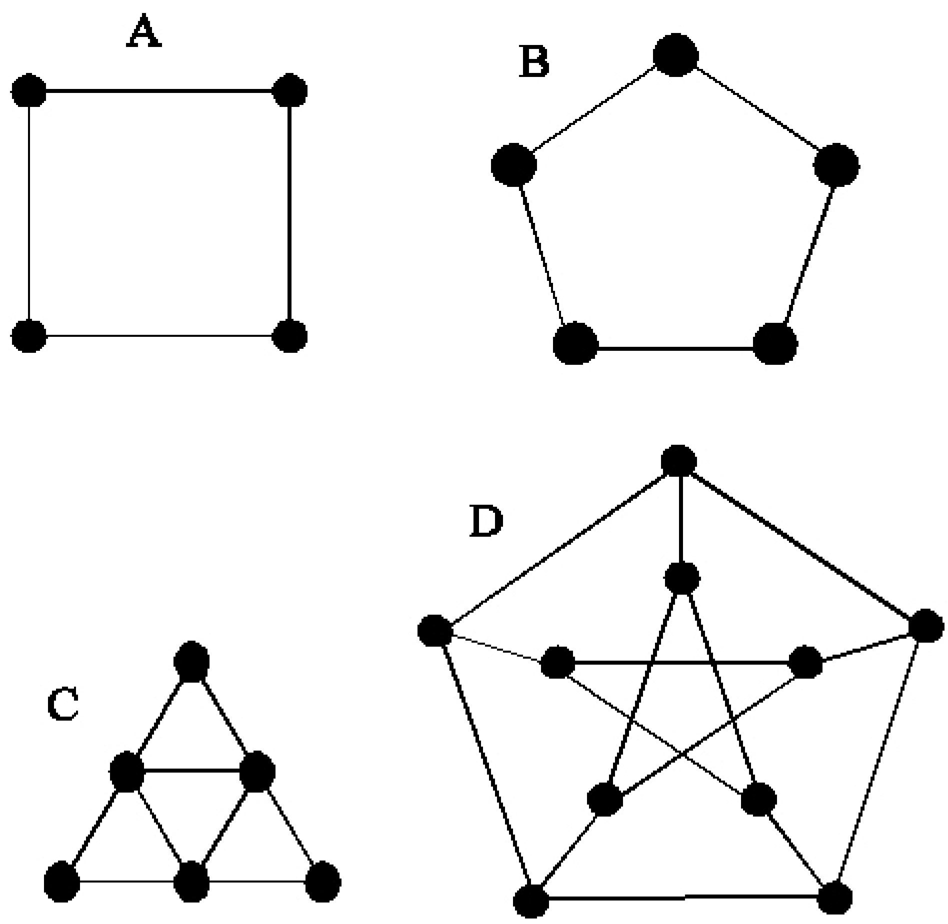
Week 3, Section 2: Matchings and Games

Joe's Game: This game can be played using any graph. Player 1 and Player 2 alternate picking vertices in the graph, subject to the following two rules:

1. Neither player can pick a vertex that has been picked earlier in the game.
2. Every vertex must be adjacent to the vertex picked just before.

The winner is the last player who can choose a vertex.

Who wins if the game is played using each of the graphs below?



Resource Book

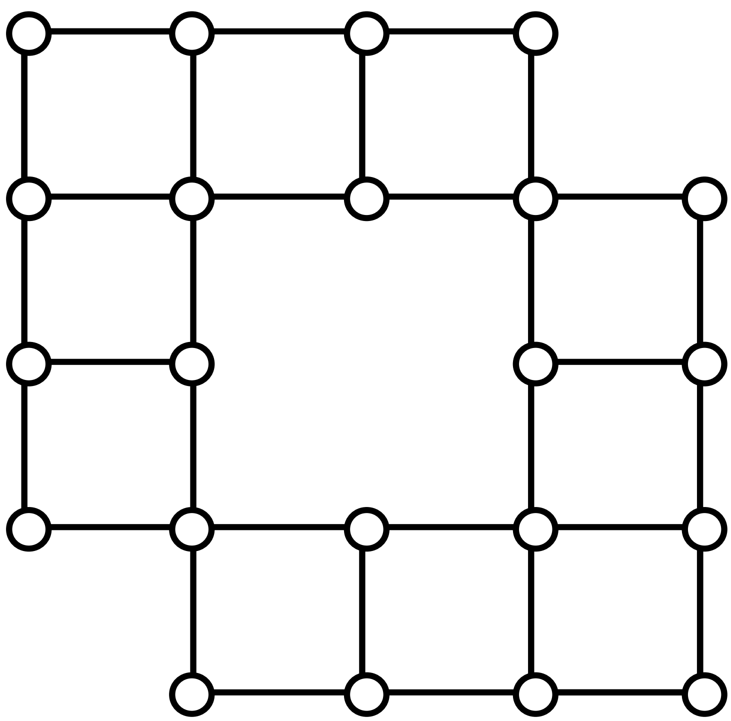
Week 3, Section 2: Matchings and Games

Joe's Game: This game can be played using any graph. Player 1 and Player 2 alternate picking vertices in the graph, subject to the following two rules:

1. Neither player can pick a vertex that has been picked earlier in the game.
2. Every vertex must be adjacent to the vertex picked just before.

The winner is the last player who can choose a vertex

Who wins if the game is played using the graph below?



Resource Book

Week 3, Section 2: Matchings and Games

The Secret of Joe's Game:

Find a maximum matching in the graph.

If the maximum matching is perfect, arrange to be Player 2. Then whatever vertex Player 1 picks, always pick the matching vertex. This way you will always win, because you will always be picking a new vertex, and it will always be adjacent to Player 1's last choice.

If the maximum matching is not perfect, arrange to be Player 1. For your first turn, select a vertex which is not matched in the maximum matching. Then whatever Player 2 does, always pick the matching vertex. This way you will always be picking a new vertex which is adjacent to Player 2's last choice.

> But what, you may ask, if Player 2 picks a vertex which is not matched in the maximum matching. Then, dear soul, you blew it — what you started with was not a maximum matching.

This secret is not to be repeated ...

... and Joe gets 20% royalties of all winnings.

Resource Book

Week 3, Section 2: Matchings and Games

The "Make 21" Game.

Starting with a zero total, Player 1 and Player 2 alternate adding either 1 or 2 to the total. Whoever makes 21 wins. Which player has a winning strategy?

> Work in pairs — one person designated as Player 1 and the other as Player 2. Your purpose, however, is not to vanquish your opponent, but to work cooperatively to figure out the game.

The "Make 25" Game.

Starting with a zero total, Player 1 and Player 2 alternate adding either 1 or 2 to the total. Whoever makes 25 wins. Which player has a winning strategy?

"Two pile Nim"

Starting with two piles of six and eight cubes, Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last object wins. Which player has a winning strategy?

> Work in pairs — one person designated as Player 1 and the other as Player 2. Your purpose, however, is not to vanquish your opponent, but to work cooperatively to figure out the game.

"Three pile Nim"

Starting with three piles of three, two, and one cubes, Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last object wins. Which player has a winning strategy?

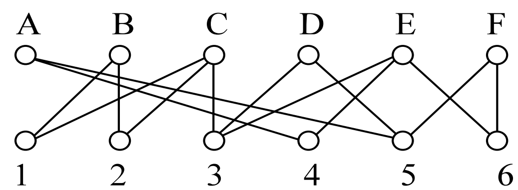
Resource Book

Week 3, Section 2: Matchings and Games

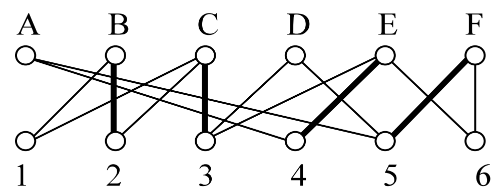
Matching and Bipartite Graphs

What advantage is gained from translating matching problems like the children and pet problem from the ordinary language of logic puzzles into the technical language of bipartite graphs? One reason is that the graphs are easier to work with than the charts in which the information is presented. But another reason is that there is a systematic method for solving such problems, called the “method of alternating chains.”

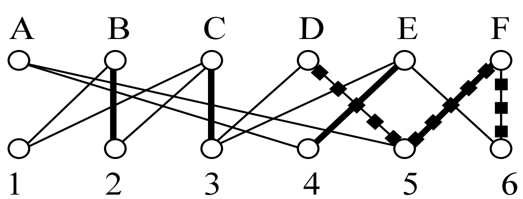
Suppose that we want to find a maximum matching in the graph below. We start by finding an initial matching (see right).

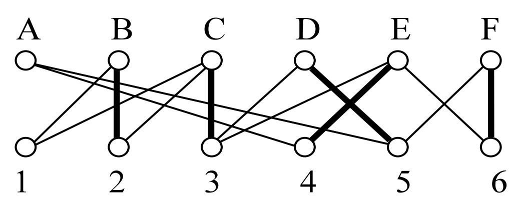


In this matching, only four of the edges in the top row are matched to edges in the bottom row. But none of the unmatched vertices can go with each other.

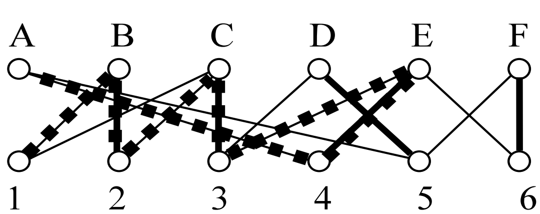


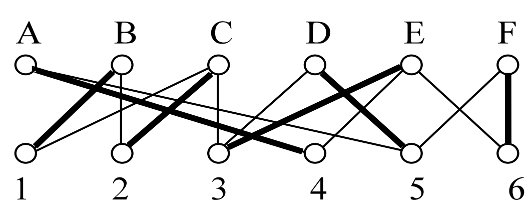
So we construct an “alternating chain,” which is a trail that starts with an unmatched vertex, that alternates edges in the matching (darker in the diagram above), and edges not in the matching, and ends at another unmatched edge. Such a chain is D to 5, 5 to F, and F to 6 (enclosed in the diagram below and to the left). We then eliminate the middle edge from the matching, and add the first and last edges to the matching, to arrive at the diagram at the right. We now have a new matching which matches two more vertices than before.





Now we carry the same procedure a step further. Here is another alternating chain: 1 to B (out), B to 2 (in), 2 to C (out), C to 3 (in), 3 to E (out), E to 4 (in), 4 to A (out) --- displayed in the diagram below and to the left. We throw out the edges labeled “in”, take in the edges labeled “out” and we end up with a perfect matching on the right.





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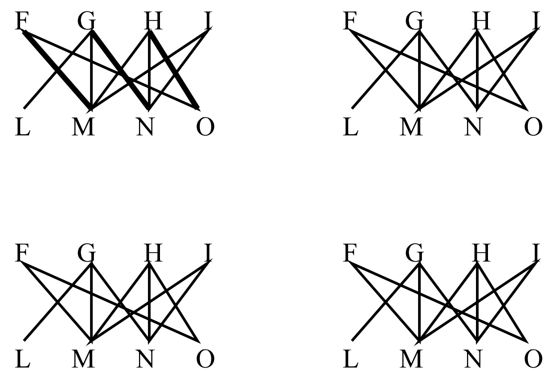
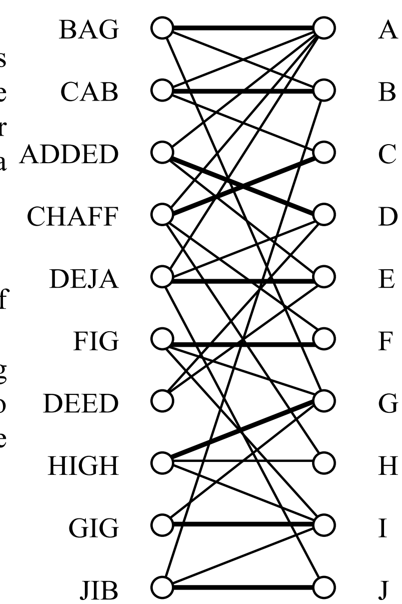
Week 3, Section 2: Matchings and Games

Practice with Alternating Chains



***** The computer center has four files, F, G, H, and I, to be stored. Each of the possible storage locations, L, M, N, and O, can hold at most one of the files. F can be stored in M or O; G can be stored in L, M, or N; H can be stored in M, N, or O; and I can be stored in M or N. Decide where the files could be stored.

Assume that the initial attempt to solve this problem led to the solution (shown graphically in the upper left figure): Store F in M, store G in N, and store H in O. Use the method of alternating chains (on the previous RES page) to identify a better solution.



***** The figure to the right shows a "first try" at solving one of yesterday's homework problems. In this initial solution, we greedily chose the alphabetically first available letter from each word. Now you want to extend our matching to a perfect matching using the method of alternating chains. Find a way to do this using as short an alternating chain as possible.

***** For each of the bipartite graphs shown below, use the method of alternating chains to extend the matching indicated by the bold lines.

For the second graph, note that your first run through with alternating chains will yield a matching with 4 matched edges instead of 3. In order to complete the solution, you should then start with that 4-edge matching and use the method of alternating chains again.