Master Document

Week 3, Day 5 — Probability and Games

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised May, 2006

Week 3, Day 5 — Probability and Games

<u>**Materials Needed**</u>	<u>**Allocated Time**</u>
Activity #1 — Homework Review	10 minutes

- no materials needed

Activity #2 — Casino Games . . . . 60 minutes

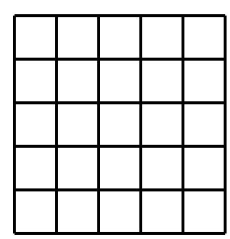
- one pair of dice per person; transparency dice (if available) for instructor

Activity #3 — Walking on Paths . . . . 35 minutes

- one pair of dice per person; transparency dice (if available) for instructor

Activity #4 — On the Boardwalk . . . . 20 minutes

- 5x5 grid drawn on floor (using thin masking tape), where the side of each square is twice the diameter of a small (about 4") plate
- 12 small plates, about 4" in diameter
- alternatively, if there is a floor with square tiling, use that as the grid, with 12 discs cut from paper plates, each with diameter equal to half the side of the squares.



. . . . TOTAL WORKSHOP TIME: 125* minutes

* In addition, 10 minutes are allocated for a break in this 2 ¼ hour workshop.

The theme of today is really gambling games. For many teachers, particularly K-8 teachers, this could be a problem because they can't discuss such topics in their classrooms directly. However, alternative formulations are frequently employed — using "number cubes" instead of dice and snack items instead of money.

A subtopic might be how the gambling industry structures its games to appeal to the ordinary (i.e., non-compulsive) gambler, by offering a variety of options (e.g., lotteries where the big prize appeals to those who seek a new life and smaller prizes — from hitting 3 out of 4 numbers — appeal to those who just want to feel like a winner once in a while), and by offering reasonable (though losing) odds (e.g., the paycheck gamble below) so that although the odds favor the house, many people win in the short run.

Activity #1. Review of Homework
(Allocated time = 10 minutes)

There may not be any need to review the homework from the previous day, in which case there will be more time for the other activities.

A. You may want to show them how to divide the trapezoid dartboard into 5 triangles each with the same height, and each of whose bases is 1/3. This is done by bisecting the shorter base of the trapezoid and drawing the median to that point within the shaded triangle, and by trisecting the longer base and drawing two lines to those points from the opposite vertex. The result is 5 triangles of equal area, two of which are shaded, yielding 2/5 for the probability that a dart will land in the shaded region. You might also point out that geometric probability is useful to review geometry.

B. You might want to reprise the "probability of rain" by asking for various probabilities given that there is a 40% chance of rain on Saturday and a 40% chance of rain on Sunday. What is the probability that it will rain on both days, on neither day, on exactly one day? All of this assumes, of course, that raining on Saturday and raining on Sunday are independent events.

Activity #2. Casino Games
(Allocated time = 70 minutes, 15 minutes per activity + 10 min break)

A. Distribute HO #1 (= TSP #1) and have participants work through the dice game on this page. In the subsequent discussion, introduce the idea of the expected value of a game — if you play the game 36 times, you expect to win 6 times, receiving

total winnings of $30; on the other hand, you pay $36 to play the game 36 times, for a net loss of $6. The expected value of the game is the net gain or loss per game; in this game, it is -$6/$36, or a loss of about $.16 per game.

At this point we are not yet giving them the "formula" for determining the expected value of a game as a sum of probabilities×winnings. In activities A and B (games 1 and 2) we determine the expected value of a game by considering some fixed number of games, calculating how many times we would win and lose to figure out our "net" winnings (or losses), and dividing by the number of games to get an "expected value of the game".

In activities C and D we will ask the class to determine the "expected value of a game" by using the sort of formula you will describe in the discussion of game 2, following activity B.

Note that we can compare this theoretical "expected value" to the actual results of our experiment. Using TSP#2, tabulate the data from all the tables and total it, find the experimental expected value for the entire class, and compare that to the theoretical expected value of minus $.16 per game.

B. Distribute HO #2 (= TSP #3). Ask participants to play Game 2. First they should roll the dice 36 times as indicated, and keep track of how many times out of 36 the numbers differed by 1. Each table will tally and then report its results; use TSP #4 to record the results, and add them up while the participants apply probability to the problem. We'll see how close to 10/36 their total is. What is the expected value of this game? In 36 games, you expect to pay $36 and to win $40, for a net gain of $4, for an expected value of about $.11 per game.

Once again, compare our probabilistic ruminations with the experimental outcome. Then analyze this game in a more formal way: On TSP#5 calculate the expected value by filling in the blanks appropriately and doing the computation.

The participants generally pick up the notion of expected value rather easily and naturally if the use of too much technical language is avoided. Keeping things simple, and slightly "cookbook," seems to work nicely. Nevertheless, it is important to be aware that the previous informal method of calculating expected value leads to ((4x10)-(1x26))/36, whereas this new method leads to (10/36)x3 + (26/36)x(-1), two expressions which are not obviously the same.

C. Ask participants to determine the expected value of Game #3 (this is on HO

#2 and TSP #6) where there are a variety of pay-offs. The expected "payoff" here is a loss of $.45 (or $.75 if you count the candy bar as worthless). Encourage the participants to calculate the expected value of this game using the "formula" given above, and then review the results using TSP #7.

[TIME FOR A 5-10 MINUTE BREAK!]

D. Distribute HO #3 (= TSP #8) and ask participants to determine the expected value of the game if your salary is $800 (or $D); it turns out that you lose, on average, 1/40 of your salary. This should be demonstrated on the board/projector with a chart similar to that for "Roulette".

Activity #3. Walking on paths.
(Allocated time = 35 minutes)

A. Show TSP #9 with the random walk on the trails diagram. Demonstrate this on the overhead projector to ensure that participants understand the instructions, and then distribute HO #4 (= TSP #10). Ask participants to take a random walk 15 times, so that we can get a picture of what the experimental probabilities and expected value are. As before, ask each table to tabulate its totals, and then add the totals for the room in a chart on the board/projector (see TSP #11). Once they finish that, work out the theoretical probabilities of ending at A and B, and ask them to complete the pattern. (It ends up being 1/81, 8/81, 24/81, 32/81, and 16/81.) Compare the theoretical and experimental results. Then calculate the expected value, and compare that with the experimental expected value.

Activity #4. On the boardwalk.
(Allocated time = 20 minutes)

A. On the floor, draw (using narrow masking tape) a grid with distance equal to twice the diameter of a small (about 4") plate. Alternatively, if there is a floor with square tiling, use discs cut from paper plates so that their diameters are one half a side of the square. (This works better, since the borders are thinner.) Designate one participant (the thrower) who throws the plates, two participants (the callers) who kneel on the ground about 5 feet in front of the thrower, eight participants (the passers) who stand 4 on each side of the thrower creating a semicircle encompassing the callers, and one participant (the recorder) who is ready with a pad and pencil. The thrower throws plates at a rate of about 1 per second onto the floor. The callers pick up the plates (after they've stopped moving), call either "Line!" or "Miss!" (depending on whether the plate was touching a line or had missed all the lines) and hand the plates

to a passer, who passes it down the line back to the thrower. The recorder uses hash marks to record the number of lines and misses that he/she hears called. In this fashion, it is possible to record about 200 data points in under 5 minutes.

Ask participants to figure out what proportion of the time the dish avoids the lines altogether (based on the data). Then work it out theoretically, and find that it is .25, using the observation that in order for the plate to lie inside the lines, its center must lie inside a smaller square within each large square whose dimensions are half those of the original square.

Hand-out #1 — Double Trouble

Game 1: It's you against the casino! It costs $1 to play the game. You roll the dice and win $5 if you get doubles, but lose otherwise. Should you play this game?

First do the following experiment.

1. Roll the dice 36 times and record the number of times you get doubles. _______

2. Calculate how much altogether you won (or lost). _____

3. Determine how much you won (or lost) per game. _____

Now add the numbers for your table.

1. What was the total number of games played at your table? ______ How many resulted in doubles? ______

2. Calculate how much altogether the people at your table won (or lost). _____

3. Determine how much was won (or lost) per game. _____

Finally, apply probability theory to tell you how much you should expect to win (or lose) if you play the game 36 times, and how much you should expect to win (or lose) per game.

Experimental Expected Value

Table	Number of Games	Number of Wins	Amount won or Lost
1			
2			
3			
4			
5			
6			
7			
8			
9			
Total			

Hand-out 2 — One's the Difference

Game 2: Again it's you against the casino! It costs \$1 to play the game. You roll the dice and you win \$4 if the numbers differ by exactly 1, but lose otherwise. Should you play this game?

First do the experiment. Roll the dice 36 times and record the number of times the two numbers differ by exactly 1. Add the numbers for your table and report the result.

Then apply probability theory to tell you how much you should expect to win or lose if you play the game 36 times.

Determine the expected value of this game.

Experimental Expected Value

Table	Number of Games	Number of Wins	Amount won or Lost
1			
2			
3			
4			
5			
6			
7			
8			
9			
Total			

Expected Value of "One's the Difference"

Probability of winning	
Amount won	
Probability of losing	
Amount "won"	

The expected value is then obtained by "adding" together

(probability)×(amount)

in each case.

________ × ________ + ________ × ________ =

If there were more cases, we would have more terms to add together.

Hand-out #2 — Roulette

Game 3: It's still you against the casino.

It costs \$10 to participate in a spin of a roulette wheel which has twenty (equally likely) sectors.

There is one spinner where the payoff is \$50, three spinners where the payoff is \$25, and four spinners where the payoff is \$15; with the other twelve spinners, you win a 50 cent candy bar (which is probably stale).

Determine the expected value of this game.

Note that the expression for expected value in this case will have 4 terms, one for each case.

Expected value of "Roulette"

Probability of winning $50 ______
Multiply by amount won ______ = ______

Probability of winning $25 ______
Multiply by amount won ______ = ______

Probability of winning $15 ______
Multiply by amount won ______ = ______

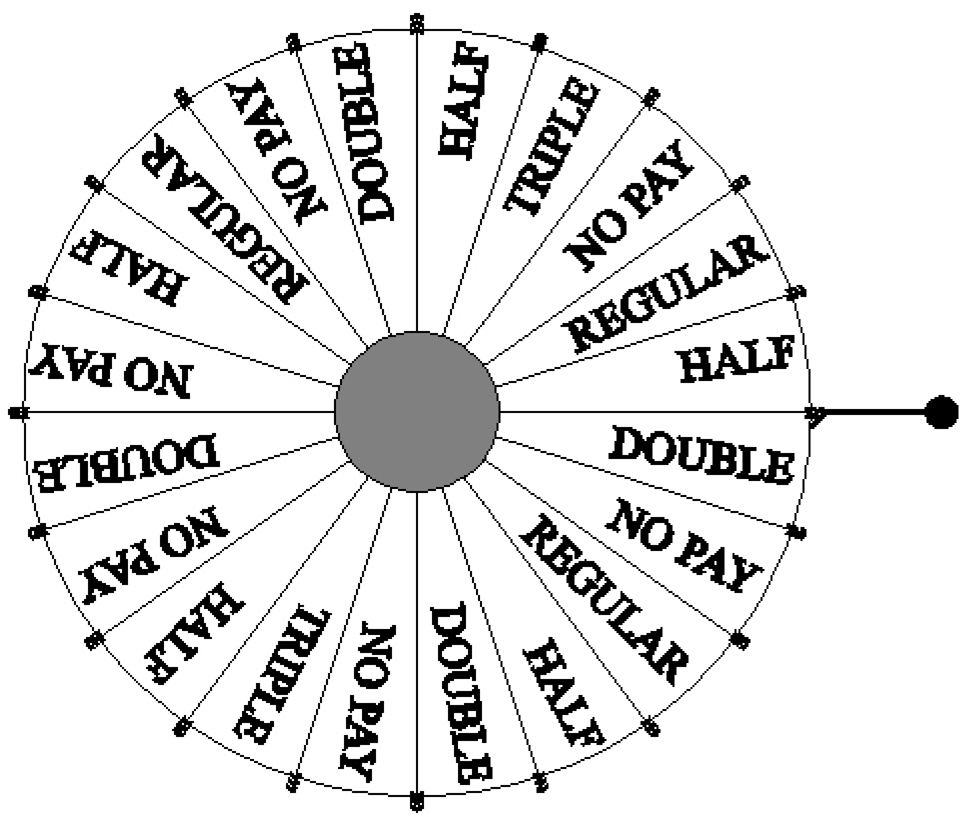
Probability of winning $0.50 ______
Multiply by amount won ______ = ______

Add together the totals
to get the expected value ______

Assuming the candy is stale,
the expected value is ______

Hand-out #3 — Salary Ride

In Las Vegas casinos there are booths where employees can go and gamble their paychecks before they are even cashed. If you play against the following roulette wheel, what is the expected value if your paycheck is $800? (What if it is $D?)



GAMBLE YOUR PAYCHECK HERE!

WIN ENOUGH TO TAKE TOMORROW OFF!

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L
I L
A I L
R A I L
T R A I L
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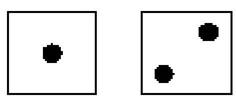
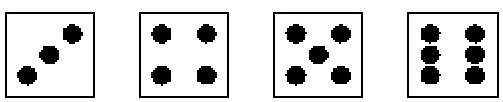
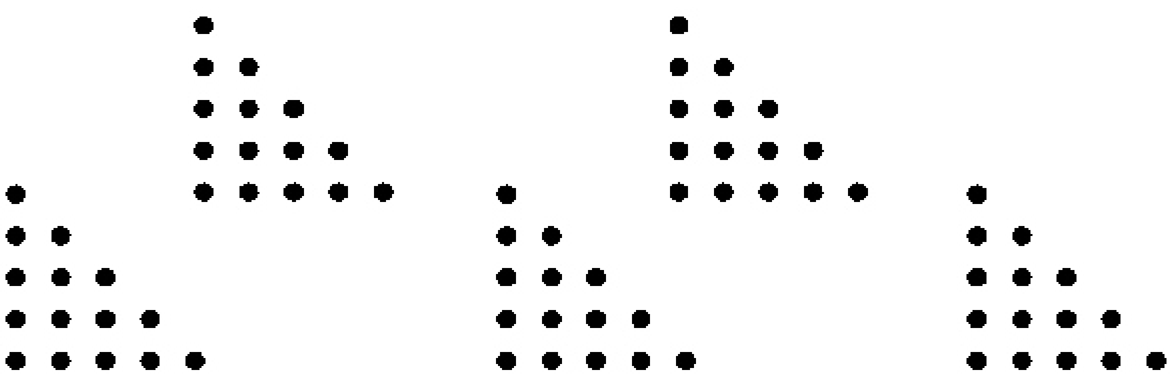
Take a random walk, starting at "T"

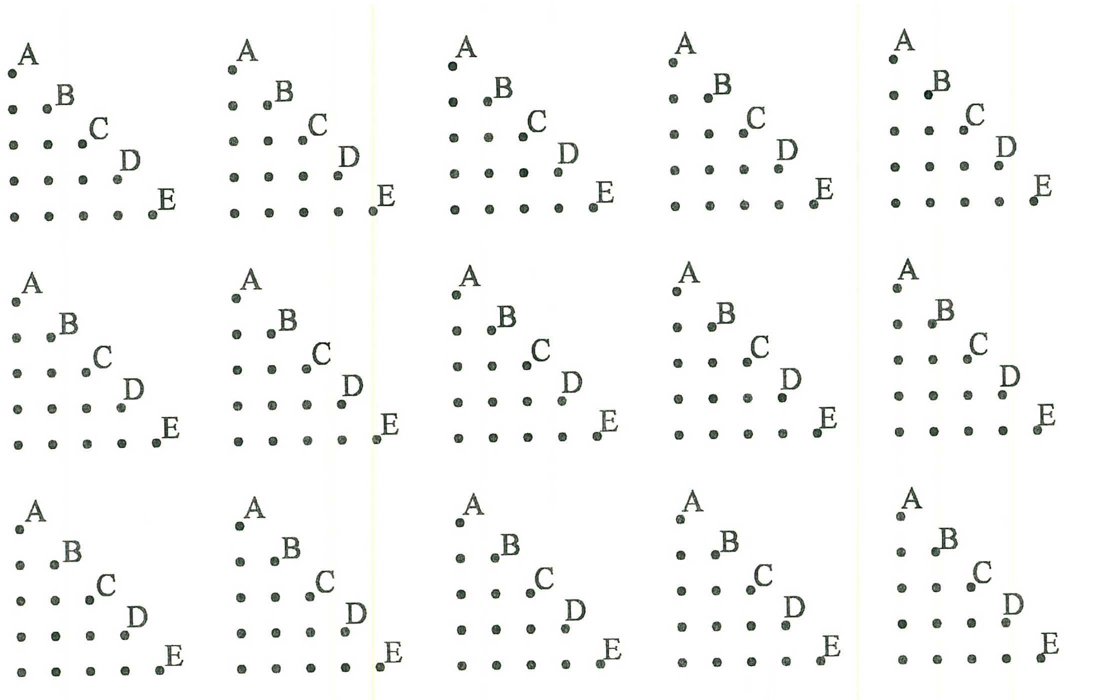
Roll a die:

 **Move North**

 **Move East**



Hand-out #4 — Take a Random Hike



Roll the dice to generate a random walk from the dot at the bottom left to the dot labeled with one of the letters A, B, C, D, or E.

Rules:	Move North — with 1 or 2
	Move East — with 3, 4, 5, or 6
You pay:	$3 for each play of the game.
You win:	$1 if you land on A
	$2 if you land on B
	$4 if you land on C
	$3 if you land on D
	$1 if you land on E

How much did you win/lose after playing 15 games?

Expected value of "Take a Random Hike"

Table	A	B	C	D	E	Won
1						
2						
3						
4						
5						
6						
7						
8						
9						
Total						

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Hand-out #2 — One's the Difference and a Roulette Game

Game 2 — One's The Difference: Again it's you against the casino! It costs $1 to play the game. You roll the dice and you win $4 if the numbers differ by exactly 1, but lose otherwise. Should you play this game?

> First do an experiment. Roll the dice 36 times and record the number of times the two numbers differ by exactly 1. Add the numbers for your table and report the results.
>
> Then apply probability theory to tell you how much you should expect to win or lose if you play the game 36 times. Determine the expected value of this game.

Game 3 — A Roulette Game: It's still you against the casino. It costs $10 to participate in a spin of a roulette wheel which has twenty (equally likely) sectors. There is one sector where the payoff is $50, three sectors where the payoff is $25, and four sectors where the payoff is $15; with the other twelve sectors, you win a 50 cent candy bar (which is probably stale). Determine the expected value of this game.

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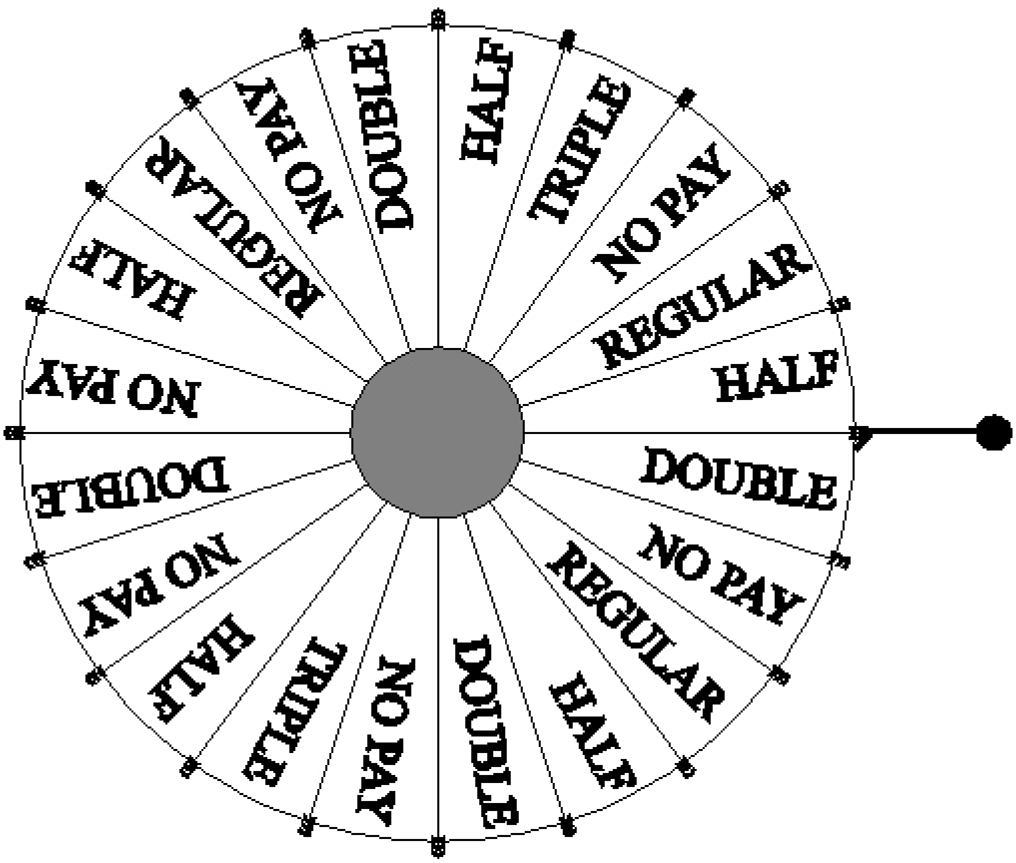
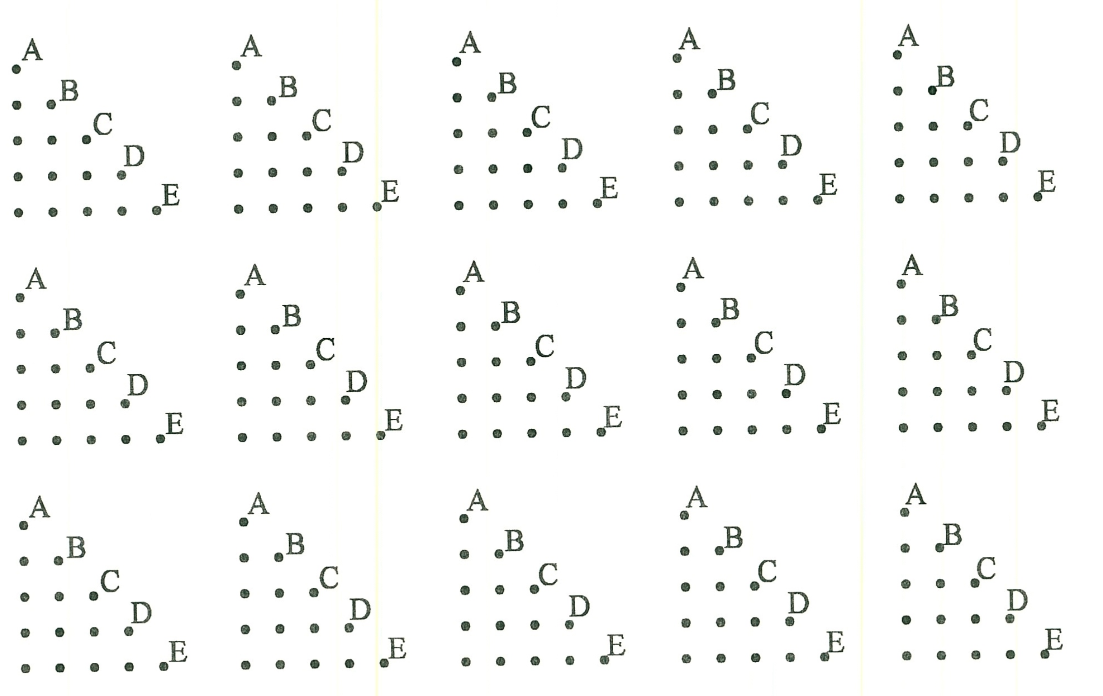


GAMBLE YOUR PAYCHECK HERE!

WIN ENOUGH TO TAKE TOMORROW OFF!

Handout #4 — Take a Random Hike



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You pay:	$3 for each play of the game.
You win:	$1 if you land on A $2 if you land on B $4 if you land on C $3 if you land on D $1 if you land on E

How much did you win/lose after playing 15 games?