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A note on the monotonicity of matrix Riccati equations

by

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ABSTRACT

In this note we show that the flow of matrix Riccati equations is monotone. Consequently, many results for Riccati equations can be obtained easily using standard as well as more recent results from the theory of monotone systems.

1 Introduction

The purpose of this short note is to illustrate the use of the theory of monotone (control) systems, to derive results concerning the asymptotic behavior of matrix Riccati equations. Reid already pointed out in [10] that the flow of these equations is monotone, see for instance lemma 6.1 and theorem 6.1 in that reference. Notice that this remark precedes the development of the theory of monotone dynamical systems [4, 5, 6, 7, 8] -and [11] for an excellent review- by more than a decade. Therefore, the full power of this theory (which may be summarized very roughly by the phrase that "almost all solutions with compact forward orbit closure converge to the set of equilibria") was not at Reid's disposal. Monotone systems theory has recently been extended to encompass systems with inputs and outputs in [1]. Equipped with this arsenal of tools we can and will prove many of the standard results concerning Riccati equations in an almost straightforward manner. This approach, rather than the novelity of the results themselves, constitutes the contribution of this paper. We note that Theorem 1 below has also been proved in [14], but it is based on a different proof technique. As an application we consider the (infinite horizon) optimal control problem for linear systems with quadratic cost. The cost matrices will be interpreted as time-varying inputs to a Riccati equation and our aim is to provide asymptotic estimates for the solutions of the differential equation.

Although we will not pursue this issue here, we note that the theory developed in [1] -in particular the results concerning cascades and feedback connections of monotone input/output systems- is readily applicable to the Riccati equations considered here. The main ingredients for such application are monotonicity of the flow and the existence of static input-state characteristics. Both concepts will be defined and shown to hold/exist for the systems under consideration.

2 Preliminaries

Let's start by introducing some terminology. The real, n^2-dimensional vector space of real $n \times n$ matrices is denoted by $\mathcal{R}$. We assume that $\mathcal{R}$ is equipped with an inner product $\langle .,. \rangle$, defined by $\langle X, Y \rangle = \text{tr}\left(XY^T\right)$. This inner product induces a norm (known as the Frobenius norm) and thus metric on $\mathcal{R}$ in the obvious way. We shall assume that all topological notions are with respect to this metric. Note that the normed vector space $\mathcal{R}$ is isometrically isomorphic to $\mathbb{R}^{n^2}$ (with the usual Euclidean norm). To see this define $T : \mathcal{R} \to \mathbb{R}^{n^2}$ by:

$$T(X) = (x_1 x_2 ... x_n)^T,$$

where x_i is the ith row of X. Then it is easily checked that T is an isomorphism and an isometry. Occasionally, it is useful to have this 'equivalence' of both spaces in mind, in particular when considering systems of differential equations or geometric objects (such as subspaces or cones which will be introduced below). The set of real symmetric matrices will be denoted by $\mathcal{S} = \{S \in \mathcal{R} \,|\, S = S^T\}$. Clearly, $\mathcal{S}$ is a linear subspace of $\mathcal{R}$ of dimension $n(n+1)/2$. Recall that if $S \in \mathcal{S}$, then the Frobenius norm of S is given by:

$$\|S\| = \sqrt{\sum_{i=1}^{n} \lambda_i^2} \tag{1}$$

where the summation runs over all eigenvalues λ_i of S. Another important property of every symmetric matrix S is the following:

$$\forall S \in \mathcal{S},\ \forall x \in \mathbb{R}^n\ :\ x^T\left(\lambda_{\min}(S)I\right)x \leq x^T S x \leq x^T\left(\lambda_{\max}(S)I\right)x \tag{2}$$

where $\lambda_{\min}(S)$ and $\lambda_{\max}(S)$ denote the minimal, respectively maximal eigenvalue of S. Let $\mathcal{P}(\mathcal{P}^+) \subset \mathcal{S}$ denote the set of symmetric positive semidefinite (definite) matrices. Then $\mathcal{P}$ is nonempty and closed (in $\mathcal{R}$ and $\mathcal{S}$), $\mathbb{R}_+\mathcal{P} \subset \mathcal{P}$, $\mathcal{P} + \mathcal{P} \subset \mathcal{P}$ and $\mathcal{P} \cap (-\mathcal{P}) = \emptyset$. Equivalently, $\mathcal{P}$ is a cone in $\mathcal{R}$ and in $\mathcal{S}$. Note that $\text{int}(\mathcal{P}) = \mathcal{P}^+$ in $\mathcal{S}$ (but obviously $\text{int}(\mathcal{P})$ would be empty in $\mathcal{R}$). We shall also need the concept of the *dual cone*. If $(X, \langle ., . \rangle)$ is a finite-dimensional real inner product space and if $K \subset X$ is a cone, then the dual cone of K is denoted by K^* and defined by:

$$K^* = \{y \in X \mid \langle y, k \rangle \geq 0,\ \forall k \in K\}.$$

Returning to the cone $\mathcal{P} \subset \mathcal{S}$, we determine its dual cone $\mathcal{P}^* \subset \mathcal{S}$ next.

Lemma 1. *$\mathcal{P} = \mathcal{P}^*$.*

Proof. First we show that $\mathcal{P} \subset \mathcal{P}^*$. Pick $P \in \mathcal{P}$. We need to show that $\langle P, Q \rangle = \text{tr}(PQ) \geq 0$ for all $Q \in \mathcal{P}$. Since $P, Q \in \mathcal{P}$, there exist $P_{1/2}, Q_{1/2} \in \mathcal{P}$ such that:

$$P = P_{1/2}P_{1/2}, \quad Q = Q_{1/2}Q_{1/2}.$$

Then we get:

$$\langle P, Q \rangle = \text{tr}\left(P_{1/2}P_{1/2}Q_{1/2}Q_{1/2}\right) = \text{tr}\left((Q_{1/2}P_{1/2})(P_{1/2}Q_{1/2})\right) = \text{tr}\left((P_{1/2}Q_{1/2})^T(P_{1/2}Q_{1/2})\right) \geq 0,$$

and we are done.

Next we show that $\mathcal{P}^* \subset \mathcal{P}$. Pick $P^* \in \mathcal{P}^*$, so $\langle P^*, P \rangle \geq 0$ for all $P \in \mathcal{P}$. Since P^* is symmetric, it has n real eigenvalues λ_i with corresponding eigenvectors $z_i \neq 0$: $P^* z_i = \lambda_i z_i$ for $i = 1, ..., n$. If we can show that all $\lambda_i \geq 0$, then we are done. To do that, we will evaluate $\langle P^*, P \rangle$ for n particular choices P_i of P, namely $P_i = z_i z_i^T$, $i = 1, ..., n$. Note that $P_i \in \mathcal{P}$ and thus that $\langle P^*, P_i \rangle \geq 0$ for all $i = 1, ..., n$. Now,

$$\langle P^*, P_i \rangle = \text{tr}(P^* z_i z_i^T) = \lambda_i \,\text{tr}(z_i z_i^T) \geq 0,$$

from which follows that $\lambda_i \geq 0$ for $i = 1, ..., n$. □

Lemma 2. *If $P \in \mathcal{P}$ and $P^* \in \mathcal{P}^*$ are such that $\langle P^*, P \rangle = 0$, then $P^*P = 0 = PP^*$.*

Proof. Since $P \in \mathcal{P}$ and $P \in \mathcal{P}^* = \mathcal{P}$ (by lemma 1), there exist $P_{1/2}, P^*_{1/2} \in \mathcal{P}$ such that:

$$P = P_{1/2}P_{1/2}, \quad P^* = P^*_{1/2}P^*_{1/2}.$$

Now, since $\langle P^*, P\rangle = 0$, this implies that

$$\mathrm{tr}(P^*_{1/2}P^*_{1/2}P_{1/2}P_{1/2}) = \mathrm{tr}\left((P_{1/2}P^*_{1/2})(P^*_{1/2}P_{1/2})\right) = \mathrm{tr}\left((P^*_{1/2}P_{1/2})^T(P^*_{1/2}P_{1/2})\right) = 0,$$

from which follows that

$$P^*_{1/2}P_{1/2} = 0 = P_{1/2}P^*_{1/2}.$$

But this in turn implies that

$$P^*P = P^*_{1/2}(P^*_{1/2}P_{1/2})P_{1/2} = 0 = P_{1/2}(P_{1/2}P^*_{1/2})P^*_{1/2} = PP^*,$$

which concludes the proof. □

Lemma 3. *Let $P, Q \in \mathcal{P}^+$ and $Q - P \in \mathcal{P}$. Then $P^{-1}, Q^{-1} \in \mathcal{P}^+$ and $P^{-1} - Q^{-1} \in \mathcal{P}$.*

Proof. The claim that $P^{-1}, Q^{-1} \in \mathcal{P}^+$ is obvious. Since $Q \in \mathcal{P}^+$, there exists $Q_{1/2} \in \mathcal{P}^+$ such that $Q = Q_{1/2}Q_{1/2}$. Multiplying $Q - P \in \mathcal{P}$ on the left and right with the inverse of $Q_{1/2}$ yields a matrix in $\mathcal{P}$:

$$I - Q^{-1}_{1/2}PQ^{-1}_{1/2} \in \mathcal{P}. \tag{3}$$

Notice that $Q^{-1}_{1/2}PQ^{-1}_{1/2} \in \mathcal{P}^+$ and therefore it can also be written as:

$$Q^{-1}_{1/2}PQ^{-1}_{1/2} = S_{1/2}S_{1/2},$$

for some $S_{1/2} \in \mathcal{P}^+$. Multiplying the matrix in (3) on the left and on the right by the inverse of $S_{1/2}$ yields:

$$S^{-1}_{1/2}S^{-1}_{1/2} - I \in \mathcal{P}.$$

Multiplying this matrix on the left and right by $Q^{-1}_{1/2}$ finally leads to:

$$Q^{-1}_{1/2}S^{-1}_{1/2}S^{-1}_{1/2}Q^{-1}_{1/2} - Q^{-1} \in \mathcal{P}.$$

The proof is concluded by noting that:

$$Q^{-1}_{1/2}S^{-1}_{1/2}S^{-1}_{1/2}Q^{-1}_{1/2} = P^{-1}.$$

□

Lemma 4. *Let $\mathcal{C}$ be a compact subset in $\mathcal{P}^+$. Then there are $\lambda_{min}, \lambda_{\max} > 0$ such that:*

$$\forall C \in \mathcal{C},\ \forall x \in \mathbb{R}^n\ :\ x^T(\lambda_{\min}I)\,x \leq x^TCx \leq x^T(\lambda_{\max}I)\,x.$$

Proof. Consider the functions $\lambda_{\min}(.)$ and $\lambda_{max}(.)$, defined on $\mathcal{S}$, which assign the minimal, respectively maximal eigenvalue to a symmetric matrix. It is well known -see e.g. Corollary $A.4.4$ on p. 458 in [13]- that both functions are continuous. Compactness of $\mathcal{C}$ implies that $\lambda_{\min}(.)$ achieves a minimum $\lambda_{\min}$ on $\mathcal{C}$ and similarly that $\lambda_{\max}(.)$ achieves a maximum on $\mathcal{C}$. The conclusion now follows from (2). □

3 Matrix Riccati differential equations

The matrix Riccati differential equation on $\mathcal{R}$ is given by the following:

$$\dot{X} = XA(t)X + B_1(t)X + XB_2^T(t) + C(t), \tag{4}$$

where $X \in \mathcal{R}$ and $A(t), B_1(t), B_2(t), C(t) : \mathbb{R} \to \mathcal{R}$ are continuous matrix-valued mappings.

Obviously, solutions exist and are unique for every $X_0 \in \mathcal{R}$. We will denote the solution starting at $t_0 \in \mathbb{R}_+$ in X_0 by $X(t, t_0, X_0)$ with $t \in \mathcal{I}$, where $\mathcal{I}$ is the maximal interval of existence. We will also consider the forward maximal interval of existence which is defined by $\mathcal{I}^+ = \mathcal{I} \cap [t_0, +\infty)$.

On the other hand, this system is not necessarily (forward) complete. (Forward) completeness means that for every solution we have that $\mathcal{I} = \mathbb{R}$ ($\mathcal{I}^+ = [t_0, +\infty)$). For instance, consider the scalar Riccati equation with $A(t) = 1$, $B_1(t) = B_2(t) = C(t) = 0$ with initial condition $X(0) = 1$. Then the corresponding solution is $X(t, 1) = 1/(1 - t)$, which is of course only defined for $t \in (-\infty, 1)$.

We will study system (4) under the following additional constraint:

$$(S) \quad A(t),\, C(t) \in \mathcal{S} \text{ and } B_1(t) = B_2(t) \text{ for all } t \in \mathbb{R}.$$

An immediate consequence is that if (S) holds, then $\mathcal{S}$ is an invariant set of (4). This follows from the fact that if $X(t)$ is a solution of (4), then $X^T(t)$ is also a solution of (4). Uniqueness of solutions then implies that if $X(0) \in S$, then $X(t) = X^T(t)$ for all t for which the solution exists. This observation justifies the restriction of the dynamics of system (4) to the invariant set $\mathcal{S}$, which will be assumed henceforth.

Our goal is to show that assuming (S), system (4) is monotone on $\mathcal{S}$ (see [11] and the appendix of this paper for the definition of monotonicity in the much more general setting of semiflows on subsets of Banach spaces). The partial order on $\mathcal{S}$ which will be preserved by the solutions is generated by the cone of positive semidefinite matrices $\mathcal{P}$ as follows: For $X_0, Y_0 \in \mathcal{S}$, we say that $X_0 \preceq Y_0$ if and only if $Y_0 - X_0 \in \mathcal{P}$. We call system (4) monotone on S if:

$$\forall X_0, Y_0 \in \mathcal{S} : \quad X_0 \preceq Y_0 \Rightarrow X(t, X_0) \preceq X(t, Y_0), \quad \forall t \in \mathcal{I}_1 \cap \mathcal{I}_2,$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are the maximal intervals of existence of the solutions $X(t, t_0, X_0)$ and $X(t, t_0, Y_0)$.

Theorem 1. *Let (S) hold. Then system (4) is monotone on $\mathcal{S}$.*

Proof. Since $\mathcal{S}$ is convex, (hence in particular p-convex) it suffices by Theorem 1.1 and 1.2 in [9] to verify that:

$$\forall (P, Q) \in \partial\mathcal{P} \times \mathcal{P}^* : \quad \langle Q, P \rangle = 0 \Rightarrow \langle Q, DF_X(P, t) \rangle \geq 0, \, \forall t \in \mathbb{R}, \tag{5}$$

where $DF_X(Y, t) = XA(t)Y + YA(t)X + B_1(t)Y + YB_1^T(t)$, the linearization of system (4) at X.

Now if the pair $(P, Q) \in \partial\mathcal{P} \times \mathcal{P}^*$, satisfies $\langle Q, P \rangle = 0$, then lemma 2 implies that $QP = PQ = 0$. From this we get:

$$\langle Q, DF_X(P) \rangle = \operatorname{tr}\left(Q[XA(t)P + PA(t)X + B_1(t)P + PB_1^T(t)]\right) = 0,$$

which concludes the proof. $\square$

The cone $\mathcal{P}$ does not only induce a partial order on $\mathcal{S}$, it is also a forward invariant set for system (4) whenever $C(t)$ is positive semidefinite. This constitutes a first simple application of monotone systems theory, based on a comparison result for solutions of differential equations due to Kamke, see for instance [14] or Appendix B in [12].

Theorem 2. *Let* (S) *hold and assume that* $C(t) \in \mathcal{P}$ *for all* $t \in \mathbb{R}$. *Then for all* $(t_0, X_0) \in \mathbb{R} \times \mathcal{P}$, $X(t, t_0, X_0) \in \mathcal{P}$, *for all* $t \in \mathcal{I}^+$.

Proof. Since $0 \preceq C(t)$ for all $t \in \mathbb{R}$, it follows that:

$$G(X,t) := XA(t)X + B_1(t)X + XB_1^T(t) \preceq F(X,t) := XA(t)X + B_1(t)X + XB_1^T(t) + C(t), \ \forall t \in \mathbb{R}.$$

Since $F(X,t)$ satisfies (5) and $X(t,t_0,0) = 0$ is a solution of the equation $\dot{X} = G(X,t)$, the result follows from proposition 1.7 and remark 1.8 in [14]. □

The following is an auxiliary result concerning boundedness of solutions and is needed later. It does not rely on monotone systems theory.

Lemma 5. *Let* (S) *hold, assume that* $B_1(t)$ *and* $C(t)$ *are bounded, and assume that* $C(t) \in \mathcal{P}$ *for all* $t \in \mathbb{R}$. *Suppose there exists* $\alpha > 0$ *such that:*

$$A(t) \preceq -\alpha I, \ \forall t \in \mathbb{R}. \tag{6}$$

Then every forward solution of (4) *starting in* $\mathcal{P}$ *remains in* $\mathcal{P}$ *and is bounded.*

Proof. By theorem 2, forward solutions starting in $\mathcal{P}$ remain in $\mathcal{P}$. Consider the following function, defined on $\mathcal{P}$:

$$V(X) = \|X\|^2$$

We will show that for all $X \in \mathcal{P}$ having sufficiently large $\|X\|$:

$$\dot{V} := 2\,\mathrm{tr}\left[X\left(XA(t)X + B_1(t)X + XB_1^T(t) + C(t)\right)\right] < 0,$$

from which the result follows.

Recall that every $X \in \mathcal{P}$ can be factored as $X = X_{1/2}X_{1/2}$ for a unique $X_{1/2} \in \mathcal{P}$. From this and using the fact that $\mathcal{P}$ is a forward invariant set (by theorem 2), we get:

$$\begin{aligned} \dot{V} &= 2\,\mathrm{tr}\left(X_{1/2}XA(t)XX_{1/2}\right) + 4\,\mathrm{tr}\left(B_1(t)X^2\right) + 2\,\mathrm{tr}\left(XC(t)\right) \\ &\leq -2\alpha\|X_{1/2}X\|^2 + 4\beta\|X^2\| + 2\gamma\|X\|, \end{aligned}$$

where we used the Cauchy-Schwarz inequality for obtaining the last two terms -and β and γ are the bounds for $\|B_1(t)\|$ respectively $\|C(t)\|$- and where we used (6) to obtain the estimate of the first term.

Now recalling the formula for the Frobenius norm of a symmetric matrix (1) and by equivalence of norms on $\mathbb{R}^n$, there exist $\alpha^* > 0$ and $\beta^* > 0$ (in fact $\beta^* = 1$ works) such that:

$$\|XX_{1/2}\|^2 \geq \alpha^*\|X\|^3, \ \ \|X^2\| \leq \beta^*\|X\|^2$$

This implies that:

$$\dot{V} \leq -\tilde{\alpha}\|X\|^3 + \tilde{\beta}\|X\|^2 + \tilde{\gamma}\|X\|$$

for suitable positive constants $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$. From this, the conclusion of the theorem is straightforward. □

4 Inputs

We specialize next to a particular Riccati equation which is studied in linear optimal control theory. There, the objective is to, given the linear time-invariant system:

$$\dot{x} = Fx + Gu$$

and initial condition $x \in \mathbb{R}^n$, find a control $u : \mathbb{R}_+ \to \mathbb{R}^m$, such that the cost:

$$\mathcal{J} = \int_0^\infty \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] dt$$

is minimized (of course, $x(t)$ denotes the solution starting at x at time 0, subject to the input $u(t)$). The cost matrices $Q(t)$ and $R(t)$ are assumed to be continuous on $\mathbb{R}_+$ and

$$(C) \quad (Q(t), R(t)) \in \mathcal{P} \times \mathcal{P}^+, \; \forall\, t \in \mathbb{R}_+.$$

It is well-known that this problem leads to the following Riccati equation:

$$\dot{X} = -XGR^{-1}(t)G^T X + F^T X + XF + Q(t) := H(X, (Q(t), R(t))), \qquad (7)$$

where $X \in \mathcal{S}$. The cost matrices $Q(t)$ and $R(t)$ will be interpreted as input signals for this equation. Usually, a single solution -namely the one corresponding to an initial condition 0 at time 0- is of interest in this context. Here, we will not make this assumption.

Note that system (7) is of the form (4) and that (S) holds. Thus solutions exist and are unique in $\mathcal{S}$. The forward solution starting in $X_0 \in \mathcal{S}$ at $t = 0$ and corresponding to the input pair $(Q(t), R(t))$ will be denoted by $X(t, X_0, (Q(t), R(t)))$, $t \in \mathcal{I}^+$ or simply by $X(t)$ when no confusion is possible.

We will assume that inputs are partially ordered in the following way: $(Q_1, R_1) \preceq (Q_2, R_2)$ if $Q_2 - Q_1, R_2 - R_1 \in \mathcal{P}$. Input signals are then partially ordered in the obvious way: $(Q_1(t), R_1(t)) \preceq (Q_2(t), R_2(t))$ if $Q_2(t) - Q_1(t), R_2(t) - R_1(t) \in \mathcal{P}$ for all $t \in \mathbb{R}_+$.

Following the definition in [1], we call system (7) monotone if for all $X_0 \preceq Y_0$ holds that:

$$(Q_1(t), R_1(t)) \preceq (Q_2(t), R_2(t)), \; \forall\, t \;\Rightarrow\; X(t, X_0, (Q_1(t), R_1(t))) \preceq X(t, X_0, (Q_2(t), R_2(t))), \forall\, t \in \mathcal{I}_1^+ \cap \mathcal{I}_2^+.$$

Theorem 3. *Let (C) hold. Then system (7) is monotone on S.*

Proof. From Theorem 1 and the remark following Theorem 2 in [1], it follows that it is sufficient to verify that:

$$\forall X \in \mathcal{S}, \;\; (Q_1, R_1), (Q_2, R_2) \in \mathcal{P} \times \mathcal{P}^+ \text{ with } (Q_1, R_1) \preceq (Q_2, R_2) : \;\; H(X, (Q_1, R_1)) \preceq H(X, (Q_2, R_2)).$$

Notice that:

$$H(X, (Q_2, R_2)) - H(X, (Q_1, R_1)) = XB(R_1^{-1} - R_2^{-1})B^T X + (Q_2 - Q_1).$$

Since $R_2, R_1 \in \mathcal{P}^+$ and $R_2 - R_2 \in \mathcal{P}$, it follows from lemma 3 that $R_1^{-1} - R_2^{-1} \in \mathcal{P}$, which implies that:

$$H(X, (Q_2, R_2)) - H(X, (Q_1, R_1)) \in \mathcal{P}.$$

This conclud the proof. □

Next we consider the asymptotic behavior of system (7) under some additional conditions for the case where inputs are assumed to be constant.

Theorem 4. *Let $(Q(t), R(t)) = (\bar{Q}, \bar{R}) \in \mathcal{P}^+ \times \mathcal{P}^+$ for all $t \in \mathbb{R}_+$ and assume that the pair (F, G) is controllable. Then $\mathcal{P}$ is forward invariant and all solutions in $\mathcal{P}$ remain bounded. Moreover, there is a unique steady state $\bar{X} \in \mathcal{P}$, and $\bar{X} \in \mathcal{P}^+$ is a global attractor.*

Proof. Forward invariance of $\mathcal{P}$ and boundedness of solutions follow from theorem 2 and lemma 5 respectively and thus (7) with constant inputs $(\bar{Q}, \bar{R})$ generates a continuous semiflow Φ on $\mathcal{P}$. The existence of a steady state in $\mathcal{P}$ will follow from theorem 6 in the Appendix. That result is applicable here, since obviously $X = \mathcal{P}$ is convex and the order boundedness condition for omega limit sets follows immediately from remark 2. Suppose now that $\Pi \in \mathcal{P}$ is a steady state. We claim that $\Pi \in \mathcal{P}^+$. Suppose not, then there is some $0 \neq x_0 \in \mathbb{R}^n$ such that $x_0^T \Pi x_0 = 0$. Also, by lemma 8.4.7 in [13] (and this lemma remains valid under the assumption that $\Pi \in \mathcal{P}$, rather than the stronger assumption that $\Pi \in \mathcal{P}^+$) it follows that $-(A_{cl}^T \Pi + \Pi A_{cl}) \in \mathcal{P}^+$, where $A_{cl} := A - GR^{-1}G^T\Pi$. Then for $h > 0$ and small we obtain that:

$$(x_0 + hA_{cl}x_0)^T \Pi (x_0 + hA_{cl}x_0) = 0 + hx_0(A_{cl}^T \Pi + \Pi A_{cl})x_0 + O(h^2) < 0,$$

which contradicts that $\Pi \in \mathcal{P}$. Every steady state in $\mathcal{P}$ therefore belongs to $\mathcal{P}^+$. Corollary 8.4.9 in [13] implies that there is exactly one steady state in $\mathcal{P}^+$. Let's denote it by $\bar{X} \in \mathcal{P}^+$. Again by theorem 6, it follows that every solution of (7) starting in $\mathcal{S}$ converges to $\bar{X}$, concluding the proof of this theorem. □

From theorem 4 follows that we can construct a map $K : \mathcal{P}^+ \times \mathcal{P}^+ \to \mathcal{P}^+$ which maps an input pair $(\bar{Q}, \bar{R}) \in \mathcal{P}^+ \times \mathcal{P}^+$ to the corresponding $\bar{X}$ mentioned in the theorem:

$$K\left(\bar{Q}, \bar{R}\right) := \bar{X}.$$

Using the terminology of [1], we call K a *static input-state characteristic*, or *characteristic* for short. A slight difference with [1] is that there, in addition to the requirement that $K\left(\bar{Q}, \bar{R}\right)$ is a global attractor, it should also be stable for each input pair $\left(\bar{Q}, \bar{R}\right)$. Although we don't need a stability assumption for what follows, we do need to establish that K is continuous, a fact which we prove next, following the arguments in remark V.3 and proposition V.4 in [1].

Lemma 6. *Under the conditions of theorem 4, the map $K : \mathcal{P}^+ \times \mathcal{P}^+ \to \mathcal{P}^+$ is continuous.*

Proof. The conclusion will follow once we establish that:

1. The graph of K is closed.
2. K is locally bounded, i.e. for every compact set $\mathcal{C} \subset \mathcal{P}^+ \times \mathcal{P}^+$, $K(\mathcal{C})$ is bounded.

The closedness of the graph follows immediately from the fact that for all $(\bar{Q}, \bar{R}) \in \mathcal{P}^+ \times \mathcal{P}^+$, there holds that:

$$H\left(K\left(\bar{Q}, \bar{R}\right), \left(\bar{Q}, \bar{R}\right)\right) = 0$$

and since H is continuous.

To show local boundedness of K, pick an arbitrary compact set $\mathcal{C} \subset \mathcal{P}^+ \times \mathcal{P}^+$. Then by lemma 4, there exist $\lambda_{\min,1}, \lambda_{\min,2} > 0$ and $\lambda_{\max,1}, \lambda_{\max,2} > 0$ such that:

$$(\lambda_{\min,1} I, \lambda_{\min,2} I) \preceq \mathcal{C} \preceq (\lambda_{\max,1} I, \lambda_{\max,2} I)\,.$$

Choose an arbitrary initial state $X_0 \in \mathcal{P}$. Monotonicity of system (7) (by theorem 3) implies that for all $t \in \mathbb{R}_+$ (by theorem 4):

$$X\left(t, X_0, (\lambda_{\min,1} I, \lambda_{\min,2} I)\right) \preceq X(t, X_0, \mathcal{C}) \preceq X\left(t, X_0, (\lambda_{\max,1} I, \lambda_{\max,2} I)\right),$$

where -slightly abusing notation- $X(t, X_0, \mathcal{C})$ denotes the solution correspondng to an arbitrary fixed input in $\mathcal{C}$.

Taking limits as $t \to \infty$ -and these limits exist by theorem 4- we arrive at:

$$K\left((\lambda_{\min,1} I, \lambda_{\min,2} I)\right) \preceq K(\mathcal{C}) \preceq K\left((\lambda_{\max,1} I, \lambda_{\max,2} I)\right).$$

This shows that $K(\mathcal{C})$ is bounded and concludes the proof. □

Remark 1. Delchamp's lemma -see for instance Excercise 8.4.12 on p. 389 in [13]- yields the much stronger conclusion that the map K is real analytic. That proof is based on the implicit function theorem, while the proof above is based on monotonicity of the flow.

As a final application of monotone systems theory, we consider (7) and obtain asymptotic estimates of its solutions. But first we introduce some terminology. The *omega limit set* of a function $f : \mathbb{R}_+ \to A$, where A is a topological space, is the (possibly empty) set $\Omega[f] := \{a \in A \,|\, \exists \{t_k\},\, t_k \to \infty \text{ as } k \to \infty :\; f(t_k) \to a\}$. We will be interested in omega limit sets of solutions of the nonautonomous system (7). If in addition a partial order $\preceq$ on A has been defined, then we define $\mathcal{L}_{\preceq}[f]$ and $\mathcal{L}_{\succeq}[f]$ (both possibly empty) as follows:

$$\mathcal{L}_{\preceq}[f] := \{m \in A \,|\, \exists \{t_k\},\, t_k \to \infty, \exists \{m_k\} \subset A,\, m_k \to m \text{ as } k \to \infty :\; m_k \preceq f(t), \forall\, t \geq t_k\}$$

and

$$\mathcal{L}_{\succeq}[f] := \{M \in A \,|\, \exists \{t_k\},\, t_k \to \infty, \exists \{M_k\} \subset A,\, M_k \to M \text{ as } k \to \infty :\; M_k \succeq f(t), \forall\, t \geq t_k\}.$$

Then we obtain the following estimates.

Theorem 5. *Let $(Q(t), R(t)) \in \mathcal{P}^+ \times \mathcal{P}^+$ be a given pair of input signals which are assumed to be continuous for all $t \in \mathbb{R}_+$ and suppose that (F, G) is a controllable pair. Then every forward solution $X(t)$ of (7), starting in $\mathcal{P}$ at $t = 0$ satisfies:*

$$K\left(\mathcal{L}_{\preceq}[(Q, R)]\right) \preceq \Omega[X] \preceq K\left(\mathcal{L}_{\succeq}[(Q, R)]\right).$$

Proof. By theorem 3 and since the characteristic K is continuous (by lemma 6), the conclusion of this theorem follows immediately from Proposition $V.7$ in [1]. □

5 Appendix

Here we review a result by Dancer [2] for monotone maps, but slightly taylor it to our needs by stating it for monotone semiflows on subsets of finite dimensional Euclidean spaces.

The setting is Euclidean space $\mathbb{R}^n$ and a closed cone $K \subset \mathbb{R}^n$ which induces a partial order $\preceq$ on $\mathbb{R}^n$ ($x \preceq y$ iff $y - x \in K$). When $\text{int}(K) \neq \emptyset$, we can define a stronger notion as follows: for $x, y \in \mathbb{R}^n$, $x \prec\prec y$ iff $y - x \in \text{int}(K)$. If $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, we will write $x \preceq, \prec\prec A$ to denote that $x \preceq, \prec\prec y$ for all $y \in A$. For every pair $x, y \in \mathbb{R}^n$, we define the *order interval* $[x, y] := \{z \in \mathbb{R}^n \,|\, x \preceq z \preceq y\}$. Let X be a subset of $\mathbb{R}^n$. We will say that a set $A \subset X$ is *order bounded in* X if there are $x, y \in X$ such that $A \subset [x, y]_X := [x, y] \cap X$.

A *semiflow* $\Phi : \mathbb{R}_+ \times X \to X$ is a continuous map, satisfying $\Phi_0 = id$ and $\Phi_t \circ \Phi_s = \Phi_{s+t}$ for all $t, s \in \mathbb{R}_+$ (for each $t \in \mathbb{R}_+$, the map $\Phi_t : X \to X$ is defined by $\Phi_t(x) := \Phi(t, x)$).

For $x \in X$, we denote the *orbit* through x by $O(x) := \{\Phi_t(x) \mid t \in \mathbb{R}_+\}$ and its *omega limit set* by $\omega(x) := \{p \in X \mid \exists \{t_k\}, t_k \to \infty \text{ as } k \to \infty : \ \Phi_{t_k}(x) \to p\}$. It is well-known that if $O(x)$ is precompact, then $\omega(x)$ is a nonempty, compact and invariant set (i.e. $\Phi_t(\omega(x)) = \omega(x)$ for all $t \in \mathbb{R}_+$) and $d(\Phi_t(x), \omega(x)) \to 0$ as $t \to \infty$, where $d(x, A) := \inf_{a \in A} d(x, a)$ and d denotes the Euclidean metric on $\mathbb{R}^n$. An equilibrium of Φ is a point $e \in X$ such that $\Phi_t(e) = e$ for all $t \in \mathbb{R}_+$.

A semiflow is called *monotone* if:

$$\forall x, y \in X : \ x \preceq y \Rightarrow \Phi_t(x) \preceq \Phi_t(y), \ t \in \mathbb{R}_+.$$

Theorem 6. *Let Φ be a monotone semiflow on X, a convex set in $\mathbb{R}^n$, such that $O(x)$ is precompact for all $x \in X$. If $\omega(x)$ is order bounded in X, then there are equilibria e_1 and e_2 of Φ such that:*

$$e_1 \preceq \omega(x) \preceq e_2.$$

Proof. Since $\omega(x)$ is order bounded in X, there is some $p \in X$ such that $\omega(x) \preceq p$. By monotonicity of Φ and invariance of $\omega(x)$, we obtain that $\omega(x) \preceq \Phi_t(p)$ for all $t \in \mathbb{R}_+$. This implies that $\omega(x) \preceq \omega(p)$. Repeating this argument yields some $q \in X$ such that $\omega(x) \preceq \omega(p) \preceq \omega(q)$. Now define $S = \{z \in X \,|\, \omega(x) \preceq z \preceq \omega(q)\}$. Note that S is nonempty since $\emptyset \neq \omega(p) \subset S$. Moreover, S is bounded -being the intersection of order intervals with X- and closed, hence compact. And S is convex, being the intersection of order intervals (which are convex) and the convex set X. Finally, by invariance of $\omega(x)$ and $\omega(q)$, we see that S is forward invariant, i.e. $\Phi_t(S) \subset S$ for all $t \in \mathbb{R}_+$. Brouwer's fixed point theorem then implies the existence of an equilibrium $e_2 \in S$ of Φ. Notice that $e_2 \in S$ implies that $\omega(x) \preceq e_2$, as desired. The existence of e_1 is proved similarly. □

Remark 2. We claim that if $X = K$ and $\text{int}(K) \neq \emptyset$, then every bounded set $C \subset X$ is order bounded in X. In this case, the condition imposed in the statement of the previous theorem that omega limit set must be order bounded in X, is trivially satisfied since omega limit sets of precompact orbits are compact, hence bounded. For instance, this happens for our particular application where $X = K = \mathcal{P}$ because clearly $\text{int}(\mathcal{P}) \neq \emptyset$ (relative to $\mathcal{S}$).

To verify the claim, pick a bounded set $C \subset K$. Then there is some $M > 0$ such that $C \subset B_M(0) := \{x \in \mathbb{R}^n \,|\, d(0,x) \leq M\}$. We will show that an $x^* \in \text{int}(K)$ exists such that $B_M(0) \prec\prec x^*$. To see this, pick an arbitrary $\tilde{x} \in \text{int}(K)$ (such an $\tilde{x}$ exists because $\text{int}(K) \neq \emptyset$) and an $\epsilon > 0$ such that $B_\epsilon(\tilde{x}) \subset \text{int}(K)$. Equivalently, $0 \prec\prec B_\epsilon(\tilde{x})$ or $B_\epsilon(-\tilde{x}) \prec\prec 0$. This implies that $B_\epsilon(0) \prec\prec \tilde{x}$ and thus by scaling that $B_M(0) \prec\prec (M/\epsilon)\tilde{x} \in \text{int}(K)$. Denoting $(M/\epsilon)\tilde{x}$ by x^*, we obtain that $C \subset B_M(0) \prec\prec \tilde{x}$. Since we assumed that $X = K$ and $C \subset X$, it is obvious that $0 \preceq C$ and hence we have shown that $C \subset [0, \tilde{x}]_X$.

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