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On graphs whose maximal cliques and stable sets intersect

by

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ABSTRACT

We say that a graph G has the CIS-property and call G a CIS-graph if each maximal clique and each maximal stable set of G intersect. By definition, G is a CIS-graph if and only if the complementary graph $\bar{G}$ is a CIS-graph too. In this paper we give some necessary and some sufficient conditions for the CIS-property to hold. In general, problems of efficient characterization and recognition of CIS-graphs remain open.

Given an integer $k \geq 2$, a *comb* (or k-*comb*) S_k is a graph with $2k$ vertices k of which, $v_1, \ldots, v_k$, form a clique C, while others, $v'_1, \ldots, v'_k$, form a stable set S, and (v_i, v'_i) is an edge for all $i = 1, \ldots, k$, and there are no other edges. The complementary graph $\bar{S}_k$ is called an *anti-comb* (or k-anti-comb). Clearly, S and C switch in the complementary graphs. Obviously, the combs and anti-combs are not CIS-graphs, since $C \cap S = \emptyset$. Hence, if a CIS-graph G contains an induced comb (respectively, anti-comb) then it must be settled, that is, G must contain a vertex v connected to all vertices of C and to no vertex of S. However, these conditions are only necessary but not sufficient for the CIS-property to hold. Our main result is the following theorem: G is a CIS-graph whenever G contains no induced 3-combs and 3-anti-combs, and every induced 2-comb is settled in G.

We also generalize the concept of CIS-graph as follows. Given integer $d \geq 2$ and a complete graph whose edges are colored by d colors $\mathcal{G} = (V; E_1, \ldots, E_d)$, we say that $\mathcal{G}$ is a CIS-d-graph (has the CIS-d-property) if $\bigcap_{i=1}^{d} C_i \neq \emptyset$ whenever C_i is a maximal color i-free subset of V, that is, $(v, v') \in E_i$ for no $v, v' \in C_i$. Clearly, in case $d = 2$ we return to the concept of CIS-graphs. (More accurately, CIS-2-graph is a pair of two complementary CIS-graphs.) We conjecture that each CIS-d-graph is a Gallai graph, that is, it contains no triangle colored by 3 distinct colors. We obtain results supporting this conjecture and also show that if it holds then characterization and recognition of CIS-d-graphs is easily reduced to characterization and recognition of CIS-graphs.

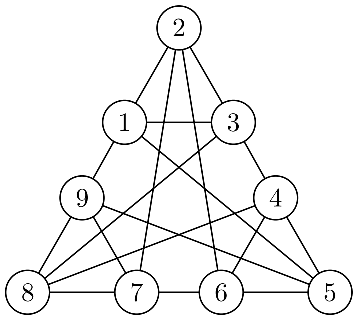
Key words: CIS-graphs, CIS-property, clique, clique-kernel intersection property, graph, independent set, stable graph, stable set

1 Introduction.

1.1 CIS-graphs

Given a graph G, we say that it has the *CIS-property*, or equivalently that G is a *CIS-graph*, if every maximal clique C and every maximal stable set S in G intersect. Obviously, they may have at most one common vertex and hence $|C \cap S| = 1$. It is convenient to represent a CIS-graph G as a *2-dimensional box partition*, that is, a matrix whose rows and columns are labeled respectively by the maximal cliques and stable sets of G and whose entries are the (unique) vertices of the corresponding intersections. For example, Figure 1 shows two CIS graphs and their intersection matrices. More examples are given in Figures 5, 6 and 9.

	258	147	369	249	537	816
123	2	1	3	2	3	1
456	5	4	6	4	5	6
789	8	7	9	9	7	8
267	2	7	6	2	7	6
591	5	1	9	9	5	1
834	8	4	3	4	3	8



	13579a	1470	369b	258a	260	48b
12b	1	1	b	2	2	b
234	3	4	3	2	2	4
456	5	4	6	5	6	4
678	7	7	6	8	6	8
890	9	0	9	8	0	8
0ab	a	0	b	a	0	b
168	1	1	6	8	6	8
249	9	4	9	2	2	4
380	3	0	3	8	0	8
46a	a	4	6	a	6	4
50b	5	0	b	5	0	b
27b	7	7	b	2	2	b

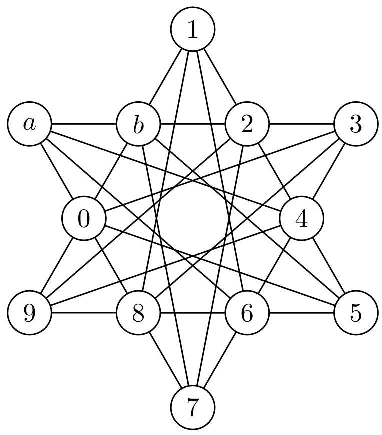


Figure 1: Two CIS graphs and their intersection matrices.

The CIS-property appears in the survey [3] (under the name *clique-kernel intersection property*) but no related results are mentioned.

The natural problems, characterizing the CIS-graphs and giving an efficient algorithm for their recognition, look difficult. One of the reasons is that the CIS-property is not hereditary. Indeed, if $C \cap S = \{v\}$ then $C \setminus \{v\}$ and $S \setminus \{v\}$ may become disjoint maximal clique and stable set after v is deleted.

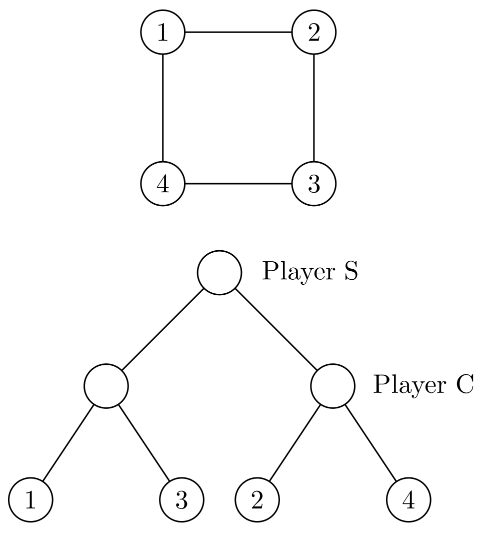
On the positive side, the CIS-property is self-complementary, that is, G is a CIS-graph if and only if the complementary graph $\bar{G}$ is a CIS-graph.

We proceed with the simple observation that every P_4-free graph is a CIS-graph, see e.g. [7, 8, 10, 12, 13, 15, 19, 24]. In fact, a stronger claim holds. We say that a set $T \subseteq V$ is a transversal of the hypergraphs $\mathcal{H} \subseteq 2^V$ if $T \cap H \neq \emptyset$ for all hyperedges $H \in \mathcal{H}$. The family of minimal transversals of $\mathcal{H}$ is denoted by $\mathcal{H}^d$ and is called the *dual* of $\mathcal{H}$. Given a graph $G = (V, E)$ we associate to it two hypergraphs, $\mathcal{C} = \mathcal{C}_G$ the collection of all maximal cliques of G, and $\mathcal{S} = \mathcal{S}_G$ the collections of all its maximal stable sets.

Proposition 1 ([12, 15, 19])**.** *A graph G has no induced P_4 if and only if the hypergraphs $\mathcal{C}$ and $\mathcal{S}$ of all maximal cliques and stable sets of G are dual hypergraphs.* □

Furthermore, P_4-free graphs are closely related to *read-once* Boolean functions and 2-person positional games, see for definitions, e.g., [11, 14, 15, 19].

Remark 1. *Read-once Boolean functions can be efficiently characterized, since their co-occurrence graphs are P_4-free, [7, 8, 12, 13, 15, 19]. Moreover, the normal forms of positional 2-person games with perfect information can be characterize by Proposition 1 too, [13, 14, 15]. Such a normal form is exactly the intersection matrix of the maximal cliques and stable sets of the corresponding graph, where the final positions (outcomes) of the game are in one-to-one correspondence with the vertices of the graph. See an example in Figure 2, where the monotone Boolean functions $F_S = 13 \vee 24$ and $F_C = (1 \vee 3)(2 \vee 4)$ corresponding to the hypergraphs $\mathcal{S} = \{(1,3),(2,4)\}$ and $\mathcal{C} = \{(1,2),(2,3),(3,4),(4,1)\}$ are read-once.*



$$\begin{aligned} \mathcal{C} &= \{(1,2)(2,3)(3,4)(4,1)\} \\ \mathcal{S} &= \{(1,3)(2,4)\} \end{aligned}$$

	12	23	34	41
13	1	3	3	1
24	2	2	4	4

Figure 2: A P_4-free graph and the corresponding positional and normal game forms

However, the absence of induced P_4s is only sufficient but not necessary for the CIS-property to hold. Let a graph G contain an induced P_4 defined by $(v_1, v_1'), (v_2, v_2'), (v_1, v_2)$. The clique $\{v_1, v_2\}$ and stable set $\{v_1', v_2'\}$ are disjoint. Hence, they can not be maximal in G

if it is a CIS-graph. In other words, G must contain a fifth vertex v_0 such that $(v_0, v_1), (v_0, v_2)$ are edges, while $(v_0, v'_1), (v_0, v'_2)$ are not. In this case we will say that P_4 *is settled* by v_0, cf. [23]. Let us note that the graph induced by $\{v_0, v_1, v_2, v'_1, v'_2\}$ is a CIS-graph, see Figure 5.

Thus, every induced P_4 in a CIS-graph must be settled. This condition is necessary, as we argued above, yet, it is not sufficient, according to the following examples.

1.2 Combs and anti-combs

Given an integer $k \geq 2$, a *comb* (or k-comb) S_k is defined as a graph with $2k$ vertices k of which form a clique $C = \{v_1, \ldots, v_k\}$, while the remaining k form a stable set $S = \{v'_1, \ldots, v'_k\}$. In addition, S_k contains the perfect matching (v_i, v'_i) for $i = 1, \ldots, k$, and there are no more edges in S_k. Let us note that graphs S_2 and P_4 are isomorphic. Furthermore, S_3 contains 3 induced S_2 and all 3 are settled. More generally, S_k contains k induced S_{k-1} and they all are settled. Figure 3 shows S_k, for $k = 2, 3$, and 4.

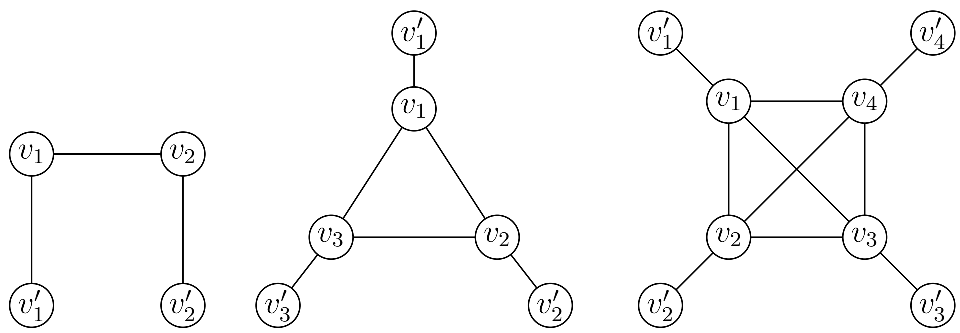


Figure 3: Combs S_k, for $k = 2, 3$ and 4

The complementary graph $\bar{S}_k$ is called an *anti-comb* (or k-anti-comb). Figure 4 shows $\bar{S}_k$ for $k = 2, 3$, and 4.

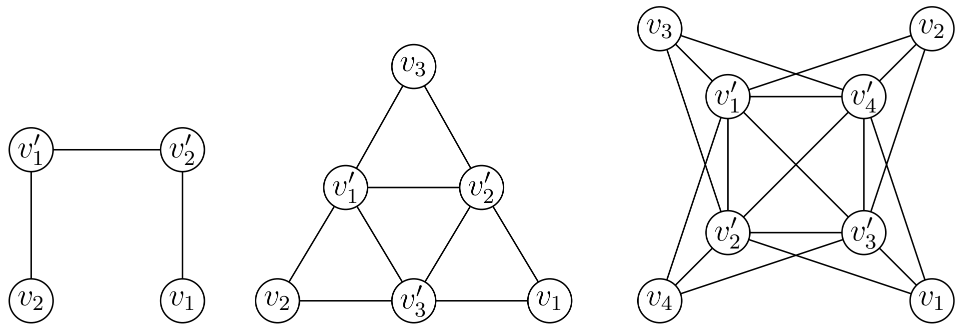
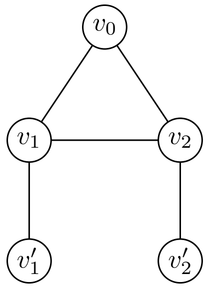


Figure 4: Anti-combs $\bar{S}_k$, for $k = 2, 3$ and 4.

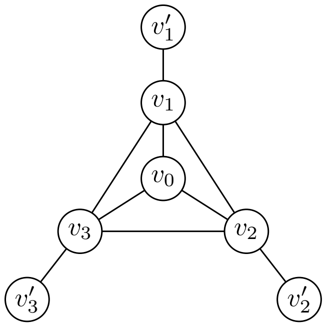
Clearly, the roles of the sets S and C are switched in S_k and $\bar{S}_k$. Obviously, combs and anti-combs are not CIS-graphs, since they contain a maximal clique C and stable set S which are disjoint. Hence, if a CIS-graph G contains an induced comb S_k (respectively, anti-comb $\bar{S}_k$) then it must be *settled*, that is, G must contain a vertex v_0 adjacent to each vertex of C and to no vertex of S. Thus, the following condition is necessary for the CIS-property to hold.

(COMB) Every induced comb and anti-comb must be settled in G.

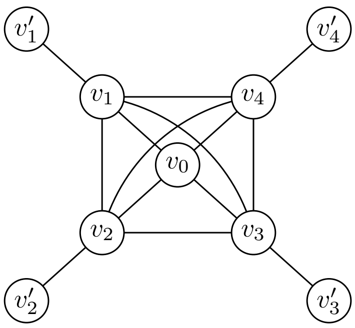
Figures 5 and 6 show settled combs and anti-combs. Notice that they are CIS graphs, as shown by the corresponding intersection matrices. Obviously, the intersection matrix for an anti-comb $\bar{S}_k$ is the transposed intersection matrix of the corresponding comb S_k.



	01′2′	12′	1′2
012	0	1	2
11′	1′	1	1′
22′	2′	2′	2



	01′2′3′	12′3′	1′23′	1′2′3
0123	0	1	2	3
11′	1′	1	1′	1′
22′	2′	2′	2	2′
33′	3′	3′	3′	3



	01′2′3′4′	12′3′4′	1′23′4′	1′2′34′	1′2′3′4
01234	0	1	2	3	4
11′	1′	1′	1′	1′	1′
22′	2′	2′	2	2′	2′
33′	3′	3′	3′	3	3′
44′	4′	4′	4′	4′	4

Figure 5: Settled combs S_k, for $k = 2, 3$ and 4.

The following obvious properties of combs and anti-combs are worth summarizing:

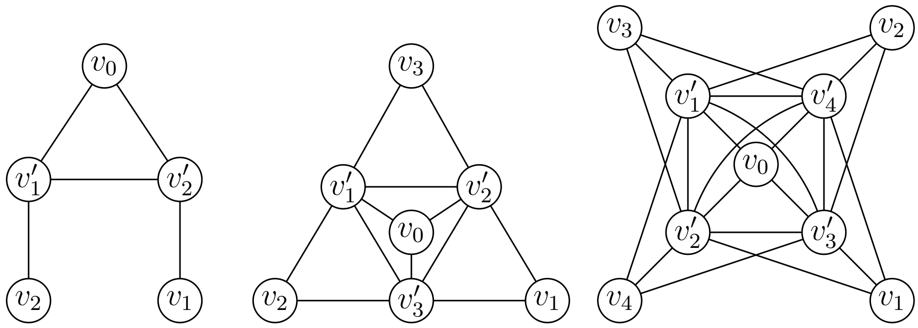


Figure 6: Settled anti-combs $\bar{S}_k$, for $k = 2, 3$ and 4.

- The 2-comb S_2 and 2-anti-comb $\bar{S}_2$ are isomorphic, while the k-comb S_k and k-anti-comb $\bar{S}_k$ are not isomorphic for $k > 2$.
- The k-comb S_k contains $\binom{k}{m}$ induced m-combs S_m which are all settled, yet, S_k contains no induced m-anti-combs $\bar{S_m}$ for $m > 2$; respectively, the k-anti-comb $\bar{S}_k$ contains $\binom{k}{m}$ induced m-anti-combs $\bar{S_m}$ which are all settled, yet, it contains no induced m-combs S_m for $m > 2$.
- A settled k-comb and a settled k-anti-comb are complementary CIS-graphs.

Obviously, **COMB** is a necessary condition for the CIS-property to hold. Yet, it is not sufficient, as we will see in Section 1.3. Let us introduce the following stronger condition.

COMB$(3,3)$ There is no induced 3-comb or 3-anti-comb, and every induced 2-comb is settled in G.

Our main result claims that this stronger condition already implies the CIS-property.

Theorem 1. *A graph G is a CIS-graph whenever it satisfies **COMB(**$3,3$**)**.*

We give the proof in Section 2. It contains a complicated case analysis in which one of the cases is especially interesting and results in a remarkable graph that is "almost" a counterexample to Theorem 1. This graph $2\mathcal{P}$ (see Figure 7) consists of two identical copies of the Petersen graph induced by the vertices $v_0, \ldots, v_9$ and $v_0', \ldots, v_9'$ respectively. Furthermore, (v_i', v_j) is an edge if and only if (v_i, v_j) is not, for all $i \neq j$. Ten remaining pairs (v_i, v_i'), $i = 0, \ldots, 9$, are uncertain, that is, configuration $2\mathcal{P}$ represents in fact 2^{10} possible graphs rather than one graph. The following properties of $2\mathcal{P}$ are easy to see.

(a) $2\mathcal{P}$ is isomorphic to its complement.

(b) $2\mathcal{P}$ is regular of "degree 9.5", that is, each vertex is incident to 9 edges and belongs to one uncertain pair.

(c) For every two vertices u, v there is an automorphism α of $2\mathcal{P}$ such that $\alpha(u) = v$.

(d) None of the 2^{10} graphs of $2\mathcal{P}$ contains an induced 3-comb or 3-anti-comb.

(e) Every induced 2-comb in all 2^{10} graphs of $2\mathcal{P}$ involves a pair v_i, v'_i for some $i = 0, \dots, 9$.

In fact, 36 induced 2-combs appear, whenever we substitute a pair v_i, v'_i by an edge (or by a non-edge). It is easy to see that none of these 2-combs can be settled by a vertex of $2\mathcal{P}$, and if it is settled by a new vertex then an unsettled 3-comb or 3-anti-comb always appears. Thus, the case under consideration does not lead to a counterexample, and a complete case analysis yields the proof of Theorem 1, see Section 2.

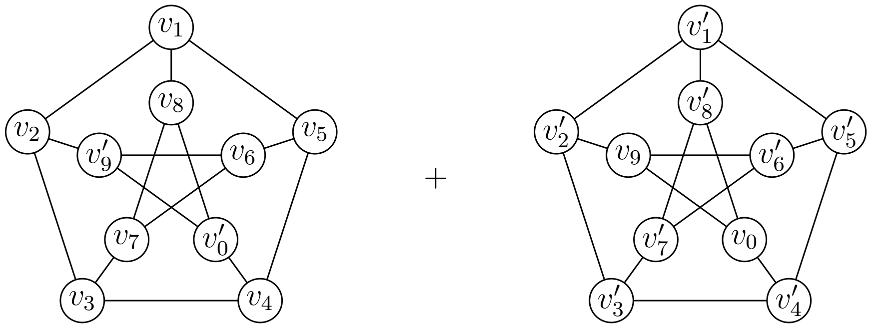


Figure 7: Graph $2\mathcal{P}$.

Examples for CIS-graphs satisfying condition $\mathbf{COMB}(3,3)$ are given in Figure 1.

It would be interesting to analyze the following relaxations of condition $\mathbf{COMB}(3,3)$ that are still stronger than **COMB**. Given integers $i, j \geq 2$, we say that a graph G satisfies condition $\mathbf{COMB}(i,j)$ if all induced combs and anti-combs in G are settled and, moreover, G contains no induced S_i and $\bar{S}_j$. By a natural convention we have $\mathbf{COMB} = \mathbf{COMB}(\infty,\infty)$.

Clearly, condition $\mathbf{COMB}(2,2)$ implies the CIS-property, since it means that the graph is P_4-free. In fact, we have $\mathbf{COMB}(2,2) \equiv \mathbf{COMB}(2,i) \equiv \mathbf{COMB}(i,2)$ for every $i \geq 2$, since the 2-comb $S_2 \equiv P_4$ is self-complementary and every comb and anti-comb contains an induced 2-comb. Furthermore, condition $\mathbf{COMB}(i,j)$ is monotone in the sense that it implies $\mathbf{COMB}(i',j')$ for all $i \leq i'$ and $j \leq j'$, and symmetric, in the sense that $\mathbf{COMB}(i,j)$ implies the CIS-property if and only if $\mathbf{COMB}(j,i)$ does (due to the fact that G is a CIS-graph if and only if its complement $\bar{G}$ is a CIS-graph).

According to Theorem 1, condition $\mathbf{COMB}(3,3)$ implies the CIS-property. However, it is not known whether $\mathbf{COMB}(4,4)$ or $\mathbf{COMB}(3,j)$ for some $j \geq 4$ imply the CIS-property, or not. Certainly, condition $\mathbf{COMB}(5,4)$ does not imply the CIS-property, as the next section shows.

1.3 (n, k, ℓ)-graphs and their complements

The following graph $G = (V, E)$ was suggested by Ron Holzman in 1994. It has $\binom{n}{1} + \binom{n}{2} = 5 + 10 = 15$ vertices, where subsets $S = \{v_1, \ldots, v_5\}$ and $C = \{v_{12}, \ldots, v_{45}\}$ induce a stable set and clique, respectively; $V = C \cup S$ (hence, G is a split graph); furthermore, every pair (v_i, v_{ij}), where $i, j = 1, \ldots, 5$ and $i \neq j$, is an edge, and there are no more edges. Let us denote this graph by $G(5, 1, 2)$, see Figure 8.

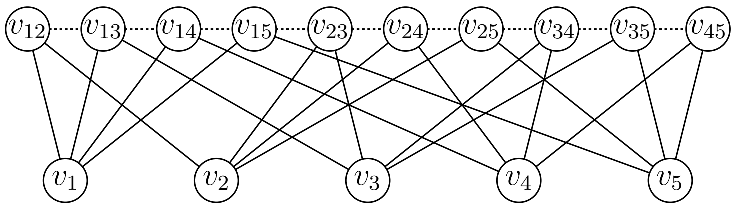


Figure 8: Graph $G(5, 1, 2)$ was constructed by Ron Holzman in 1994.

It is easy to verify that $G(5, 1, 2)$ contains no induced 5-combs and 4-anti-combs. In section 3 we will show that all induced combs and anti-combs in $G(5, 1, 2)$ are settled. For example, the 4-comb induced by vertices $(v_{12}, v_{13}, v_{14}, v_{15}, v_2, v_3, v_4, v_5)$ is settled by v_1 and the 3-anti-comb induced by $(v_{12}, v_{13}, v_{23}, v_1, v_2, v_3)$ is settled by v_{45}, etc. Thus, the graph $G(5, 1, 2)$ satisfies condition **COMB**$(5, 4)$, however, it is not a CIS-graph, since $C \cap S = \emptyset$. Let us note that the settled extension of $G(5, 1, 2)$ is a CIS-graph, see Figure 9.

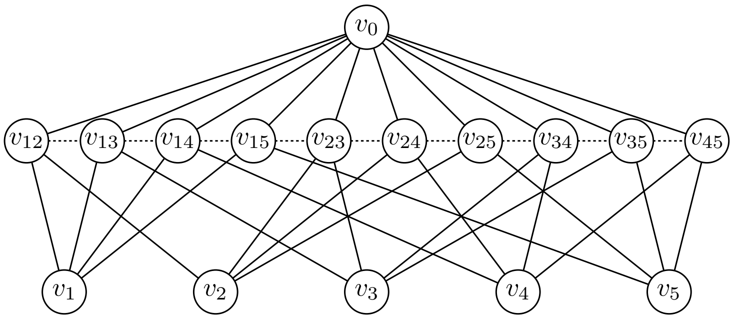
We generalize the above example as follows. Given integers n, k, ℓ such that $n > k \geq 1$ and $n > \ell \geq 1$, consider a set S (respectively, C) consisting of $\binom{n}{k}$ (respectively, $\binom{n}{\ell}$) vertices labeled by k-subsets (respectively, by ℓ-subsets) of a ground n-set. Let us introduce the graph $G(n, k, \ell)$ on the vertex-set $C \cup S$ such that S is a stable set, C is a clique, and a vertex of S is adjacent to a vertex of C if and only if the corresponding k-set is either a subset or a superset of the corresponding ℓ-set. Obviously, $G(n, k, \ell)$ is not a CIS-graph, since $C \cap S = \emptyset$. However, some of these graphs satisfy the condition **COMB**, for example, $G(5, 1, 2)$. Moreover, $G(5, 1, 2)$ satisfies the stronger condition **COMB**$(5, 4)$.

By definition, $G(n, 1, 1) = S_n$ is an n-comb and $G(n, n-1, 1) = \bar{S_n}$ is an n-anti-comb. Furthermore, it is easy to see that

(i) the graphs $G(n, k, \ell)$ and $G(n, n-k, n-\ell)$ are isomorphic.

Hence, without loss of generality we can assume that $k \leq \ell$ and even that $k < \ell$, since $G(n, k, k)$ is just a comb $S_{\binom{n}{k}}$. Then, from the simple fact that a set contains an element if and only if the complementary set does not contain it, we derive

(ii) the graphs $G(n, k, 1)$ and $G(n, 1, n-k)$ are complementary.



	0 12 13 14 15 23 24 25 34 35 45	1 12 13 14 15	2 12 23 24 25	3 13 23 34 35	4 14 24 34 45	5 15 25 35 45
0 1 2 3 4 5	0	1	2	3	4	5
12 3 4 5	12	12	12	3	4	5
13 2 4 5	13	13	2	13	4	5
14 2 3 5	14	14	2	3	14	5
15 2 3 4	15	15	2	3	4	15
23 1 4 5	23	1	23	23	4	5
24 1 3 5	24	1	24	3	24	5
25 1 3 4	25	1	25	3	4	25
34 1 2 5	34	1	2	34	34	5
35 1 2 4	35	1	2	35	4	35
45 1 2 3	45	1	2	3	45	45

Figure 9: Settled $G(5, 1, 2)$.

Thus, the graphs $G(n, k, n-1)$ and $G(n, n-k, 1)$ are isomorphic by (i) and complementary to $G(n, 1, k)$ by (ii). Hence, without loss of generality we can assume that $\ell \le n-2$. Summarizing, we will assume in the sequel that

$$1 \le k < \ell \le n-2. \tag{1.1}$$

In section 3 we will prove the following two claims analyzing the existence of unsettled anti-combs and combs in $G(n, k, \ell)$.

Theorem 2.

(i) Each induced anti-comb in $G(n, k, \ell)$ *is settled whenever*

$$n > \frac{k+1}{k}\ell.$$

(ii) An unsettled induced anti-comb exists in $G(n, k, \ell)$ *whenever*

$$k + \ell \le n \le \frac{k+1}{k}\ell.$$

Theorem 3.

(a) Each induced comb is settled in $G(n,1,\ell)$*, and it is settled in* $G(n,2,\ell)$ *whenever*

$$n < 2\ell - 3.$$

(b) An unsettled induced comb exists in $G(n,k,\ell)$ *for* $k \geq 2$ *whenever*

$$n \geq \frac{k}{k-1}\ell - \frac{r}{k-1} \quad \text{or} \quad n = \frac{k}{k-1}\ell - \frac{r}{k-1} - 1 \text{ and } \ell > r + k^2 - k,$$

where $r \equiv \ell \pmod{k-1}$ *and* $r \in \{2, 3, ..., k\}$.

Let us denote by $\mathbf{G}$ the subfamily of graphs $G(n,k,\ell)$ whose induced combs and anti-combs are all settled and n,k,ℓ satisfy (1.1).

Corollary 1. *For* $k = 1$ *and* $k = 2$ *the membership in* $\mathbf{G}$ *is characterized as follows:*

$$G(n,1,\ell) \in \mathbf{G} \quad \text{iff} \quad n > 2\ell$$
$$G(n,2,\ell) \in \mathbf{G} \quad \text{iff} \quad 2\ell - 3 > n > (3/2)\ell.$$

Proof. By (1.1) we have $n \geq \ell + 2 \geq \ell + k$, whenever $k \leq 2$, and thus, by Theorem 2, all induced anti-combs are settled in $G(n,k,\ell)$ for $k \leq 2$ if and only if $n > \frac{k+1}{k}\ell$. This and (a) of Theorem 3 then implies the claim for $k = 1$.

If $k = 2$ then $G(n,2,\ell)$ has an unsettled comb, by (b) of Theorem 3, if $n \geq 2\ell - 2$ or if $n = 2\ell - 3$ and $\ell > 4$, since $r = 2$ in this case. However, if $n = 2\ell - 3$ then $\ell \geq 5$ by (1.1). Hence, the second condition holds automatically, and therefore by (a) and (b) of Theorem 3, we can conclude that $G(n,2,\ell)$ has an unsettled comb if and only if $n \geq 2\ell - 3$. □

Thus, for $k = 1$ we get $\{G(5,1,2), G(6,1,2), G(7,1,2), G(7,1,3), \ldots\} \subseteq \mathbf{G}$ and for $k = 2$ we get $\{G(14,2,9), G(16,2,10), G(17,2,11), G(18,2,11), G(19,2,12), G(20,2,13), \ldots\} \subseteq \mathbf{G}$.

Remark 2. *Notice that conditions (i) and (ii) of Theorem 2 provide an almost complete characterization of the existence of unsettled anti-combs in* $G(n,k,\ell)$*. However, it is not clear if condition* $n \geq k + \ell$ *in part (ii) is necessary. Note that if* $k \leq 2$*, then this condition holds automatically by* (1.1)*. For instance, we do not know if* $G(8,3,6)$ *has an unsettled anti-comb. Computer experiments show that there are no unsettled* m*-anti-combs for* $m \leq 10$*. In any case,* $G(8,3,6)$ *has an unsettled* 6*-comb, by Theorem 3.*

Let us also note that we know much less about combs. For instance, we could only treat the case of $k \leq 2$ *in (a) of Theorem 3, though we conjecture that a similar claims can hold for all* k*. For example,* $G(10,3,8)$ *is the smallest graph for which we do not know if it contains an unsettled comb or anti-comb.*

Based on the proofs of the above theorems and on several numerical examples we conjecture that membership in $\mathbf{G}$ *can be characterized by inequalities of the approximate form*

$$\frac{k}{k-1}\ell + O(k) \;\geq\; n \;\geq\; \frac{k+1}{k}\ell - O(k).$$

This is certainly the case for $k \leq 2$*, by Corollary 1.*

By definition, in a graph $G = G(n, k, \ell) \in \mathbf{G}$, as well as in its complement $\bar{G}$, all induced combs and anti-combs are settled, that is, both G and $\bar{G}$ satisfy the condition **COMB**. Let us note however that $\bar{G}$ is not an (n, k, ℓ)-graph unless $k = 1$. (Recall that $G(n, 1, \ell)$ and $G(n, n - \ell, 1)$ are complementary.)

It seems that every non-CIS-graph satisfying **COMB** contains either an induced $G(n, k, \ell) \in \mathbf{G}$ or its complement. At least, we have no counterexample for this claim.

Let us add that, unlike the case of combs and anti-combs, one graph from $\mathbf{G}$ may contain another as an unsettled induced subgraph. For example, $G(6, 1, 2)$ contains an unsettled induced $G(5, 1, 2)$, while in $G(7, 1, 2)$ all induced $G(5, 1, 2)$ are settled. Yet, in $G(7, 1, 2)$ there is an unsettled induced $G(6, 1, 2)$. Vice versa, in $G(7, 1, 3)$ each induced $G(6, 1, 2)$ is settled but there are unsettled induced $G(5, 1, 2)$. Further, in $G(8, 1, 3)$, all induced $G(5, 1, 2)$ and $G(7, 1, 2)$ are settled but there are unsettled induced $G(6, 1, 2)$ and $G(7, 1, 3)$. Due to this "non-transitivity", in order to enforce the CIS-property for a graph G, it seems easier to assume that all induced subgraphs from $\mathbf{G}$ as well as their complements are settled in G. Of course, it is even simpler to assume that G does not contain these subgraphs at all.

Conjecture 1. *If* G *contains no induced* $G(5, 1, 2)$ *nor its complement* $G(5, 3, 1)$ *and all induced combs and anti-combs are settled in* G *then* G *is a CIS-graph.*

We should remark here that $G(n, k, l)$ contains an induced $G(n', k', l')$ whenever $n' \leq n$, $k' \leq k$, and $l' \leq l$.

Remark 3. *Finally, let us note that CIS-graphs and perfect graphs look somewhat similar. Both classes are self-complementary. Odd holes and anti-holes are similar to combs and anti-combs. The following two tests look similar too: whether* G *contains an induced odd hole or anti-hole and whether* G *contains an induced unsettled comb or anti-comb. It seems that CIS-graphs, like perfect graphs, may allow a simple characterization and an efficient recognition algorithm (which may be very difficult to obtain, though). However, there are dissimilarities, too. The property of perfectness is hereditary, unlike the CIS-property. Also, there are non-CIS-graphs in which all induced combs and anti-combs are settled. Up to now, every minimal such graphs that we know (or its complement) belongs to family* $\mathbf{G}$*, however, other examples may exist.*

CIS-graphs were recently mentioned (under the name of stable *graphs) in [25]. It is shown that recognition of stable graphs is a special case of a difficult problem (strongly bipartite bihypergraph recognition problem) considered in this paper. Based on this observation, the authors conjecture that recognition of stable graphs is co-NP-complete. However, we conjecture that this problem is polynomial.*

1.4 CIS-d-graphs.

Let us generalize the concept of a CIS-graph as follows. For a given integer $d \geq 2$, a complete graph whose edges are colored by d colors $\mathcal{G} = (V; E_1, \ldots, E_d)$ is called a *d-graph.* To a given d-graph $\mathcal{G}$ let us assign a family of d hypergraphs $\mathcal{C} = \mathcal{C}(G) = \{\mathcal{C}_i \mid i = 1, \ldots, d\}$ on the common vertex set V, where the hyperedges of $\mathcal{C}_i$ are all inclusion maximal subsets of V containing no edges of color i. We say that $\mathcal{G}$ is a *CIS-d-graph* (has the *CIS-d-property*) if $\bigcap_{i=1}^{d} C_i \neq \emptyset$ for all selections $C_i \in \mathcal{C}_i$ for $i = 1, ..., d$. Note that such an intersection can contain at most one vertex for any d-graph, by the definitions. Clearly, if $d = 2$ then we obtain the original concept of CIS-graphs. (More accurately, CIS-2-graph is a pair of two complementary CIS-graphs.) Similarly to CIS-graphs, CIS-d-graphs also satisfy a natural requirement that can be considered as a generalization of settling. Assume that X_i is a clique in the subgraph $\cup_{j \neq i} E_j$ for $i = 1, ..., d$, and that $\cap_{i=1}^{d} X_i = \emptyset$. Then, these cliques cannot all be maximal in a CIS-d-graph, and hence, we must have a vertex $x \in V$ such that $(x, y) \notin E_i$ for all $y \in X_i$, for $i = 1, ..., d$. We shall say in this case that $\{X_1, X_2, ..., X_d\}$ are *settled* by x.

Given a CIS-d-graph $\mathcal{G}$, let us assign to it a d-dimensional table $g = g(\mathcal{G})$, that is, a mapping $g : \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_d \to V$, defined by $g(C_1, C_2, ..., C_d) = v$ if $\{v\} = \cap_{i=1}^{d} C_i$. Let us observe that this d-dimensional array is partitioned by the elements of V into $n = |V|$ sub-arrays that are called *boxes*, since the following implication holds. If $g(C'_1, ..., C'_d) = g(C''_1, ..., C''_d) = v$, then v belongs to all these sets, and thus, $g(C_1, ..., C_d) = v$ for all 2^d choices $C_i \in \{C'_i, C''_i\}$, $i = 1, ..., d$.

Let us further introduce two special edge colored graphs. Let Π denote the 2-colored graph whose both chromatic components form a P_4, that is, $V = \{v_1, v_2, v_3, v_4\}$; $E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, and $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$. Furthermore, let Δ denote the 3-colored triangle, for which $V = \{v_1, v_2, v_3\}$, $E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, and $E_3 = \{(v_3, v_1)\}$. Figure 11 illustrates these graphs.

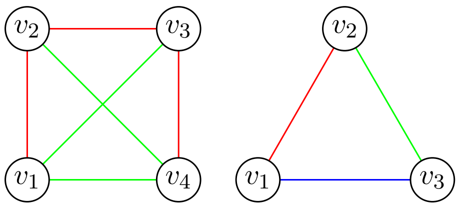


Figure 10: Colored Π and Δ.

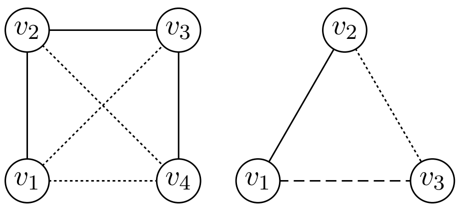


Figure 11: Colored Π and Δ (in black and white for printing).

Let us proceed now with the following observation.

Proposition 2 ([13, 15]). *Every* Π- *and* Δ-*free* d-*graph is a CIS*-d-*graph.*

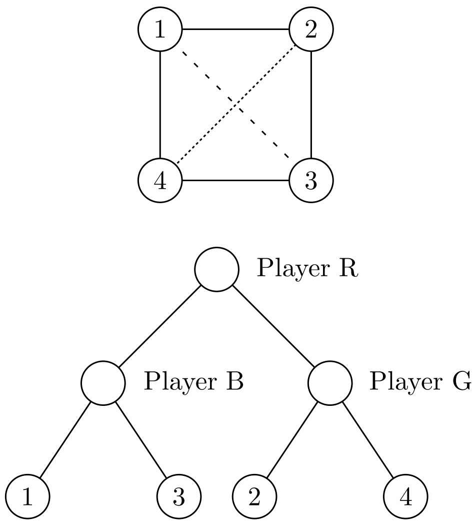
□

In fact, a stronger claim holds.

Proposition 3 ([13, 14, 15]). *A* d-*graph* $\mathcal{G}$ *is* Π- *and* Δ-*free if and only if the corresponding mapping* $g(\mathcal{G})$ *defines a positional* d-*person game with perfect information whose final positions (outcomes of the game) are in one-to-one correspondence with the vertices of* $\mathcal{G}$.

□

For example, let us consider the Π- and Δ-free 3-graph $\mathcal{G}$ given in Figure 12. For this graph we have $\mathcal{C}_1 = \{(1,3),(2,4)\}$, $\mathcal{C}_2 = \{(1,2,4),(2,3,4)\}$, and $\mathcal{C}_3 = \{(1,2,3),(1,3,4)\}$. The mapping $g(\mathcal{G})$, and the positional game corresponding to it are also shown in Figure 12.



$$\begin{aligned} \mathcal{C}_1 &= \{(13)(24)\} \\ \mathcal{C}_2 &= \{(124)(234)\} \\ \mathcal{C}_3 &= \{(123)(134)\} \end{aligned}$$

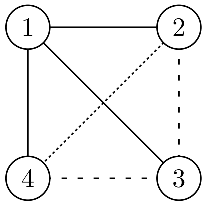
13			24	
1	1	124	2	4
3	3	234	2	4
1	1		1	1
2	3		2	3
3	4		3	4

Figure 12: A Π- and Δ-free 3-graph and the corresponding positional and normal game forms.

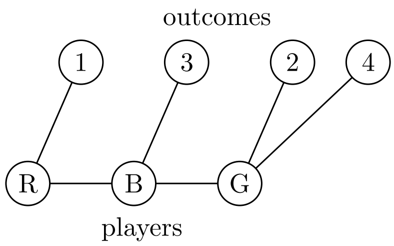
Another example of a Π- and Δ-free 3-graph is given in Figure 13. In this case we have $\mathcal{C}_1 = \{(1),(2,3,4)\}$, $\mathcal{C}_2 = \{(1,3),(1,2,4)\}$, and $\mathcal{C}_3 = \{(1,2,3),(1,3,4)\}$. The mapping $g(\mathcal{G})$, and the positional game corresponding to it are also shown in Figure 13.

Of course, the condition that a d-graph $\mathcal{G}$ must be Π- and Δ-free is only sufficient but not necessary for the CIS-d-property to hold. On the other hand, the following condition is clearly necessary. Given a d-graph $\mathcal{G} = (V; E_1, ..., E_d)$ and a partition $P_1 \cup \cdots \cup P_\delta = \{1, ..., d\}$ of its colors, we can define a δ-graph $\mathcal{G}' = (V; E'_1, ..., E'_\delta)$ by setting $E'_i = \cup_{j \in P_i} E_j$, $i = 1, ..., \delta$. Let us call $\mathcal{G}'$ a δ-*projection* of $\mathcal{G}$.

Proposition 4. *For every CIS*-d-*graph* $\mathcal{G}$ *and every partition of its set of colors* $\{1, \ldots, d\}$ *into* δ *non-empty subsets* ($2 \leq \delta \leq d$), *the corresponding* δ-*graph* $\mathcal{G}'$ *is a CIS*-δ-*graph.*



$$\mathcal{C}_1 = \{(1)(234)\}$$
$$\mathcal{C}_2 = \{(13)(124)\}$$
$$\mathcal{C}_3 = \{(123)(134)\}$$



1			234	
1	1	13	3	3
1	1	124	2	4
1	1		1	1
2	3		2	3
3	4		3	4

Figure 13: Another Π- and Δ-free 3-graph and the corresponding positional and normal game forms.

In particular, in case $\delta = 2$ we get two complementary CIS-graphs.

The following conjecture is open since 1978.

Conjecture 2. *([13]) Every CIS-d-graph is Δ-free.*

According to Proposition 4, it would suffice to prove the conjecture for $d = 3$. In this case it was verified up to $n = 12$ vertices by a computer code written by Steven Jaslar in 2003. We will consider this conjecture in Section 4 and show that, similarly to combs and anti-combs, all Δs in a CIS-d-graph must be settled and it takes two vertices to settle a Δ (see Section 4.2). Although there are d-graphs in which all Δs are settled, yet it seems impossible to have settled simultaneously all combs and anti-combs in 2-projections of these d-graphs, a condition that is necessary by Proposition 4.

In the literature Δ-free d-graphs are called *Gallai's graphs*, since they were introduced by Gallai in [10]. The above conjecture means that CIS-d-graphs form a subfamily of the Gallai graphs. Gallai's graphs are well studied, see e.g. [1, 4, 5, 6, 9, 18, 20, 21]. The subfamily of the Gallai CIS-d-graphs is characterized below in terms of CIS-graphs. Hence, to characterize CIS-d-graphs it would suffice to do it for $d = 2$ and to prove Conjecture 2.

First, let us note that both Gallai's and CIS d-graphs are closed under substitution. (For Gallai's graphs this is well known, see e.g., [18].) Moreover, the inverse claims hold too.

Proposition 5. *Let us substitute a vertex v of a d-graph $\mathcal{G}'$ by a d-graph $\mathcal{G}''$ and denote the obtained d-graph by $\mathcal{G}$. Then $\mathcal{G}$ is a Gallai (respectively, CIS-) d-graph if and only if both $\mathcal{G}'$ and $\mathcal{G}''$ are Gallai (respectively, CIS-) d-graphs.*

In case $d = 2$ this proposition implies the similar property for CIS-graphs.

Proposition 6. *Let us substitute a vertex v of a graph G' by a graph G'' and denote the obtained graph by G. Then G is a CIS-graph if and only if both G' and G'' are CIS-graphs.*

□

Let us recall however that, in contrast to this claim, an induced subgraph of a CIS-graph may not have the CIS-property.

Here and in the sequel we assume that the set of colors $[d] = \{1, \ldots, d\}$ is the same for all considered d-graphs, while some chromatic components may be empty. For example, by a 2-graph we mean a d-graph with only 2 non-empty chromatic components.

It is known that every Gallai d-graph can be obtained from 2-graphs by recursive substitutions. More precisely, the following claim holds.

Proposition 7 (Gyárfás and Simonyi, [18])**.** *For every Gallai's d-graph $\mathcal{G}$ there exist a 2-graph $\mathcal{G}_0$ with n vertices and n Gallai's d-graphs $\mathcal{G}_1, \ldots, \mathcal{G}_n$ such that $\mathcal{G}$ is obtained by substituting n vertices of $\mathcal{G}_0$ by $\mathcal{G}_1, \ldots \mathcal{G}_n$.*

In [18], this claim is derived from the following Lemma.

Lemma 1 ([10] and [18])**.** *Every Gallai d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ with $d \geq 3$ has a color $i \in [d]$ that does not span V, or in other words, the graph $G_i = (V, E_i)$ is not connected.*

Gyárfás and Simonyi remark that this Lemma "is essentially a content of Lemma (3.2.3) in [10]" and they derive Proposition 7 from it as follows. If $d = 2$ we are done. Otherwise, we have a color $i \in [d]$ such that the graph $G_i = (V, E_i)$ has at least two connected components. It is easy to see that for any two of these components all edges between them are of the same color. Collapsing these components into vertices we get a smaller Gallai $(d-1)$-graph that, by induction, can be generated as required. □

Clearly, this Lemma provides a linear time algorithm for decomposing Gallai's d-graphs into 2-graphs. Let us note however that such a decomposition may be not unique, since several chromatic components may be disconnected.

Remark 4. *It is interesting to compare Lemma 1 with the following Lemma proved in [13, 15]. If a d-graph $\mathcal{G}$ is Π- and Δ-free and $d \geq 3$ then there exists a unique color $i \in [d]$ such that the complement of the chromatic component i is disconnected.*

Obviously, by Propositions 5, Proposition 7 can be extended as follows.

Proposition 8. *A Gallai d-graph $\mathcal{G}$ has the CIS-d-property if and only if all $n+1$ d-graphs $\mathcal{G}_1, \ldots, \mathcal{G}_n$ and $\mathcal{G}_0$ from Proposition 7 have this property.*

□

In other words, all Gallai's CIS-d-graphs can be obtained from CIS-2-graphs by recursive substitutions and hence, a characterization or polynomial recognition algorithm of CIS-graphs would provide one for Gallai's CIS-d-graphs too.

From Propositions 7 and 8 we will derive the following claim.

Proposition 9. *Given a Gallai d-graph $\mathcal{G}$ such that at least $d-1$ of its chromatic components are CIS-graphs, then $\mathcal{G}$ is a CIS-d-graph.*

In particular, this means that the remaining chromatic component defines a CIS-graph too. Cameron, Edmonds, and Lovasz proved the similar claim for perfect graphs, [5]. Later, in [4], Cameron and Edmonds shows that in fact the claim holds for any class of graphs that is closed under (i) substitution, (ii) complementation, and (iii) taking induced subgraphs. Let us recall that CIS-graphs satisfy only (i) and (ii) but not (iii). Nevertheless, the claim holds for CIS-graphs. It also holds for Π- and Δ-free d-graphs, [13].

For example, let us consider a 3-graph $\mathcal{G}$ in Figure 14. Graphs G_1 and G_2 are isomorphic, each of them is a settled 2-comb with one isolated vertex. Hence, they are CIS-graphs. Yet, G_3 is not, since $S = \{2, 3, 5, 6\}$ and $C = \{1, 4\}$ are disjoint. However, $\mathcal{G}$ is not Gallai's 3-graph, because, e.g. $\{1, 2, 3\}$ is a Δ.

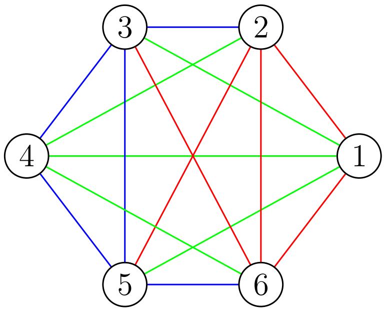


Figure 14: A non-Gallai 3-graph in which G_1 and G_2 are CIS-graphs, while G_3 is not.

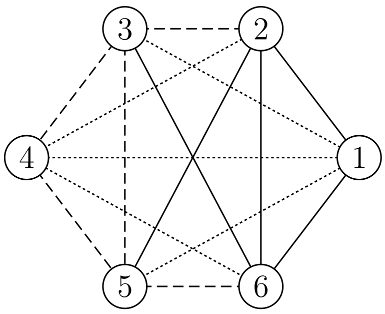


Figure 15: A non-Gallai 3-graph in which G_1 and G_2 are CIS-graphs, while G_3 is not (in black and white for printing).

Finally, let us remark that, by Propositions 4 and 9, if at least $d - 1$ of the chromatic components of a Gallai d-graph $\mathcal{G}$ have the CIS-property then all 2^d possible projections of $\mathcal{G}$ have this property too.

2 Proof of Theorem 1

In this section we prove Theorem 1 which claims that graphs satisfying condition COMB(3, 3) are CIS-graphs. First we describe the structure of our proof and a few main lemmas, then we give the complete proofs which are technical, long, and partially computer assisted.

2.1 Plan of the proof of Theorem 1

Let us assume by contradiction that there is a graph G such that

(i) it contains no induced 3-combs and 3-anti-combs,

(ii) each induced 2-comb is settled in G, and

(iii) there exist a maximal clique C and a maximal stable set S in G such that $S \cap C = \emptyset$.

First, we will prove that G must contain an induced subgraph G_{10}, shown in Figure 16.

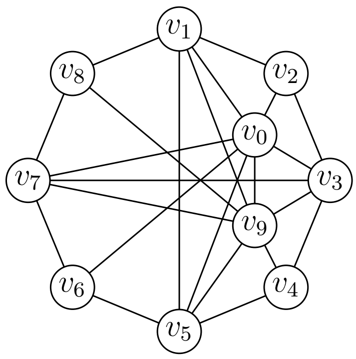


Figure 16: Graph G_{10}.

Lemma 2. *If G satisfies conditions (i), (ii), and (iii), then G must contain an induced G_{10}.*

Graph G_{10} contains no induced 3-combs and 3-anti-combs, yet it contains several unsettled induced 2-combs. To settle them we have to introduce 10 new vertices that, somewhat surprisingly, induce a graph isomorphic to G_{10} itself (since otherwise an induced 3-comb or 3-anti-comb would appear). Moreover, the obtained 20-vertex graph is the sum of two Petersen graphs, that is, the graph $2\mathcal{P}$ described in section 1.2, Figure 7.

Lemma 3. *If G contains an induced G_{10} and satisfies conditions (i) and (ii), then G must contain an induced $2\mathcal{P}$.*

Let us recall that $2\mathcal{P}$ contains 10 uncertain pairs of vertices each of which can be either an edge or non-edge. Hence in fact, $2\mathcal{P}$ represent $2^{10} = 1024$ graphs. We will show that all these 1024 graphs contain no induced 3-combs and 3-anti-combs and, moreover, each induced 2-comb in $2\mathcal{P}$ (that contains no uncertain pair) is settled. However, 36 induced 2-combs appear in $2\mathcal{P}$ whenever we fix any uncertain pair either as an edge or as a non-edge. In ie easy to see that none of these 2-combs are settled in $2\mathcal{P}$. We will show that they cannot be settled in G either, because if a vertex of G were settling one of them then an induced 3-comb or 3-anti-comb would exist in G. We can reformulate this result as follows.

Lemma 4. *If G satisfies conditions (i) and (ii), then it can not contain an induced $2\mathcal{P}$.*

Obviously, the above 3 lemmas prove Theorem 1 by contradiction. We will prove Lemmas 2, 3, and 4 below in Sections 2.2,2.3, and 2.4, respectively.

The last two proofs are computer assisted. We use two procedures, one for generating all induced 2-combs, 3-combs, and 3-anti-combs of a given graph G, and a second one for testing if all induced 2-combs are settled in G, and outputting all non-settled ones.

2.2 Proof of Lemma 2

Let us consider a pair of disjoint maximal clique C and maximal stable set S of G, as in condition (iii). Let $N_S(v)$ be the set of neighbors of v in S. Notice that

$$\bigcap_{v \in C} N_S(v) = \emptyset, \tag{2.2}$$

because C is maximal. Moreover,

$$N_S(v) \neq \emptyset \quad \text{for all} \quad v \in C, \tag{2.3}$$

because S is maximal.

We assume that G satisfies conditions (i), (ii), and (iii). The following series of claims will imply the lemma.

Claim 3.1. *Given a maximal clique C and a (not necessarily maximal) stable set S in G such that $C \cap S = \emptyset$, there exists vertices $u, v \in C$ such that $N_S(u) \cap N_S(v) = \emptyset$.*

Proof. Assume by contradiction that for all pairs of vertices $u, v \in C$, we have $N_S(u) \cap N_S(v) \neq \emptyset$. By this assumption, $|C| \geq 3$, otherwise C would not be maximal.

So let $I = \{v_1, v_2, \ldots, v_k\}$ be a minimal subset of C such that $\bigcap_{v \in I} N_S(v) = \emptyset$. Such a minimal subset of C exists according to (2.2). Furthermore, by our assumption $|I| \geq 3$.

Now, define $u_i \in \bigcap_{j \neq i}^{k} N_S(v_j)$ for $i = 1, ..., k$. Note that $u_i \neq u_j$, due to the minimality of I. Thus, any 3 vertices $v_1, v_2, v_3 \in I$ with the corresponding u_1, u_2, u_3 form an $\bar{S}_3$ (see Figure 17), contradicting condition (i). □

Note that for this claim we only need that G is S_3-free.

From Claim 3.1, it follows that there are some pairs of vertices $u, v \in C$ such that $N_S(u) \cap N_S(v) = \emptyset$. Hence, there exist $x \in N_S(u)$ and $y \in (N_S(v))$ such that x, u, v, y form an S_2 not settled by any vertex of S. The following claim states a useful property of any vertex $w \in V(G)$ settling such an S_2.

Claim 3.2. *We have $N_S(w) \subseteq N_S(u) \cup N_S(v)$.*

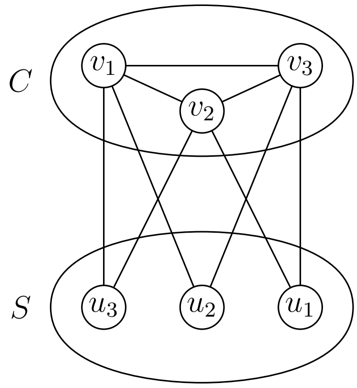


Figure 17: Illustration of the proof of Claim 3.1.

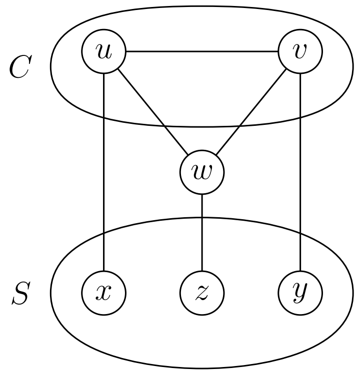


Figure 18: Illustration of the proof of Claim 3.2.

Proof. First notice that $x, y \notin N_S(w)$ because w is a settling vertex. Then, assume by contradiction that there is a vertex $z \in N_S(w) \setminus (N_S(u) \cup N_S(v))$. Then, vertices u, v, w, x, y, z form an S_3 (see Figure 18), contradicting condition (i). □

For the remainder of the proof we fix a maximal clique C, a maximal stable set S, and vertices $u, v \in C$ such that

(iv) $C \cap S = \emptyset$, $N_S(u) \cap N_S(v) = \emptyset$, and $N_S(u) \cup N_S(v)$ is minimal,

among all possible choices of such sets C, S and vertices $u, v \in C$ satisfying the conditions of (iv). Let us note that by (2.2) and (2.3), we have such a selection of C, S, u, and v for which $N_S(u) \neq \emptyset$, $N_S(v) \neq \emptyset$, and hence $u \neq v$.

Claim 3.3. *Let $x \in N_S(u)$, $y \in N_S(v)$, and w be a vertex of $V(G)$ that settles $S_2 = \{x, u, v, y\}$. Then, $N_S(w) \cap N_S(u) \neq \emptyset$ and $N_S(w) \cap N_S(v) \neq \emptyset$.*

Proof. From Claim 3.2, we know that $N_S(w) \subseteq N_S(u) \cup N_S(v)$. Assume by contradiction that e.g., $N_S(w) \cap N_S(u) = \emptyset$. This implies that $N_S(w) \subseteq N_S(v) \setminus \{y\}$ (since w is settling S_2).

Then, consider a maximal clique $C' \supseteq \{u, w\}$. Notice that $C' \cap S = \emptyset$ because $N_S(w) \cap N_S(u) = \emptyset$. But $N_S(u) \cup N_S(w) \subsetneq N_S(u) \cup N_S(v)$, since $y \notin N_S(u) \cup N_S(w)$, contradicting property (iv), that is, the minimality of $N_S(u) \cup N_S(v)$. □

We define next a minimal collection of settling vertices $\mathcal{W}$. Given a maximal clique C, a maximal stable set S, and vertices $u, v \in C$ satisfying property (iv), let us consider all possible 2-combs induced by $\{x, u, v, y\}$ in G, where $x \in N_S(u)$ and $y \in N_S(v)$. Let us call a *settling vertex* a vertex w of G that settles such a 2-comb. If w is a settling vertex, then we have by Claims 3.2 and 3.3 that $X(w) = N_S(w) \cap N_S(u)$ and $Y(w) = N_S(w) \cap N_S(v)$ are subsets, uniquely defined by w, satisfying the following properties:

$$X(w) \neq \emptyset \quad Y(w) \neq \emptyset \quad \text{and} \quad N_S(w) = X(w) \cup Y(w). \tag{2.4}$$

Note that we may have $X(w) = X(w')$ and $Y(w) = Y(w')$ for two distinct settling vertices. Note further that if $X(w) \subseteq X(w')$ and $Y(w) \subseteq Y(w')$ hold for two vertices w and w', then the set of S_2 subgraphs settled by w' are also settled by w.

Let us consider now all pairs of subsets (X, Y) such that $X = X(w)$ and $Y = Y(w)$ for some settling vertex w. Let us call such a pair (X, Y) *minimal*, if for there is no settling vertex w' such that $X(w') \subseteq X$, $Y(w') \subseteq Y$ and $X(w') \cup Y(w') \subsetneq X \cup Y$, and let $\mathcal{XY}$ denote the collection of all such minimal pairs. For each pair $(X, Y) \in \mathcal{XY}$ let us choose one settling vertex $w = w_{XY}$ for which $X = X(w)$ and $Y = Y(w)$, and denote by $\mathcal{W} = \{w_{XY} | (X, Y) \in \mathcal{XY}\}$ the collection of these vertices.

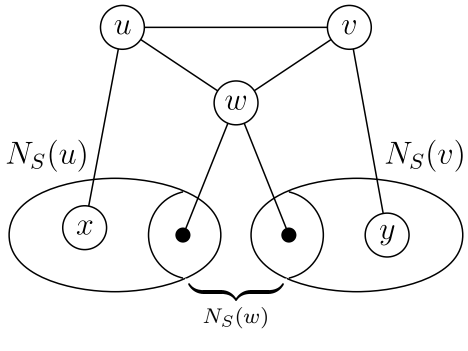


Figure 19

Claim 3.4. *There are at least two distinct vertices in $\mathcal{W}$.*

Proof. The statement follows from the definition of $\mathcal{W}$ and (2.4). Indeed, if $w_{XY} \in \mathcal{W}$, then by (2.4) there are vertices $x \in X$ and $y \in Y$, and hence the 2-comb S_2 induced by $\{x, u, v, y\}$ is not settled by w_{XY}. Let w be a vertex settling this 2-comb. By the minimality of (X, Y) the pair $(X(w), Y(w))$ is not comparable to (X, Y), and hence we must have a pair $(X', Y') \in \mathcal{XY}$ such that $X' \subseteq X$ and $Y' \subseteq Y$. Consequently, $w_{X'Y'} \in \mathcal{W}$ and $w_{XY} \neq w_{X'Y'}$. □

In the sequel we consider pairs of vertices from $\mathcal{W}$ and derive some containment relations for the corresponding sets. First we consider pairs which are edges of G.

Claim 3.5. *If* $(w_{XY}, w_{X'Y'}) \in E(G)$ *and* $X \cap X' \neq \emptyset$*, then* $Y \subseteq Y'$ *or* $Y' \subseteq Y$*.*

Proof. Assume by contradiction that there is a vertex $x \in X \cap X'$, but $Y \not\subseteq Y'$ and $Y' \not\subseteq Y$, that is, there are vertices $y_1 \in Y \setminus Y'$ and $y_2 \in Y' \setminus Y$. Then, an $\bar{S_3}$ is formed by $w_{XY}, w_{X'Y'}, v, x, y_1, y_2$ (see Figure 20), in contradiction to (i). □

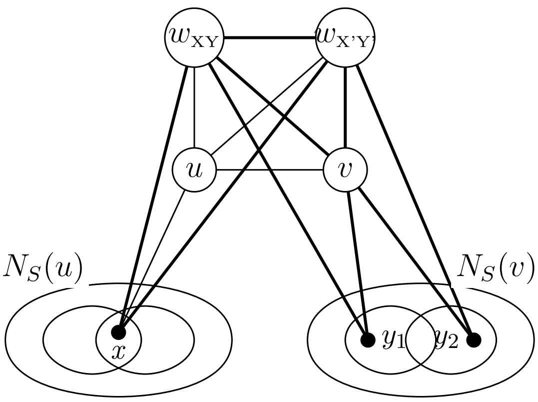


Figure 20: Illustration of the proof of Claim 3.5.

We next show a stronger version of the above claim, by proving proper containments.

Claim 3.6. *If* $(w_{XY}, w_{X'Y'}) \in E(G)$ *and* $X \cap X' \neq \emptyset$*, then either* $Y \subsetneq Y'$ *or* $Y' \subsetneq Y$*.*

Proof. Assume by contradiction that $X \cap X' \neq \emptyset$ and $Y = Y'$. By this assumption $Y \cap Y' \neq \emptyset$. Hence, we can apply Claim 3.5 (with the roles of X and Y exchanged), and conclude that $X \subseteq X'$ or $X' \subseteq X$.

Say e.g., that $X \subseteq X'$. Then, $X \cup Y \subseteq X' \cup Y'$, and consequently we would not have both $w_{X,Y}$ and $w_{X',Y'}$ in $\mathcal{W}$, by its definitions. □

Claim 3.7. *If* $(w_{XY}, w_{X'Y'}) \in E(G)$*, then exactly one of the following holds:*

(a) $X \cap X' = Y \cap Y' = \emptyset$,

(b) $(X \subsetneq X'$ *and* $Y' \subsetneq Y)$,

(c) $(X' \subsetneq X$ *and* $Y \subsetneq Y')$.

Proof. This follows from Claim 3.6 by applying it twice: once directly and once exchanging the roles of X and Y. Since X, Y, X' and Y' are nonempty sets by (2.4), cases (a), (b) and (c) are pairwise exclusive. □

Next we consider pairs of settling vertices that are not edges of G.

Claim 3.8. *If* $(w_{XY}, w_{X'Y'}) \notin E(G)$*, then either* $X \subseteq X'$ *or* $Y \subseteq Y'$.

Proof. If not, then there are vertices $x \in X \backslash X'$ and $y \in Y \backslash Y'$ such that $\{w_{XY}, u, v, x, y, w_{X'Y'}\}$ form a 3-anti-comb $\bar{S}_3$ (see Figure 21), in contradiction to condition (i).

Note that we cannot have both containments in the claim, because of the minimality of pairs in $\mathcal{XY}$. □

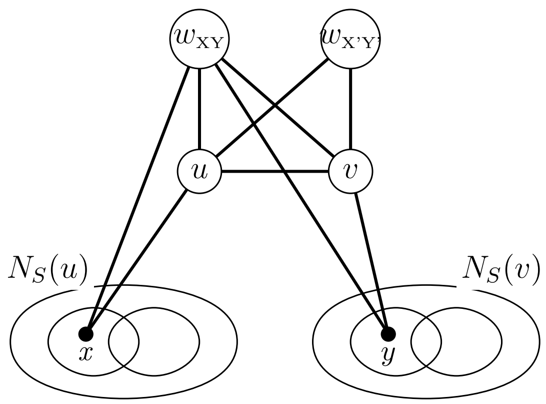


Figure 21: Illustration of the 3-anti-comb $\bar{S}_3$ induced by $\{w_{XY}, u, v, x, y, w_{X'Y'}\}$.

Claim 3.9. *If* $(w_{XY}, w_{X'Y'}) \notin E(G)$*, then exactly one of the following must hold:*

(a) $X \subsetneq X'$ *and* $Y' \subsetneq Y$,

(b) $X' \subsetneq X$ *and* $Y \subsetneq Y'$,

(c) $X = X'$,

(d) $Y = Y'$.

Proof. Since the roles of (X, Y) and (X', Y') are symmetric, it follows directly by Claim 3.8 that one of (a), (b), (c), or (d) holds. To see that exactly one of them holds, it is enough to note that (c) and (d) together would contradict the minimality of the pairs $(X, Y) \in \mathcal{XY}$. □

We are going to show next that if (c) or (d) holds in the previous claim for some vertices $w_{XY}, w_{X'Y'} \in \mathcal{W}$, then G contains an induced G_10, as claimed in Lemma 2. For this end, let us first observe that if e.g., (d) holds, then we cannot have $X \subseteq X'$ or $X' \subseteq X$, by the minimality and uniqueness of pairs in $\mathcal{XY}$. Consequently, we can choose vertices $x \in X \setminus X'$, and $x' \in X' \setminus X$. Let us also choose an arbitrary vertex $y \in Y = Y'$ (which exists by (2.4)), and consider first the 2-comb S_2 induced by $\{x, u, v, y\}$. This 2-comb is settled by neither w_{XY} nor $w_{X'Y'}$, and therefore there must be a vertex $w_{AB} \in \mathcal{W}$ settling it, since all 2-combs, containing (u, v) as their middle edge, are settled by some vertices in $\mathcal{W}$.

Claim 3.10. *If $Y = Y'$, then $(w_{AB}, w_{XY}) \in E(G)$.*

Proof. Since $x \notin A$ and $y \notin B$ we have

$$X \not\subseteq A \quad \text{and} \quad Y \not\subseteq B \tag{2.5}$$

implied. Assume indirectly that $(w_{AB}, w_{XY}) \notin E(G)$, then the previous observation implies that in Claim 3.9 applied to w_{XY} and w_{AB} none of (a), (b), (c) or (d) could hold. This contradiction proves the claim. □

Claim 3.11. *If $Y = Y'$, then $A \cap X = B \cap Y = \emptyset$, $A \cup X = N_S(u)$ and $B \cup Y = N_S(v)$.*

Proof. Due to (2.5) only (a) of Claim 3.7 is possible, that is $A \cap X = B \cap Y = \emptyset$ is implied. Therefore the neighborhoods of w_{AB} and w_{XY} within S are disjoint, and since they are subsets of the neighborhoods of u and v, they cannot be proper subsets by property (iv), implying the statement. □

Claim 3.12. *If $Y = Y'$, then $(w_{AB}, w_{X'Y'}) \notin E(G)$.*

Proof. Since $y \in Y' \setminus B$ and $x \in X \setminus A$ (since w_{AB} is settling $\{x, u, v, y\}$), cases (b) and (c) of Claim 3.7 cannot hold for the pair w_{AB} and $w_{X'Y'}$. Thus, if $(w_{AB}, w_{X'Y'}) \in E(G)$ then $A \cap X' = B \cap Y' = \emptyset$ would follow by Claim 3.7. Therefore, the neighborhoods of w_{AB} and $w_{X'Y'}$ in S are disjoint, and their union is a proper subset of $N_S(u) \cup N_S(v)$, in contradiction with property (iv). This contradiction proves the claim. □

Claim 3.13. *If $Y = Y'$, then $A = X' = N_S(u) \setminus X$ and $Y = Y' = N_S(v) \setminus B$.*

Proof. Claim 3.11 and Claim 3.9 applied to w_{AB} and $w_{X'Y'}$ implies that only (c) of Claim 3.9 can hold. Thus, the statement implied by Claim 3.11 and (c) of Claim 3.9. □

Let us still assume $Y = Y'$ and consider next the 2-comb induced by $\{x', u, v, y\}$ (where $x' \in X' \setminus X$). None of the vertices w_{XY}, $w_{X'Y'}$ and w_{AB} settle this 2-comb, hence, there is a vertex $w_{A'B'} \in \mathcal{W}$ that settles it. By exchanging the roles of w_{XY} and $w_{X'Y'}$ in Claims 3.10 - 3.13, we can conclude that

$$(w_{A'B'}, w_{XY}) \notin E(G), \quad (w_{A'B'}, w_{X'Y'}) \in E(G), \quad A' = X' \quad \text{and} \quad B = B'. \tag{2.6}$$

Claim 3.14. *If $Y = Y'$ or $X = X'$, then G contains an induced G_{10}.*

Proof. Note that the roles of conditions (c) and (d) in Claim 3.9 are perfectly symmetric, thus we could arrive to the same conclusions from both assumptions. Starting with $Y = Y'$ we arrived to the equalities of Claim 3.13 and (2.6). Choosing one vertex from each of the sets X, Y, A, and B, these four vertices together with u, v, w_{XY}, $w_{X'Y'}$, w_{AB}, and $w_{A'B'}$ form an induced G_{10} by the above claims and definitions (see Figure 22). □

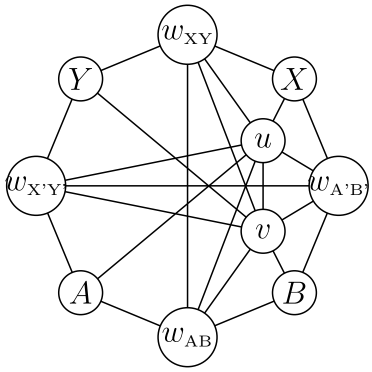


Figure 22: Illustration of the induced G_{10} that appears by adding the settling vertices $w_{XY}, w_{X'Y'}, w_{AB}, w_{A'B'}$.

For the rest of the proof, we assume that (a) or (b) of Claim 3.9 holds for every non-edge $(w_{XY}, w_{X'Y'}) \notin E(G)$. We are going to derive a contradiction from this assumption, completing the proof of Lemma 2.

First, we show that under the above assumption, case (a) of Claim 3.7 never holds.

Claim 3.15. *If* $(w_{XY}, w_{X'Y'}) \in E(G)$*, then either* $X \cap X' \neq \emptyset$ *or* $Y \cap Y' \neq \emptyset$*.*

Proof. Assume by contradiction that (a) of Claim 3.9 holds, that is that $X \cap X' = Y \cap Y' = \emptyset$. Then, by the minimality of $N_S(u) \cup N_S(v)$ as stated in property (iv), and by Claim 3.2, we know that $N_S(u) = X \cup X'$ and $N_S(v) = Y \cup Y'$.

Let us consider vertices $x \in X$ and $y \in Y'$ such that the set $\{x, u, v, y\}$ forms a 2-comb. This 2-comb is settled neither by w_{XY} nor by $w_{X'Y'}$. Since every 2-comb with (u, v) as a middle edge is settled by a vertex of $\mathcal{W}$, this 2-comb is also settled by one, say by a vertex $w_{AB} \in \mathcal{W}$. Let us now check the connections of this vertex to to w_{XY} and $w_{X'Y'}$. We consider two cases:

Case 1. If $(w_{AB}, w_{XY}) \notin E(G)$, then by Claim 3.9 we must have $A \subset X$ and $Y \subset B$, because $x \notin A$, and because we assumed that only cases (a) or (b) are possible in Claim 3.9.

If $(w_{AB}, w_{X'Y'}) \notin E(G)$, then by similar reasoning based on by Claim 3.9 and the fact that $y \notin B$ we can conclude that $X' \subset A$ and $B \subset Y'$. This however leads to a contradiction, since $A \subseteq X$ and $X \cap X' = \emptyset$.

Hence, we must have $(w_{AB}, w_{X'Y'}) \in E(G)$ in this case. Then by Claim 3.7 either $X' \cap A = Y' \cap B = \emptyset$ or A, X' and B, Y' are inversely nested. However, the latter is not possible, since $A \subset X$ and $X \cap X' = \emptyset$. In this case the neighborhoods of w_{AB} and $w_{X'Y'}$ are disjoint in S, and their union is a proper subset of $N_S(u) \cup N_S(v)$ (since $x \notin A$), in contradiction with property (iv).

Case 2. If $(w_{AB}, w_{XY}) \in E(G)$, then (b) of Claim 3.7 is not possible, since $x \in X \setminus A$. If (a) holds, that is if $X \cap A = Y \cap B = \emptyset$, then the neighborhoods of w_{AB} and w_{XY} are disjoint in S, and their union is a proper subset of $N_S(u) \cup N_S(v)$ (since $y \in Y' \setminus B$), contradicting to property (iv). Consequently, case (c) holds, that is $A \subset X$ and $Y \subset B$, and consequently we can proceed as in Case 1.

In both cases we arrived to a contradiction, completing the proof of the claim. □

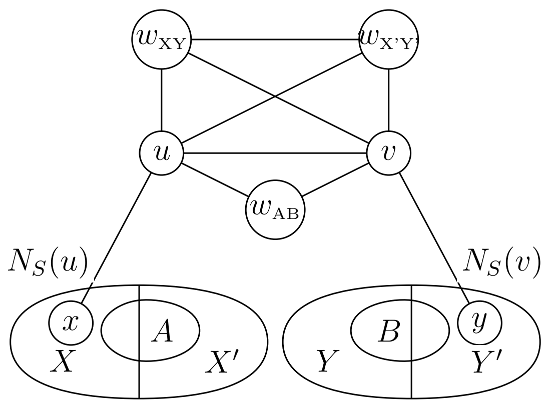


Figure 23

The above claim implies that if $(w_{XY}, w_{X'Y'}) \in E(G)$, then the sets X, X' and Y, Y' are inversely nested (cases (b) or (c) in Claim 3.7). Since we also assumed that only cases (a) or (b) are possible in Claim 3.9, we can conclude that for all pairs of settling vertices $w_{XY}, w_{X'Y'} \in \mathcal{W}$ we have

$$\text{either } \ X \subset X' \text{ and } Y' \subset X \quad \text{ or } \quad X' \subset X \text{ and } Y \subset Y'. \tag{2.7}$$

Now we are ready to complete the proof of the lemma.

Let us consider an arbitrary vertex $w_{XY} \in \mathcal{W}$. Since w_{XY} is settling a 2-comb with (u, v) as its middle edge, we must have $Y \neq N_S(v)$, and consequently we can choose a vertex $y \in N_S(v) \setminus Y$. Furthermore, we have $X \neq \emptyset$ by (2.4), thus we can also choose a vertex $x \in X$.

Then, the 2-comb S_2 induced by $\{x, u, v, y\}$ is not settled by w_{XY}, and therefore there is a vertex $w_{X'Y'} \in \mathcal{W}$ settling this 2-comb. Then, by (2.7) we must have $X' \subseteq X \setminus \{x\}$ and $Y \subset Y'$, since $x \notin X'$.

Then, $X' \neq \emptyset$ by (2.4), so we can choose a vertex $x' \in X' \subsetneq X$. The 2-comb induced by $\{x', u, v, y\}$ is not settled by either w_{XY} or $w_{X'Y'}$, and therefore there is a vertex $w_{X''Y''} \in \mathcal{W}$ settling this 2-comb.

Clearly, we can repeat the same arguments, and choose a vertex $x'' \in X'' \subsetneq X' \subsetneq X$, etc., resulting in an infinite chain $X \supsetneq X' \supsetneq X'' \supsetneq \cdots$ of strictly nested nonempty subsets, contradicting the finiteness of G. This concludes the proof of the lemma. $\square$

2.3 Proof of Lemma 3

In this section we present the proof of Lemma 3, claiming that if G contains G_{10} as an induced subgraph and satisfies conditions (i) and (ii) of Section 2.1, then it must have an induced $2\mathcal{P}$ configuration (see Figures 16 and 7).

The proof is a case analysis that was assisted by a computer program. We assume by contradiction that there is a graph that has an induced G_{10}, has all 2-combs settled and does not contain 3-combs and 3-anti-combs. The graph G_{10} itself contains neither 3-combs nor 3-anti-combs, but it has several 2-combs that are not settled in it. For instance, such 2-combs are induced by $\{v_2, v_1, v_5, v_4\}$, $\{v_6, v_7, v_3, v_4\}$, $\{v_1, v_2, v_3, v_7\}$, etc. Therefore, some other vertices of G must settle these 2-combs.

We show that in order to settle all 2-combs of G_{10}, the graph G must contain a disjoint copy of G_{10} such that the 20 vertices of these two G_{10} subgraphs form an induced $2\mathcal{P}$ configuration. Since we do not know G, we try to extend G_{10}, and we show that this can be done essentially in a unique way.

We use a computer program to find all unsettled 2-combs of G_{10}. For each, one by one, we introduce a new vertex to settle it. After adding a settling vertex $v' \notin V(G_{10})$, we consider the pairs (v', v_j) for all $v_j \in V(G_{10})$. Some of these pairs are forced to be edges or non-edges, since G contains no induced 3-combs and 3-anti-combs. Some other pairs, however, may remain *uncertain*, that is those pairs may be either edges or non-edges of G. Surprisingly, all but one of the pairs are forced. We can discover the forced edge assignments by excluding all other possible assignments. This can be accomplished by exhibiting an induced 3-comb or 3-anti-comb. This task is also assisted by a computer program.

Another property which simplifies our case analysis is the symmetry of G_{10}. In particular, we reduce significantly the number of cases in our proof by means of the following three automorphisms:

A_1: $(3)(7)(1,5)(2,4)(6,8)(0,9)$

A_2: $(1)(5)(2,8)(3,7)(4,6)(0,9)$

A_3: $(7,5,3,1)(8,6,4,2)(0,9)$

They are given in the *cycle* notation, that is $(i_1, i_2, \ldots, i_n)$ means the cyclic mapping $i_1 \mapsto i_2$, $i_2 \mapsto i_3$, …, $i_n \mapsto i_1$. Figure 24 shows the graphs after the application of these automorphisms.

From now on we will choose some of the unsettled 2-combs to be settled, and try to fix as many edges and non-edges as possible. Even though the order that we pick the 2-combs may seem arbitrary, we follow an order that reduces the number of cases to be considered.

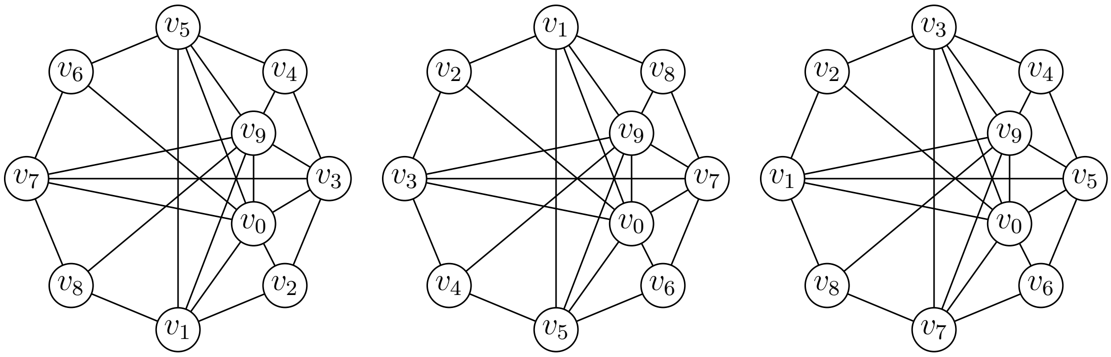


Figure 24: Graphs $A_1(G_{10})$, $A_2(G_{10})$, and $A_3(G_{10})$.

Let us choose first the 2-comb induced by $\{v_2, v_3, v_7, v_8\}$, and denote by v_1' the vertex that settles it. The pairs (v_1', v_3) and (v_1', v_7) are forced to be edges, while (v_1', v_2) and (v_1', v_8) are forced to be non-edges, by the definition of settling. There are six more pairs, connecting v_1' with v_0, v_1, v_4, v_5, v_6 and v_9, that remain uncertain.

Let us note first that (v_1', v_5) has to be a non-edge, since otherwise the vertices $\{v_3, v_7, v_1', v_2, v_8, v_5\}$ form a 3-comb. Unlike (v_1', v_5), the pairs (v_1', v_0), (v_1', v_4), (v_1', v_6), (v_1', v_9) cannot be fixed if treated individually. But analyzing them together, we conclude that (v_1', v_4) and (v_1', v_6) are edges, while (v_1', v_0) and (v_1', v_9) are non-edges. Table 1 shows that in any other case there is an induced 3-comb or 3-anti-comb.

Only one pair (v_1', v_1) remains uncertain, since no induced S_3 nor $\bar{S}_3$ appears whether this pair is an edge or not.

(v_1', v_4)	(v_1', v_6)	(v_1', v_0)	(v_1', v_9)	S_3 or $\bar{S}_3$
0	0	0	0	$S_3 : \{v_3, v_0, v_9, v_1', v_6, v_8\}$
0	0	0	1	$\bar{S}_3 : \{v_2, v_5, v_1', v_3, v_0, v_9\}$
0	0	1	0	$\bar{S}_3 : \{v_4, v_8, v_1', v_3, v_7, v_9\}$
0	0	1	1	$\bar{S}_3 : \{v_4, v_6, v_1', v_5, v_0, v_9\}$
0	1	0	0	$S_3 : \{v_5, v_6, v_0, v_2, v_4, v_1'\}$
0	1	0	1	$S_3 : \{v_3, v_9, v_1', v_2, v_6, v_8\}$
0	1	1	0	$\bar{S}_3 : \{v_4, v_8, v_1', v_3, v_7, v_9\}$
0	1	1	1	$S_3 : \{v_3, v_9, v_1', v_2, v_6, v_8\}$
1	0	0	0	$S_3 : \{v_3, v_0, v_9, v_6, v_8, v_1'\}$
1	0	0	1	$\bar{S}_3 : \{v_2, v_5, v_1', v_3, v_0, v_9\}$
1	0	1	0	$S_3 : \{v_4, v_5, v_9, v_6, v_8, v_1'\}$
1	0	1	1	$S_3 : \{v_7, v_0, v_1', v_2, v_4, v_8\}$
1	1	0	0	**none**
1	1	0	1	$S_3 : \{v_3, v_9, v_1', v_2, v_6, v_8\}$
1	1	1	0	$S_3 : \{v_7, v_0, v_1', v_2, v_4, v_8\}$
1	1	1	1	$S_3 : \{v_3, v_9, v_1', v_2, v_6, v_8\}$

Table 1: Case analysis for the pairs (v_1', v_0), (v_1', v_4), (v_1', v_6), (v_1', v_9).

Table 2 shows the connections between v_1' and the vertices of G_{10}.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v_1'	$*$	0	1	1	0	1	1	0	0	0

Table 2: Connections between v_1' and G_{10}. An entry 1 for v_i means that there is an edge between v_1' and v_i, while 0 means that there is no edge between them. Finally, $*$ means an uncertain pair.

Next, we use automorphisms to simplify case analysis for the three 2-combs induced by $\{v_4, v_3, v_7, v_6\}$, $\{v_6, v_5, v_1, v_8\}$, and $\{v_2, v_1, v_5, v_4\}$ respectively, and not settled by v_1'.

Let us denote by v_5' the vertex that settles $\{v_4, v_3, v_7, v_6\}$. By applying the automorphism A_1 to G_{10}, the 2-comb $\{v_2, v_3, v_7, v_8\}$ settled by v_1' becomes $\{v_4, v_3, v_7, v_6\}$. Consequently, v_5' should have the same connections as v_1' has after applying A_1. Table 3 shows the connections between v_5' and G_{10}.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v_5'	0	1	1	0	$*$	0	1	1	0	0

Table 3: Connections between v_5' and G_{10}.

Analogously, let us denote by v_3' the vertex that settles $\{v_2, v_1, v_5, v_4\}$. By applying A_3 to G_{10}, $\{v_2, v_3, v_7, v_8\}$ becomes $\{v_2, v_1, v_5, v_4\}$. Therefore, v_3' should have the same connections as v_1' after transformation A_3. Table 4 shows the connections between v_3' and G_{10}.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v_3'	1	0	$*$	0	1	1	0	1	0	0

Table 4: Connections between v_3' and G_{10}.

Next, let us denote by v_7' the vertex that settles $\{v_8, v_1, v_5, v_6\}$. By applying A_3 then A_2 to G_{10}, $\{v_2, v_3, v_7, v_8\}$ becomes $\{v_8, v_1, v_5, v_6\}$. Thus, v_3' should have the same connections as v_1' after transformations A_3 then A_2 (or the same connections as v_3' after A_2). Table 5 shows the connections between v_7' and G_{10}.

Let us next consider four 2-combs induced by $\{v_5, v_1, v_2, v_3\}$, $\{v_1, v_5, v_4, v_3\}$, $\{v_7, v_3, v_4, v_5\}$, and $\{v_1, v_2, v_3, v_7\}$. They are not settled by any of the vertices of G_{10}, nor by v_1', v_3', v_5', v_7'.

Let v_2' denote the vertex settling $\{v_3, v_4, v_5, v_1\}$. By definition of settling, the pairs (v_2', v_4) and (v_2', v_5) are edges, while (v_2', v_1) and (v_2', v_3) are non-edges. The pair (v_2', v_9) must be an edge, since otherwise $\{v_1, v_3, v_2', v_4, v_5, v_9\}$ forms a 3-anti-comb. Table 6 shows the case analysis for the pairs (v_2', v_6), (v_2', v_7), (v_2', v_8), and (v_2', v_0). The only possible configuration is that (v_2', v_6), (v_2', v_7), (v_2', v_8) are edges, and (v_2', v_0) is not. The pair (v_2', v_2) remains uncertain. Table 7 shows the connections between v_2' and the vertices of G_{10}.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v'_7	1	1	0	1	1	0	$*$	0	0	0

Table 5: Connections between v'_7 and G_{10}.

(v'_2, v_6)	(v'_2, v_7)	(v'_2, v_8)	(v'_2, v_0)	S_3 or $\bar{S}_3$
0	0	0	0	$S_3 : \{v_1, v_5, v_0, v_3, v_8, v'_2\}$
0	0	0	1	$\bar{S}_3 : \{v_6, v_8, v'_2, v_7, v_0, v_9\}$
0	0	1	0	$S_3 : \{v_4, v_5, v'_2, v_3, v_6, v_8\}$
0	0	1	1	$S_3 : \{v_4, v_5, v'_2, v_3, v_6, v_8\}$
0	1	0	0	$S_3 : \{v_1, v_5, v_0, v_3, v_8, v'_2\}$
0	1	0	1	$\bar{S}_3 : \{v_1, v_4, v_7, v_5, v_0, v'_2\}$
0	1	1	0	$S_3 : \{v_4, v_5, v'_2, v_3, v_6, v_8\}$
0	1	1	1	$S_3 : \{v_4, v_5, v'_2, v_3, v_6, v_8\}$
1	0	0	0	$S_3 : \{v_1, v_5, v_0, v_3, v_8, v'_2\}$
1	0	0	1	$\bar{S}_3 : \{v_1, v_4, v_6, v_0, v_9, v'_2\}$
1	0	1	0	$\bar{S}_3 : \{v_1, v_7, v'_2, v_5, v_6, v_0\}$
1	0	1	1	$\bar{S}_3 : \{v_1, v_4, v_6, v_0, v_9, v'_2\}$
1	1	0	0	$S_3 : \{v_1, v_5, v_0, v_3, v_8, v'_2\}$
1	1	0	1	$\bar{S}_3 : \{v_1, v_4, v_6, v_0, v_9, v'_2\}$
1	1	1	0	**none**
1	1	1	1	$\bar{S}_3 : \{v_1, v_4, v_6, v_0, v_9, v'_2\}$

Table 6: Case analysis for the pairs (v'_2, v_6), (v'_2, v_7), (v'_2, v_8), (v'_2, v_0).

Let v'_4 denote the vertex settling $\{v_5, v_1, v_2, v_3\}$. By applying A_1 to G_{10}, the subgraph $\{v_1, v_5, v_4, v_3\}$ becomes $\{v_5, v_1, v_2, v_3\}$. Therefore, vertex v'_4 must have the same connections as v'_2 after transformation A_1. Table 8 shows the connections between v'_4 and G_{10}.

Next, let v'_6 denote the vertex settling $\{v_7, v_3, v_4, v_5\}$. By applying transformations, first A_1 and then A_3, to G_{10}, the subgraph $\{v_1, v_5, v_4, v_3\}$ becomes $\{v_7, v_3, v_4, v_5\}$. Thus, v'_6 must have the same connections as v'_2 after the transformation $A_3 \circ A_1$. Table 9 shows the connections between v'_6 and G_{10}.

Let us next denote by v'_8 the vertex that settles $\{v_1, v_2, v_3, v_7\}$. By applying A_3^{-1}, to G_{10}, the subgraph $\{v_1, v_5, v_4, v_3\}$ becomes $\{v_1, v_2, v_3, v_7\}$. Therefore, v'_8 should have the same connections as v'_2 after A_3^{-1}. Table 10 shows the connections between v'_8 and G_{10}.

At this point, all S_2 subgraphs of G_{10} are settled by some of the vertices v'_1, v'_2, $\ldots$, v'_8. Yet, nothing was said about the connections between those vertices. Nevertheless, all 3-combs and 3-anti-combs that appeared to indicate contradictions were independent from those connections; in other words, each of those subgraphs contains only one vertex v'_i and the remaining five vertices are in G_{10}.

Interestingly, the connections between these eight vertices are uniquely implied. Table 11 shows the only possible assignments of edges and non-edges between the vertices v'_i and v'_j, for

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v_2'	0	$*$	0	1	1	1	1	1	1	0

Table 7: Connections between v_2' and G_{10}.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v_4'	1	1	0	$*$	0	1	1	1	0	1

Table 8: Connections between v_4' and G_{10}.

$i, j = 1, \ldots, 8$, $i \neq j$. Each entry of the table contains the assignment, and the corresponding 3-comb or 3-anti-comb that would appear if the entry was reversed.

Let us notice that the pairs (v_i, v_i') still remain uncertain. This means that all 2^8 possible graphs have no induced 3-combs and 3-anti-combs. Yet, they contain some unsettled induced 2-combs.

Next, we introduce the automorphism A_4 of the current configuration, induced by the 18 vertices $V(G_{10}) \cup \{v_1', \ldots, v_8'\}$.

A_4: $(1, 3, 5, 7)(2, 4, 6, 8)(0, 9)(1', 3', 5', 7')(2', 4', 6', 8')$.

Let us further consider the unsettled 2-comb induced by $\{v_2, v_1', v_5', v_6\}$, and denote by v_0' the vertex that settles it. By definition, (v_0', v_1') and (v_0', v_5') are edges, while (v_0', v_2) and (v_0', v_6) are non-edges. The pair (v_0', v_9) cannot be an edge, since otherwise $\{v_1', v_5', v_0', v_2, v_6, v_9\}$ forms a 3-comb. Table 16 shows that (v_0', v_4) and (v_0', v_8) must be edges, while (v_0', v_1), (v_0', v_3), (v_0', v_5) and (v_0', v_7), must be non-edges. Furthermore, the pairs (v_0', v_2'), (v_0', v_3'), (v_0', v_6') and (v_0', v_7') must be edges, since otherwise one of the following 3-combs would appear: $\{v_4, v_5, v_2', v_1, v_7, v_0'\}$, $\{v_1, v_8, v_3', v_2, v_6, v_0'\}$, $\{v_1, v_8, v_6', v_3, v_5, v_0'\}$, or $\{v_1, v_8, v_9, v_3, v_7', v_0'\}$. The pairs (v_0', v_4') and (v_0', v_8') cannot be edges, since otherwise the 3-combs induced by $\{v_1, v_2, v_4', v_3, v_5, v_0'\}$ and $\{v_2, v_3, v_8', v_1, v_7, v_0'\}$ would appear. Finally, the pair (v_0', v_0) remains uncertain. Table 12 shows the connections between v_0' and G_{10}.

Next, let us consider the 2-comb induced by $\{v_3', v_7', v_4, v_8\}$ and denote by v_9' the vertex settling it. Notice that this 2-comb can be obtained from $\{v_1', v_5', v_2, v_6\}$ by applying transformation A_4. Therefore v_9' must have the same connections as v_0' after applying A_4. Table 13 shows the connections between v_9' and G_{10}.

We summarize the connections between vertices $v_1', \ldots, v_9', v_0'$ in Table 14, and between $v_1, \ldots, v_9, v_0$ and $v_1', \ldots, v_9', v_0$ in Table 15.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v_6'	1	1	1	1	0	$*$	0	1	1	0

Table 9: Connections between v_6' and G_{10}.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v'_8	0	1	1	1	1	1	0	$*$	0	1

Table 10: Connections between v'_8 and G_{10}.

Edge	S_3 or $\bar{S}_3$	Edge	S_3 or $\bar{S}_3$
$(v'_1, v'_2) = 1$	$S_3 : \{v_4, v_5, v'_2, v_8, v_0, v'_1\}$	$(v'_1, v'_3) = 0$	$S_3 : \{v_6, v'_1, v'_3, v_4, v_8, v_0\}$
$(v'_1, v'_4) = 0$	$S_3 : \{v_6, v'_1, v'_4, v_3, v_5, v_8\}$	$(v'_1, v'_5) = 1$	$\bar{S}_3 : \{v_2, v_8, v'_1, v_3, v_7, v'_5\}$
$(v'_1, v'_6) = 0$	$S_3 : \{v_4, v'_1, v'_6, v_2, v_5, v_7\}$	$(v'_1, v'_7) = 0$	$S_3 : \{v_4, v'_1, v'_7, v_2, v_6, v_9\}$
$(v'_1, v'_8) = 1$	$S_3 : \{v_5, v_6, v'_8, v_2, v_9, v'_1\}$	$(v'_2, v'_3) = 1$	$S_3 : \{v_7, v_8, v'_2, v_4, v_0, v'_3\}$
$(v'_2, v'_4) = 0$	$\bar{S}_3 : \{v_1, v_3, v'_2, v_7, v_0, v'_4\}$	$(v'_2, v'_5) = 0$	$S_3 : \{v_8, v'_5, v'_2, v_1, v_3, v_6\}$
$(v'_2, v'_6) = 0$	$\bar{S}_3 : \{v_1, v_4, v_7, v_8, v'_2, v'_6\}$	$(v'_2, v'_7) = 0$	$S_3 : \{v_4, v'_7, v'_2, v_1, v_3, v_6\}$
$(v'_2, v'_8) = 0$	$\bar{S}_3 : \{v_1, v_3, v'_2, v_5, v_0, v'_8\}$	$(v'_3, v'_4) = 1$	$S_3 : \{v_6, v_7, v'_4, v_2, v_9, v'_3\}$
$(v'_3, v'_5) = 0$	$S_3 : \{v_8, v'_5, v'_3, v_2, v_6, v_9\}$	$(v'_3, v'_6) = 0$	$S_3 : \{v_8, v'_3, v'_6, v_2, v_5, v_7\}$
$(v'_3, v'_7) = 1$	$\bar{S}_3 : \{v_2, v_4, v'_3, v_1, v_5, v'_7\}$	$(v'_3, v'_8) = 0$	$S_3 : \{v_6, v'_3, v'_8, v_1, v_4, v_7\}$
$(v'_4, v'_5) = 1$	$S_3 : \{v_1, v_2, v'_4, v_6, v_9, v'_5\}$	$(v'_4, v'_6) = 0$	$\bar{S}_3 : \{v_3, v_5, v'_4, v_1, v_9, v'_6\}$
$(v'_4, v'_7) = 0$	$S_3 : \{v_2, v'_7, v'_4, v_3, v_5, v_8\}$	$(v'_4, v'_8) = 0$	$\bar{S}_3 : \{v_1, v_3, v_6, v_2, v'_4, v'_8\}$
$(v'_5, v'_6) = 1$	$S_3 : \{v_1, v_8, v'_6, v_4, v_0, v'_5\}$	$(v'_5, v'_7) = 0$	$S_3 : \{v_2, v'_5, v'_7, v_4, v_8, v_0\}$
$(v'_5, v'_8) = 0$	$S_3 : \{v_2, v'_5, v'_8, v_1, v_4, v_7\}$	$(v'_6, v'_7) = 1$	$S_3 : \{v_3, v_4, v'_6, v_8, v_0, v'_7\}$
$(v'_6, v'_8) = 0$	$\bar{S}_3 : \{v_1, v_7, v'_8, v_3, v_9, v'_6\}$	$(v'_7, v'_8) = 1$	$S_3 : \{v_2, v_3, v'_8, v_6, v_9, v'_7\}$

Table 11: Case analysis for the connections between $v'_1, \ldots, v'_8$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v'_0	0	0	0	1	0	0	0	1	0	$*$

Table 12: Connections between v'_0 and G_{10}.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v'_9	0	1	0	0	0	1	0	0	0	$*$

Table 13: Connections between v'_9 and G_{10}.

	v'_1	v'_2	v'_3	v'_4	v'_5	v'_6	v'_7	v'_8	v'_9	v'_0
v'_1	–	1	0	0	1	0	0	1	1	1
v'_2	1	–	1	0	0	0	0	0	0	1
v'_3	0	1	–	1	0	0	1	0	1	1
v'_4	0	0	1	–	1	0	0	0	1	0
v'_5	1	0	0	1	–	1	0	0	1	1
v'_6	0	0	0	0	1	–	1	0	0	1
v'_7	0	0	1	0	0	1	–	1	1	1
v'_8	1	0	0	0	0	0	1	–	1	0
v'_9	1	0	1	1	1	0	1	1	–	1
v'_0	1	1	1	0	1	1	1	0	1	–

Table 14: Connections between vertices $v'_1, \ldots, v'_9, v'_0$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_0
v'_1	$*$	0	1	1	0	1	1	0	0	0
v'_2	0	$*$	0	1	1	1	1	1	1	0
v'_3	1	0	$*$	0	1	1	0	1	0	0
v'_4	1	1	0	$*$	0	1	1	1	0	1
v'_5	0	1	1	0	$*$	0	1	1	0	0
v'_6	1	1	1	1	0	$*$	0	1	1	0
v'_7	1	1	0	1	1	0	$*$	0	0	0
v'_8	0	1	1	1	1	1	0	$*$	0	1
v'_9	0	1	0	0	0	1	0	0	$*$	0
v'_0	0	0	0	1	0	0	0	1	0	$*$

Table 15: Connections between vertices $v_1, \ldots, v_9, v_0$ and $v'_1, \ldots, v'_9, v'_0$.

Table 16: Case analysis for the pairs (v_0', v_1), (v_0', v_3), (v_0', v_4), (v_0', v_5), (v_0', v_7) and (v_0', v_8).

(v_0', v_1)	(v_0', v_3)	(v_0', v_4)	(v_0', v_5)	(v_0', v_7)	(v_0', v_8)	S_3 or $\bar{S}_3$
0	0	0	0	0	0	$S_3 : \{v_2, v_3, v_5', v_1, v_4, v_0'\}$
0	0	0	0	0	1	$S_3 : \{v_1, v_8, v_9, v_2, v_4, v_0'\}$
0	0	0	0	1	0	$S_3 : \{v_2, v_3, v_5', v_1, v_4, v_0'\}$
0	0	0	0	1	1	$S_3 : \{v_1, v_8, v_9, v_2, v_4, v_0'\}$
0	0	0	1	0	0	$S_3 : \{v_1, v_5, v_9, v_2, v_7, v_0'\}$
0	0	0	1	0	1	$S_3 : \{v_1, v_5, v_9, v_2, v_7, v_0'\}$
0	0	0	1	1	0	$S_3 : \{v_2, v_3, v_5', v_1, v_4, v_0'\}$
0	0	0	1	1	1	$S_3 : \{v_1, v_8, v_9, v_2, v_4, v_0'\}$
0	0	1	0	0	0	$S_3 : \{v_3, v_4, v_9, v_2, v_8, v_0'\}$
0	0	1	0	0	1	**none**
0	0	1	0	1	0	$S_3 : \{v_3, v_4, v_9, v_2, v_8, v_0'\}$
0	0	1	0	1	1	$S_3 : \{v_3, v_7, v_9, v_2, v_5, v_0'\}$
0	0	1	1	0	0	$S_3 : \{v_1, v_5, v_9, v_2, v_7, v_0'\}$
0	0	1	1	0	1	$S_3 : \{v_1, v_5, v_9, v_2, v_7, v_0'\}$
0	0	1	1	1	0	$S_3 : \{v_3, v_4, v_9, v_2, v_8, v_0'\}$
0	0	1	1	1	1	$S_3 : \{v_4, v_5, v_0', v_3, v_6, v_8\}$
0	1	0	0	0	0	$S_3 : \{v_3, v_7, v_9, v_1, v_6, v_0'\}$
0	1	0	0	0	1	$S_3 : \{v_1, v_8, v_9, v_2, v_4, v_0'\}$
0	1	0	0	1	0	$\bar{S}_3 : \{v_4, v_8, v_0', v_3, v_7, v_9\}$
0	1	0	0	1	1	$S_3 : \{v_1, v_8, v_9, v_2, v_4, v_0'\}$
0	1	0	1	0	0	$S_3 : \{v_1, v_5, v_9, v_2, v_7, v_0'\}$
0	1	0	1	0	1	$S_3 : \{v_1, v_5, v_9, v_2, v_7, v_0'\}$
0	1	0	1	1	0	$S_3 : \{v_3, v_7, v_0', v_2, v_5, v_8\}$
0	1	0	1	1	1	$S_3 : \{v_1, v_8, v_9, v_2, v_4, v_0'\}$
0	1	1	0	0	0	$S_3 : \{v_3, v_7, v_9, v_1, v_6, v_0'\}$
0	1	1	0	0	1	$S_3 : \{v_3, v_4, v_0', v_2, v_5, v_8\}$
0	1	1	0	1	0	$S_3 : \{v_4, v_5, v_9, v_6, v_8, v_0'\}$
0	1	1	0	1	1	$S_3 : \{v_3, v_4, v_0', v_2, v_5, v_8\}$
0	1	1	1	0	0	$S_3 : \{v_1, v_5, v_9, v_2, v_7, v_0'\}$
0	1	1	1	0	1	$S_3 : \{v_1, v_5, v_9, v_2, v_7, v_0'\}$
0	1	1	1	1	0	$S_3 : \{v_3, v_7, v_0', v_2, v_5, v_8\}$
0	1	1	1	1	1	$S_3 : \{v_3, v_1', v_0', v_2, v_6, v_8\}$
1	0	0	0	0	0	$S_3 : \{v_1, v_5, v_9, v_3, v_6, v_0'\}$
1	0	0	0	0	1	$S_3 : \{v_1, v_5, v_9, v_3, v_6, v_0'\}$
1	0	0	0	1	0	$S_3 : \{v_1, v_5, v_9, v_3, v_6, v_0'\}$
1	0	0	0	1	1	$S_3 : \{v_1, v_5, v_9, v_3, v_6, v_0'\}$
1	0	0	1	0	0	$S_3 : \{v_3, v_7, v_1', v_2, v_8, v_0'\}$
1	0	0	1	0	1	$S_3 : \{v_3, v_7, v_5', v_4, v_6, v_0'\}$
1	0	0	1	1	0	$S_3 : \{v_1, v_5, v_0', v_2, v_4, v_7\}$

(v_0', v_1)	(v_0', v_3)	(v_0', v_4)	(v_0', v_5)	(v_0', v_7)	(v_0', v_8)	S_3 or $\bar{S}_3$
1	0	0	1	1	1	$S_3 : \{v_1, v_5, v_0', v_2, v_4, v_7\}$
1	0	1	0	0	0	$S_3 : \{v_1, v_5, v_9, v_3, v_6, v_0'\}$
1	0	1	0	0	1	$S_3 : \{v_1, v_5, v_9, v_3, v_6, v_0'\}$
1	0	1	0	1	0	$S_3 : \{v_1, v_5, v_9, v_3, v_6, v_0'\}$
1	0	1	0	1	1	$S_3 : \{v_1, v_5, v_9, v_3, v_6, v_0'\}$
1	0	1	1	0	0	$S_3 : \{v_3, v_4, v_9, v_2, v_8, v_0'\}$
1	0	1	1	0	1	$S_3 : \{v_1, v_8, v_0', v_2, v_4, v_7\}$
1	0	1	1	1	0	$S_3 : \{v_3, v_4, v_9, v_2, v_8, v_0'\}$
1	0	1	1	1	1	$S_3 : \{v_4, v_5, v_0', v_3, v_6, v_8\}$
1	1	0	0	0	0	$S_3 : \{v_6, v_7, v_1', v_5, v_8, v_0'\}$
1	1	0	0	0	1	$S_3 : \{v_3, v_1', v_0', v_2, v_6, v_8\}$
1	1	0	0	1	0	$S_3 : \{v_3, v_7, v_0', v_1, v_4, v_6\}$
1	1	0	0	1	1	$S_3 : \{v_3, v_7, v_0', v_1, v_4, v_6\}$
1	1	0	1	0	0	$S_3 : \{v_1, v_5, v_0', v_3, v_6, v_8\}$
1	1	0	1	0	1	$S_3 : \{v_3, v_1', v_0', v_2, v_6, v_8\}$
1	1	0	1	1	0	$S_3 : \{v_1, v_5, v_0', v_2, v_4, v_7\}$
1	1	0	1	1	1	$S_3 : \{v_1, v_5, v_0', v_2, v_4, v_7\}$
1	1	1	0	0	0	$S_3 : \{v_4, v_5, v_9, v_6, v_8, v_0'\}$
1	1	1	0	0	1	$S_3 : \{v_1, v_8, v_0', v_2, v_4, v_7\}$
1	1	1	0	1	0	$S_3 : \{v_4, v_5, v_9, v_6, v_8, v_0'\}$
1	1	1	0	1	1	$S_3 : \{v_3, v_4, v_0', v_2, v_5, v_8\}$
1	1	1	1	0	0	$S_3 : \{v_1, v_5, v_0', v_3, v_6, v_8\}$
1	1	1	1	0	1	$S_3 : \{v_1, v_8, v_0', v_2, v_4, v_7\}$
1	1	1	1	1	0	$S_3 : \{v_1, v_5, v_0', v_3, v_6, v_8\}$
1	1	1	1	1	1	$S_3 : \{v_3, v_1', v_0', v_2, v_6, v_8\}$

Interestingly, the graph induced by $v_1', \ldots, v_9', v_0'$ is an isomorphic copy of G_{10}. Moreover, (v_i, v_j') for $i \neq j$ is an edge if and only if (v_i, v_j) is not an edge, while the pairs (v_i, v_i'), $i = 0, 1, ..., 9$ are uncertain. Thus, this configuration is the sum of two copies of G_{10}, that is the graph $2G_{10}$'s (see Figure 25). Let us recall that to any graph G we can apply the same operation and obtain the sum $G + G = 2G$.

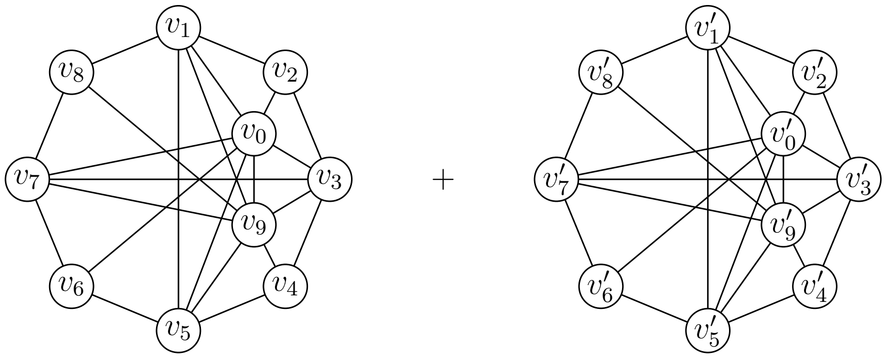


Figure 25: The sum of two G_{10}'s (or $2G_{10}$).

Another remarkable property of the obtained configuration is as follows: if we exchange v_0 with v'_0 and v_9 with v'_9 then the resulting graph becomes the sum of two Petersen graphs, that is, $2\mathcal{P} \equiv 2G_{10}$, as shown in Figure 26.

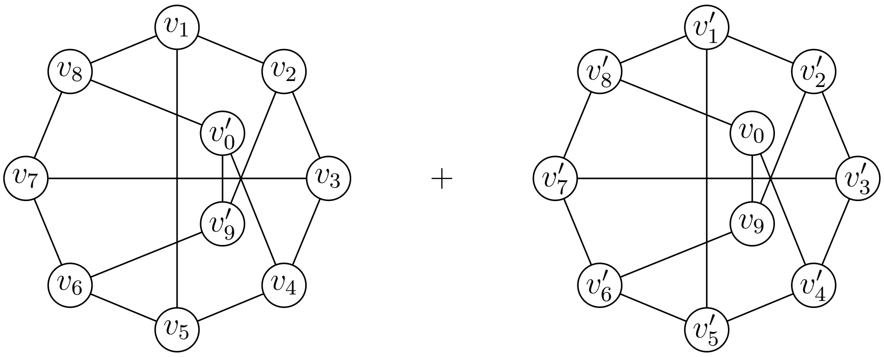


Figure 26: The graph $2\mathcal{P}$, isomorphic to $2G_{10}$ by exchanging v_0, v'_0 and v_9, v'_9.

This completes the proof of Lemma 3. □

2.4 Proof of Lemma 4

We prove that if a graph G contains an induced $2\mathcal{P}$, then it must have either an unsettled 2-comb, or an induced 3-comb or 3-anti-comb.

Let us recall that $2\mathcal{P}$ still has 10 uncertain edges. Hence, it gives us in fact 1024 possible graphs, one of which is an induced subgraph of G. Since we do not know which one, we will prove the statement by considering each such possible subgraphs.

Remarkably, none of these 1024 graphs contains an induced 3-comb or 3-anti-comb, as verified by computer.

Furthermore, $2\mathcal{P}$ itself contains no induced 2-combs either. (Since $2\mathcal{P}$ contains uncertain pairs, we call a subgraph of $2\mathcal{P}$ an induced one only if it does not involve any uncertain pair.) However, each of the 1024 graphs obtained from $2\mathcal{P}$ contains many 2-combs each of which involves exactly one pair of vertices v_i and v'_i for some index i.

Now we will fix one of the uncertain pairs (once as an edge and once as a non-edge), while keeping all others uncertain. Several (36) unsettled induced 2-combs appear that contain the fixed uncertain pair. Each of these 2-combs must be settled in G by our assumption (i), thus there exists a vertex x settling it. There are 16 pairs (x, y), where y is a vertex of $2\mathcal{P}$, not belonging to the unsettled 2-comb. We check all 2^{16} possible edge/non-edge assignments to these 16 pairs, and find by computer search that for each of them an induced 3-comb or 3-anti-comb exists.

More precisely, let us fix the uncertain pair (v_0, v'_0) and consider two cases:

1. If (v_0, v'_0) is an edge then the 2-comb induced by the vertices $\{v_1, v_0, v'_0, v_4\}$ is unsettled in $2\mathcal{P}$, because no vertex in $2\mathcal{P}$ is connected to both v_0 and v'_0 by the definition of the sum of two graphs.

Let x be a settling vertex. Then, by definition, (x, v_0), (x, v_0') must be edges of G, and the pairs (x, v_1) and (x, v_4) must be non-edges. There are 16 other pairs of the form (x, y), where y is a vertex of $2\mathcal{P}$. Hence, there are 2^{16} possible assignments of edges/non-edges between x and $2\mathcal{P}$. We check by computer all 2^{16} possible assignments and find that in each 2^{16} graphs there is an induced (without uncertain pairs) 3-comb or 3-anti-comb.

2. If (v_0, v_0') is not an edge of G then the 2-comb induced by the vertices $\{v_0, v_1, v_8, v_0'\}$ is not settled in $2\mathcal{P}$. Since it must be settled in G by condition (i), there is a vertex x of G that settles it. Similarly to the previous case, we again consider all 2^{16} graphs, and find by computer search that all of them contain an induced 3-comb or 3-anti-comb.

This concludes the proof of Lemma 4. □

3 Proof of Theorems 2 and 3

Proof of Theorem 2: Recall by (1.1) that we can reduce the case analysis by assuming that $1 \le k < \ell \le n-2$.

We start by proving (i). Assume by contradiction that there exists an unsettled $\bar{S_m} = \{B_1, \ldots, B_m, A_1, \ldots, A_m\}$, $|B_i| = k$, $|A_i| = \ell$. Then, by assumption we must have

$$A_i \supset B_j \ \text{ for all } \ j \neq i \quad \text{ and } \quad A_i \not\supset B_i. \tag{3.8}$$

Let us recall that $\bar{S_m}$ is settled by a k-set K iff $K \subseteq \bigcap_{j=1}^m A_j$, and it is settled by an ℓ-set L iff $L \not\supseteq B_i$ for $i = 1, \ldots, m$.

Let $\mathcal{B} = \{B_1, \ldots, B_m\}$, and let $X \subseteq [n]$ be the set that contains the elements that are in more that one of the B_i's, i.e. $X = \{x \in [n] \mid \deg_{\mathcal{B}}(x) > 1\}$. Notice that $X \subseteq \bigcap_{j=1}^m A_j$ because by (3.8) we have that every vertex belonging to two or more of the sets from $\mathcal{B}$ must belong to all sets A_i, $i = 1, ..., m$. Clearly $|X| < k$, otherwise $\bar{S_m}$ would be settled by a k-set in X.

In the following steps of the proof, we will derive some inequalities, to arrive to a contradiction. First, we need some more definitions.

Let a_p, $p = 0, 1, \ldots q \le |X| < k$, be the number of sets $B_i \in \mathcal{B}$ for which $|B_i \cap X| = p$, and let $\mathcal{H} = \{B_i \cap X | i = 1, \ldots, m\}$. Let us observe first that $\tau(\mathcal{B}) \le \tau(\mathcal{H}) + a_0$, where τ denotes the size of a minimum vertex cover. To see this inequality, let us first cover the intersecting hyperedges of $\mathcal{B}$ optimally by $\tau(\mathcal{H})$ vertices, and then cover the rest by choosing one vertex from each remaining set outside of X (i.e., by at most a_0 additional vertices). Moreover, we have $\tau(\mathcal{B}) > n - \ell$, since otherwise there exists an ℓ-set settling $\bar{S_m}$. Thus, we can conclude that

$$\tau(\mathcal{H}) + a_0 \ge n - \ell + 1 \tag{3.9}$$

Assume w.l.o.g. that $|B_1 \cap X| \le |B_2 \cap X| \le \ldots \le |B_m \cap X|$. Since we know by (3.8) that $\bigcup_{j=1}^{m-1} B_j \subseteq A_m$, we have:

$$|\bigcup_{i=1}^{m-1} B_j| = |X| + \sum_{p=0}^{q}(k-p)a_p - (k-q) \leq \ell \tag{3.10}$$

Let us now take away k times equation (3.9) from (3.10) and obtain

$$|X| + \sum_{p=0}^{q}(k-p)a_p - (k-q) - k(\tau(\mathcal{H}) + a_0) \;\leq\; \ell - k(n-\ell+1)$$

which can be simplified to

$$|X| + \sum_{p=1}^{q}(k-p)a_p + q - k\tau(\mathcal{H}) \;\leq\; (k+1)\ell - kn \tag{3.11}$$

Notice that the right had side of (3.11) is negative by our initial assumption of $kn > (k+1)\ell$. Thus, to arrive to a contradiction, it is enough to prove that

$$k\tau(\mathcal{H}) \leq |X| + \sum_{p=1}^{q}(k-p)a_p + q. \tag{3.12}$$

Let us observe next that $\sum_{p=1}^{q}(k\ a_p) = k|\mathcal{H}|$, and that $\sum_p (p\ a_p) = \sum_{H\in\mathcal{H}} |H|$. Thus, we can equivalently rewrite inequality (3.12) as:

$$k(|\mathcal{H}| - \tau(\mathcal{H})) \;\geq\; \sum_{H\in\mathcal{H}} |H| - |X| - q \tag{3.13}$$

To show (3.13), let us construct a cover C of $\mathcal{H}$ as follows. First we choose into C a vertex of the highest degree in $\mathcal{H}$. This vertex covers at least $\frac{\sum_{H\in\mathcal{H}}|H|}{|X|}$ hyperedges of $\mathcal{H}$. We cover the remaining edges by choosing one vertex from each. This simple procedure shows that

$$\tau(\mathcal{H}) \;\leq\; |C| \;\leq\; |\mathcal{H}| - \frac{\sum_{H\in\mathcal{H}}|H|}{|X|} + 1. \tag{3.14}$$

From this simple inequality we can derive the following:

$$\begin{aligned} k(|\mathcal{H}| - \tau(\mathcal{H})) \;&\geq\; \tfrac{k}{|X|} \textstyle\sum_{H\in\mathcal{H}} |H| \; - k \\ &=\; \textstyle\sum_{H\in\mathcal{H}} |H| + \tfrac{k-|X|}{|X|} \sum_{H\in\mathcal{H}} |H| \; - k \\ &\geq\; \textstyle\sum_{H\in\mathcal{H}} |H| \; - |X| \end{aligned}$$

where, the second inequality follows from $|X| \leq \sum_{H\in\mathcal{H}} |H|$, which is true, since every vertex of X has degree at least 2 in $\mathcal{B}$. The above inequalities then prove (3.13), since $q \geq 0$, which then yields the desired contradiction, completing the proof of (i). □

We prove next (ii). We will show, by a construction that an unsettled $\bar{S_m}$ exists in $G(n,k,\ell)$, whenever $kn \le (k+1)\ell$ and $n \ge k+\ell$.

For this let us set $r \equiv \ell \pmod{k}$, $0 \le r < k$, $m = \frac{\ell+k-r}{k}$, and let $B_1, \ldots, B_m$, and R be pairwise disjoint subsets of $[n] = \{1,2,...,n\}$, such that $|R| = r$ and $|B_i| = k$ for $i = 1,...,m$. Notice that

$$|R \cup B_1 \cup \cdots \cup B_m| = km + r = \ell + k. \tag{3.15}$$

Thus, it is possible to choose such pairwise disjoint subsets, since $k+\ell \le n$ by our assumption. Let us further define

$$A_i \;=\; R \cup \left(\bigcup_{j \ne i} B_j\right) \qquad \text{for} \qquad i = 1, ..., m.$$

With these definitions, we have $|A_i| = r + k(m-1) = r + (\ell - r) = \ell$ for all $i = 1,...,m$. Furthermore, $A_i \supseteq B_j$ if and only if $i \ne j$. Thus, the sets A_1, ..., A_m, and B_1, ..., B_m are vertices of $G(m,k,\ell)$ forming an $\bar{S_m}$.

We show that this $\bar{S_m}$ is unsettled in $G(n,k,\ell)$. For this, observe first that $|\bigcap_{i=1}^m A_i| = |R| = r < k$, and consequently, no k-set can settle $\bar{S_m}$.

Next, let us assume indirectly that there is an ℓ-set L which settles $\bar{S_m}$. Hence, L cannot be connected in $G(n,k,\ell)$ to any of the B_i's. In other words, $L \not\supseteq B_i$ for $i = 1,...,m$. It follows that $|L \cap B_i| \le k-1$ for all $i = 1,...,m$, implying

$$|L| \le m(k-1) + r + (n-k-\ell). \tag{3.16}$$

That is, we can take at most $k-1$ elements from each of the k-sets, and the remaining $r+n-k-\ell$ elements of $[n]$, as implied by (3.15). It is now enough to show that $|L| < \ell$, because this contradicts the assumption that L is an ℓ-set. To do this, let us rewrite (3.16) as

$$|L| \le m(k-1) + r + (n-k-\ell) = \frac{\ell+k-r}{k}(k-1) + r - n - k - \ell,$$

which implies

$$\begin{aligned} k|L| + \ell &\le (\ell+k-r)(k-1) + k(r-n-k-\ell) + \ell \\ &= k\ell - \ell + k^2 - k - kr + r + kr + kn - k^2 - k\ell + \ell \\ &= kn - (k-r) \; < \; kn \; \le \; (k+1)\ell \end{aligned}$$

where the last two inequalities follow by $k > r$ and our assumption that $kn \le (k+1)\ell$. Thus, $|L| < \ell$ follows, completing the proof of (ii). □

This completes the proof of Theorem 2. □

Proof of Theorem 3:

We prove first (a). Even though this claim is only for $k \le 2$, let us first disregard this restriction. Assume by contradiction that there exists an unsettled S_m in $G(m,k,\ell)$ defined by the sets $\{B_1, \ldots, B_m, A_1, \ldots, A_m\}$, where $|B_i| = k$, $|A_i| = \ell$, for $i = 1, \ldots, m$, and $B_j \subseteq A_i$, iff $i = j$. Set $\mathcal{B} = \{B_1, \ldots, B_m\}$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$.

By definitions, an ℓ-set L can settle S_m only if $[n] \setminus L$ is a vertex cover of the hypergraph $\mathcal{B}$. Furthermore, a k-set K can settle S_m, only if $K \subseteq A_i$ for all $i = 1, ..., m$. Since S_m is assumed to be unsettled in $G(n, k, \ell)$, we must have the following properties.

(i) $\tau(\mathcal{B}) \geq n - \ell + 1$, since otherwise the complement of a minimum vertex cover of $\mathcal{B}$ would contain a settling ℓ-set.

(ii) $|\bigcap_{i=1}^{m} A_i| < k$, since otherwise the intersection of the sets of $\mathcal{A}$ would contain a settling k-set.

Let us also observe that $B_j \subseteq A_i$ if and only if $i = j$ implies that $\bar{A}_i = [n] \setminus A_i$ is a vertex cover for $\mathcal{B} \setminus B_i$, implying $|\bar{A}_i| = n - \ell \geq \tau(\mathcal{B} \setminus \{B_i\}) \geq \tau(\mathcal{B}) - 1$. This, together with (i), implies that

$$n - \ell \;=\; \tau(\mathcal{B}) - 1 \;=\; \tau(\mathcal{B} \setminus \{B_i\}) \tag{3.17}$$

for all $i = 1, ..., m$.

Let us now consider the subset

$$X \;=\; [n] \setminus \bigcup_{i=1}^{m} B_i.$$

Equations (3.17) imply that $X \subseteq A_i$ for all $i = 1, ..., m$. Thus, by property (ii) we must have

$$|X| \;\leq\; k - 1 \tag{3.18}$$

Another consequence of (3.17) is that the hypergraph $\mathcal{B}$ is τ-critical, i.e., the minimum vertex cover size strictly decreases whenever we remove a hyperedge from $\mathcal{B}$. This also implies that $\mathcal{B}$ is α-critical, where $\alpha(\mathcal{B})$ is the size of the largest *independent set* of $\mathcal{B}$, i.e., the largest set not containing a hyperedge of $\mathcal{B}$. This is because $\alpha(\mathcal{B}) + \tau(\mathcal{B}) = n$ for all hypergraphs $\mathcal{B}$.

Let us now consider the case of $k = 1$. In this case we have $|\mathcal{B}| = \tau(\mathcal{B})$ and by (3.18) $X = \emptyset$, implying that $|\mathcal{B}| = n$, which together with the previous equality and (3.17) imply

$$n = |\mathcal{B}| = \tau(\mathcal{B}) = n - \ell + 1$$

from which $\ell = 1$ follows, contradicting (1.1).

Let us next consider the case of $k = 2$. In this case $\mathcal{B}$ is an α-critical graph G on vertex set $V = [n] \setminus X$, with $\alpha(G) = \alpha(\mathcal{B}) - |X| = \ell - 1 - |X|$.

We apply a result attributed to Erdös and Gallai (see Excercise 8.20 in [22]; see also the proof of Exercise 8.10 by Hajnal), stating that in an α-critical graph G with no isolated vertices we have $|V| \geq 2\alpha(G)$. This implies for our case that $n - |X| \geq 2(\ell - 1 - |X|)$, from which

$$n \geq 2\ell - 2 - |X|$$

follows. Since by (3.18) we have $|X| \le k-1 = 1$, the above inequality implies

$$n \ge 2\ell - 3$$

contradicting (a) of Theorem 3, according to which we have $n < 2\ell - 3$. □

Remark 5. *We could extend the above line of arguments for $k \ge 3$, if the inequality $n \ge \frac{k}{k-1}\alpha(\mathcal{B})$ were valid for α-critical k-uniform hypergraphs, in general. However, this is not the case, as the following exampleshows: let $n = 10$, $k = 3$ and $\mathcal{B} = \{\{1,2,3\}, \{3,4,5\}, \{5,6,7\}, \{7,8,9\}, \{9,10,1\}\}$. In this case we have $\alpha(\mathcal{B}) = 7$, and $10 \not\ge (3/2)7 = 21/2$.*

We prove finally (b). We will now provide a construction for an unsettled S_m. Let $L = \{2,3,\ldots,k\}$, and choose $r \in L$, such that $r \equiv \ell \pmod{k-1}$ (for instance, if $k = 2$ then we have $r = 2$).

Let us next partition $[n]$ as

$$[n] \;=\; X \cup \bigcup_{j=1}^{p} Q_j,$$

where $|X| = r-1$, $p = \frac{\ell-r}{k-1}$, and where the sets $Q_1,\ldots,Q_p$ are almost equal, i.e., $|Q_i| \sim \frac{n-r+1}{p}$.

Then, we construct an unsettled $S_m = \{B_1,\ldots,B_m,\, A_1,\ldots,A_m\}$ as follows. We define $m = \sum_{j=1}^{p}\binom{|Q_i|}{k}$, and the sets B_i, $i = 1,...,m$ are the$k$-subsets of the Q_j-s, i.e.,

$$\{B_1,..,B_m\} \;=\; \bigcup_{i=1}^{p}\binom{Q_i}{k}.$$

Finally, we set for $i = 1,...,m$

$$A_i \;=\; X \cup B_i \cup \bigcup_{\substack{1\le j\le p\\ j\ne j^*}} R_{ij},$$

where $B_i \subseteq Q_{j^*}$ and $R_{ij} \subseteq Q_j$, $|R_{ij}| = k-1$ for all $j \ne j^*$. In other words, each A_i contains X, the corresponding set B_i, and $k-1$ points from each set Q_j not containing B_i.

It is easy to see that $|A_i| = \ell$. Indeed,

$$\begin{aligned}|A_i| &= k + r - 1 + (p-1)(k-1)\\ &= k + r - 1 + \left(\frac{\ell - r}{k-1} - 1\right)(k-1)\\ &= r + \ell - r = \ell\end{aligned}$$

Let us observe first that by the above calculations no ℓ-set can settle S_m. This is because all ℓ-sets must intersect at least one of the Q_j's in k or more points, therefore any ℓ-set contains at least one of the B_i's.

Furthermore, we can show that $|Q_j| \geq k$, for $j = 1, ..., p$. By our assumption we have $n(k-1) \geq \ell k - r - k + 1$ from which we can derive the following chain of inequalities:

$$\begin{aligned} n &\geq \ell\frac{k}{k-1} - \frac{k+r-1}{k-1} \\ n(k-1) &\geq k\ell - k - r + 1 \\ n(k-1) - kr + k + r - 1 &\geq k\ell - kr \\ (n-r+1)(k-1) &\geq k\ell - kr \\ (n-r+1) &\geq k\frac{\ell - r}{k-1} = kp \\ \frac{n-r+1}{p} &\geq k, \end{aligned}$$

which implies that $|Q_j| \geq \lfloor \frac{n-r+1}{p} \rfloor \geq k$.

Finally we have to prove that no k-set can settle S_m. For this, as we remarked earlier, it is enough to show that $|\bigcap_{i=1}^m A_i| < k$, which will follow from

$$\left(\bigcap_{i=1}^m A_i\right) \cap Q_j = \emptyset \tag{3.19}$$

for $j = 1, \ldots, p$, since then $(\bigcap_{i=1}^m A_i) \subseteq X$ is implied, and we have $|X| = k - 1$.

To see (3.19) let us consider the following cases:

Case 1. If $|Q_j| > k$ then for all $v \in Q_j$, there is an index i such that $B_i \subset Q_j \setminus \{v\}$, implying by the definitions that $v \notin A_i$. Hence, (3.19) follows.

Case 2. If $|Q_j| = k$ and $m \geq k + 1$, then we have $Q_j = B_{i^*}$ for exactly one index $i^* \in \{1, ..., m\}$. For all other indices i we have $Q_j \cap A_i = R_{ij}$ of size $k - 1$. Thus, since $m \geq k + 1$, we can choose for each $v \in Q_j$ an index $i \neq i^*$ such that $v \notin A_i$, implying (3.19).

Case 3. If $m \leq k$ then we must have $|Q_1| = |Q_2| = \ldots = |Q_p| = k$, $m = p \leq k$, since we already know that $|Q_j| \geq k$ for all $j = 1, ..., p$, and if $|Q_j| > k$ for at least one index j, then $m \geq k + 1$ would be implied. Thus, we have

$$\begin{aligned} n &= |X| + \sum_{i=1}^{p} |Q_i| \\ &= r - 1 + pk \\ &= r - 1 + k\frac{\ell - r}{k-1} \\ &= \ell\frac{k}{k-1} - \frac{r}{k-1} - 1, \end{aligned}$$

and hence, by our assumption, we must have $\ell \geq r + k^2 - k + 1$. However, $p \leq k$ implies that $p = \frac{\ell - r}{k-1} \leq k$ from which $\ell \leq r + k^2 - k$ follows. □

This completes the proof of Theorem 3. □

4 More about CIS-d-graphs

4.1 Proof of Propositions 4, 5, and 9

Proof of Proposition 4. Obviously, every partition of colors can be realized by successive identification of two colors. Hence, the following Lemma implies Proposition 4.

Given a $(d+1)$-graph $\mathcal{G} = (V; E_1, \ldots, E_d, E_{d+1})$, let us identify the last two colors d and $d+1$ and consider the d-graph $\mathcal{G}' = (V; E_1, \ldots, E_{d-1}, E_{\mathbf{d}})$ where $E_{\mathbf{d}} = E_d \cup E_{d+1})$.

Lemma 5. *If $\mathcal{G}$ is a CIS-$(d+1)$-graph then $\mathcal{G}'$ is a CIS-d-graph.*

Proof. Suppose that $\mathcal{G}'$ does not have the CIS-d-property, that is there are d vertex-sets $C_1, \ldots, C_{d-1}, C_{\mathbf{d}} \subseteq V$ such that they have no vertex in common, and C_i is a maximal subset of V avoiding color i for $i = 1, \ldots, d-1$ and $C_{\mathbf{d}}$ is a maximal subset of V avoiding both colors d and $d+1$. Clearly, there exist maximal vertex-sets C_d and C_{d+1} avoiding colors d and $d+1$ respectively and such that $C_d \cap C_{d+1} = C_{\mathbf{d}}$. Then $C_1, \ldots, C_{d-1}, C_d, C_{d+1} \subseteq V$ are maximal vertex-sets avoiding colors $1, \ldots, d-1, d, d+1$ respectively and with no vertex in common. Hence, the $(d+1)$-graph $\mathcal{G}'$ does not have the CIS-$(d+1)$-property, either. $\square$

Proof of Proposition 5. It follows by a routine case analysis from the definitions.

First, let us consider Gallai's property. Suppose that $\mathcal{G}$ has a Δ. Clearly, it can not contain exactly one edge in $\mathcal{G}''$, since then two remaining edges are of the same color. If this Δ contains 2 edges in $\mathcal{G}''$ then the third one is there, too, and hence G'' contains a Δ. If all 3 edges are in $\mathcal{G}'$ then G' contains a Δ.

If G'' contains a Δ then clearly this Δ is in $\mathcal{G}$ too. Let G' contain a Δ. If it does not contain the vertex v substituted by $\mathcal{G}''$ then this Δ remains in $\mathcal{G}$. If it contains v then two other vertices with any vertex of $\mathcal{G}''$ form a Δ in $\mathcal{G}$.

Now let us consider the CIS-property. To simplify the notation we restrict ourselves by the case $d = 2$, though exactly the same arguments work in general. It is easy to see that any maximal cliques (respectively, stable sets) of $\mathcal{G}'$ which do not contain v remain unchanged in $\mathcal{G}$, while a maximal clique C' (respectively, a maximal stable set S') of $\mathcal{G}'$ which contains v and for every maximal clique C'' (respectively, every maximal stable set S'') of $\mathcal{G}''$ the set $C = C' \cup C'' \setminus \{v\}$ (respectively, $S = S' \cup S'' \setminus \{v\}$ is a maximal clique (respectively, a maximal stable set) of $\mathcal{G}$ and moreover, there are no other maximal cliques (respectively, maximal stable sets) in $\mathcal{G}$.

It is not difficult to verify that every maximal clique $C = C' \cup C'' \setminus \{v\}$ and every maximal stable set $S = S' \cup S'' \setminus \{v\}$ in $\mathcal{G}$ intersect if and only if every maximal clique C' intersects every a maximal stable set S' of G' and every maximal clique C'' intersects every a maximal stable set S'' of G''. Indeed, if $C' \cap S' = \{v'\} \neq \{v\}$ then $C \cap S = \{v'\}$ for any C'' and S''. If $C' \cap S' = \{v\}$ then $C \cap S = C'' \cap S''$ and hence $C \cap S \neq \emptyset$ if and only if $C'' \cap S'' \neq \emptyset$. If $C \cap S \neq \emptyset$ then both $C' \cap S'$ and $C'' \cap S''$ must be non-empty. $\square$

Proof of Proposition 9. Let us recall that a d-graph with only two non-empty chromatic components is called a 2-graph. Clearly, for 2-graphs the claim holds, since G is a CIS-graph whenever $\bar{G}$ is, and vice versa.

In general, we proceed by induction on the number of vertices. Let $\mathcal{G}$ by a Gallai d-graph whose at least $d-1$ chromatic components are CIS-graphs. By Proposition 7, $\mathcal{G}$ can be realized by substituting a Gallai d-graph $\mathcal{G}''$ for a vertex v of a Gallai d-graph $\mathcal{G}'$. (Moreover, we could further assume that one of these two d-graphs is a 2-graph, yet this assumption is not needed.) By Proposition 6, the same $d-1$ chromatic components form CIS-graphs in both $\mathcal{G}'$ and $\mathcal{G}''$. (Recall that, in contrast to this claim, an induced subgraph of a CIS-graph may not have the CIS-property.) Hence, by induction hypothesis, both $\mathcal{G}'$ and $\mathcal{G}''$ are CIS-d-graphs and, by Proposition 5, $\mathcal{G}$ is a CIS-d-graph too. ☐

4.2 Settling Δ

Let $V = \{v_1, v_2, v_3\}$ and assume that $E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, and $E_3 = \{(v_3, v_1)\}$ form a Δ, see Figure 27. Obviously, Δ is not a CIS-3-graph. Indeed, let us consider $C_1 = \{v_2, v_3\}$, $C_2 = \{v_3, v_1\}$, and $C_3 = \{v_1, v_2\}$. There is no edge from E_i in C_i for $i = 1, 2, 3$ and $C_1 \cap C_2 \cap C_3 = \emptyset$. Hence, if a CIS-3-graph $\mathcal{G} = (V; E_1, E_2, E_3)$ contains a Δ then it must contain a vertex v_4 such that the sets $C_1' = \{v_2, v_3, v_4\}$, $C_2' = \{v_3, v_1, v_4\}$, and $C_3' = \{v_1, v_2, v_4\}$ contain no edges from E_1, E_2, and E_3, respectively.

Similarly, let us consider the sets $C_1 = \{v_3, v_1\}$, $C_2 = \{v_1, v_2\}$, and $C_3 = \{v_2, v_3\}$. Again, there is no edge from E_i in C_i for $i = 1, 2, 3$ and $C_1 \cap C_2 \cap C_3 = \emptyset$. Hence, if a CIS-3-graph $\mathcal{G} = (V; E_1, E_2, E_3)$ contains a Δ then it must contain a vertex v_5 such that $C_1' = \{v_3, v_1, v_5\}$, $C_2' = \{v_1, v_2, v_5\}$, and $C_3' = \{v_2, v_3, v_5\}$ contain no edges from E_1, E_2, and E_3, respectively.

It is easy to check that $v_4 \neq v_5$ and that we must have $(v_4, v_1), (v_1, v_2), (v_2, v_5) \in E_1$, $(v_4, v_2), (v_2, v_3)$, $(v_3, v_5) \in E_2$, $(v_4, v_3), (v_3, v_1), (v_1, v_5) \in E_3$, see Figure 28. This leaves only one pair (v_4, v_5) whose color is not implied. Yet, let us note that for any coloring of (v_4, v_5) a new Δ appears. For example, if $(v_4, v_5) \in E_1$ then vertices (v_3, v_4, v_5) form a Δ'.

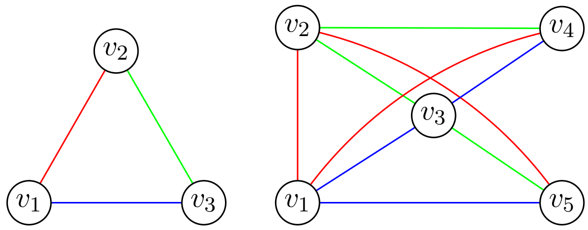


Figure 27: Settling Δ.

4.3 A stronger conjecture

We say that two vertices v_4 and v_5 settle Δ. Note however that v_1 and v_2 do not settle Δ'. So we need more vertices to settle it. Nevertheless, there are d-graphs whose all Δs are settled. First such example was given by Andrey Gol'berg in 1984, see Figure 30.

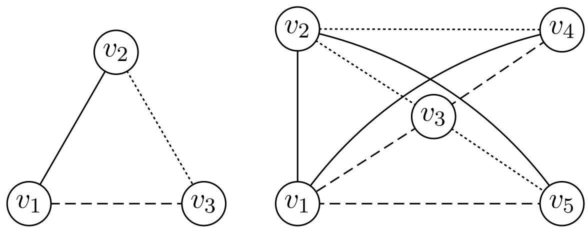


Figure 28: Settling Δ (in black and white for printing).

We call this construction a 4-cycle. It has 4 Δs and they are all settled. Yet, if we partition its three colors into two sets we will get 44 2-combs none of which is settled. Hence, by Proposition 4, the 4-cycle is not a CIS-3-graph.

Moreover, in the next section we give examples of 3-graphs whose all Δs and 2-combs are settled, however, their 2-projections have unsettled induced 3-combs or 3-anti-combs.

Conjecture 3. *Let $\mathcal{G}$ be a non-Gallai 3-graph with chromatic components G_1, G_2, G_3, then there is an unsettled Δ in $\mathcal{G}$ or G_i has an unsettled induced comb or anti-comb for some* $i = 1, 2, 3$.

Obviously, Proposition 4 and Conjecture 3 imply Conjecture 2.

4.4 Even cycles and flowers

In this section we describe some interesting 3-graphs in support of Conjecture 3. They have all Δs settled, and sometimes even all 2-combs are settled in their 2-projections. However, then unsettled 3-combs, or 3-anti-combs, or 4-combs appear.

Let us consider four Δs in Figure 29. They form a cycle.

This construction can be extended (uniquely) to a 3-graph, shown in Figure 31, in which all four Δs are settled "counterclockwise" (i.e., Δs induced by the triplets $\{0, 1, 2\}$, $\{2, 3, 4\}$, $\{4, 5, 6\}$, and $\{6, 7, 0\}$ are settled by the pairs $\{3, 4\}$, $\{5, 6\}$, $\{7, 1\}$, and $\{1, 2\}$, respectively), and no new Δ appears. However, 2-projections of this 3-graph contain 44 unsettled 2-combs (induced by the quadrupples $\{0, 5, 1, 4\}$, $\{3, 2, 6, 7\}$, $\{4, 1, 2, 3\}$, $\{0, 5, 6, 7\}$, etc.) as shown in Figure 31.

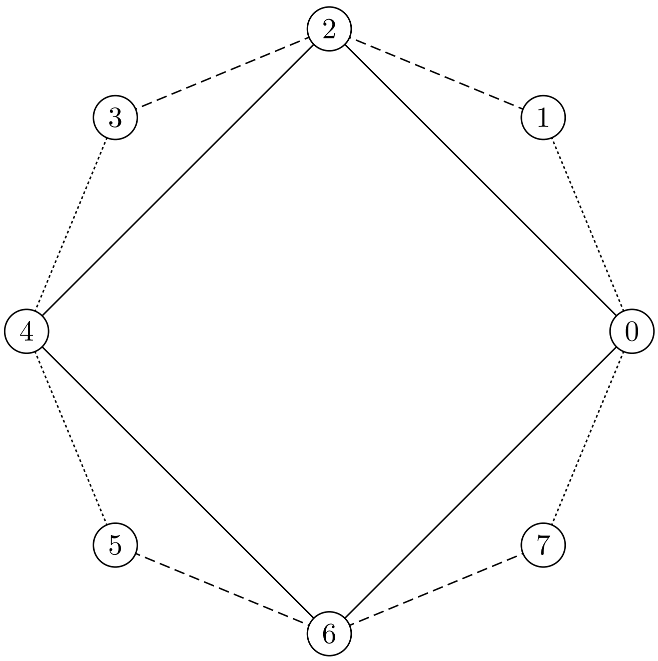


Figure 29: Initial 4-cycle structure.

Level 1: GBBGGBBG
Level 2: RGRBRGRB
Level 3: RBRGRBRG
Level 4: GBBRGBBR

4 settled Δs
44 S_2: 0 settled

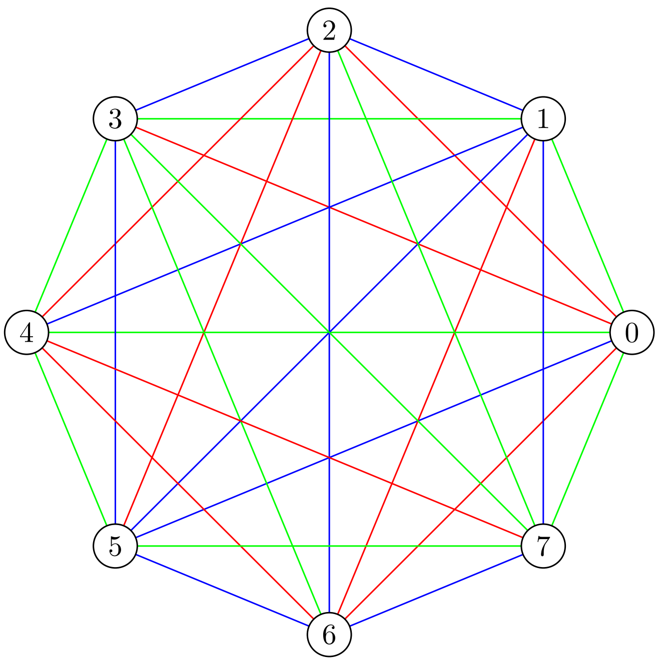


Figure 30: 4-cycle with all Δs settled.

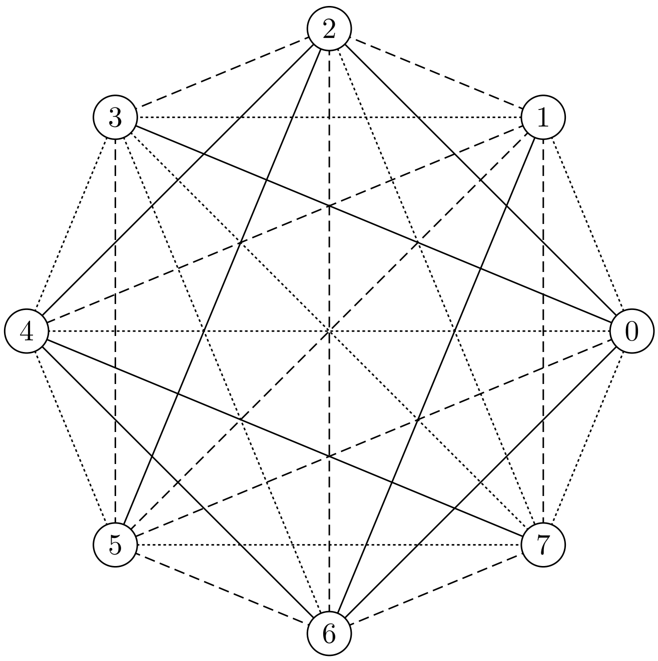


Figure 31: 4-cycle (in black and white for printing). This 3-graph was constructed by Andrey Gol'berg in 1984.

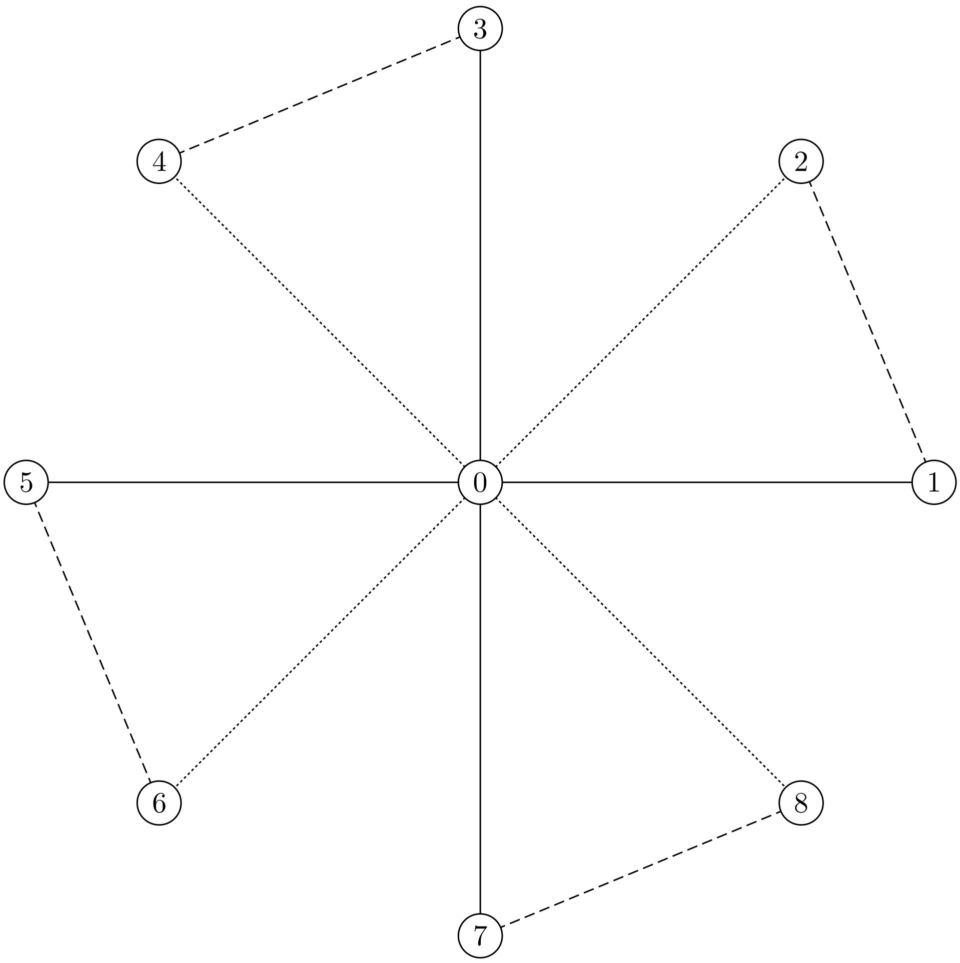


Figure 32: Initial 4-flower structure.

Now, let us consider four Δs with one common vertex as shown in Figure 32. This construction we call a 4-*flower*. It can be extended to a 3-graph, as shown in Figure 34, in which all four Δs are settled "counterclockwise" (i.e., Δs induced by the triplets $\{0,1,2\}$, $\{0,3,4\}$, $\{0,5,6\}$, and $\{0,7,8\}$ are settled by the pairs $\{3,4\}$, $\{5,6\}$, $\{7,8\}$, and $\{1,2\}$, respectively). Although four more Δs (induced by the triplets $\{0,1,6\}$, $\{0,2,5\}$, $\{0,4,7\}$, and $\{0,3,8\}$) appear in this extension), yet they are settled too. Moreover, 2-projections of this 3-graph contain twenty induced 2-combs that are all settled. However, there exist also eight induced 3-combs that are not settled.

Using a computer, we analyzed also some larger flowers (namely, $2j$-flowers for $j = 3, 4, 5,$ and 6) shown below. In all these examples all Δs are settled. However, in agreement with Conjecture 3, for each of these 3-graphs always there is a 2-projection that contains an unsettled comb or anti-comb.

We have to explain the notation used in the figures. The three colors are red R, blue B, and green G, and we denote them by solid, dashed, and dotted lines, respectively.

In a $2j$-flower we denote the central vertex by 0 and other vertices are labeled by $1, 2, \ldots, 2j-1, 2j$. Due to the symmetry, we can describe this 3-graph in terms of a list of colors L present in *level* i, where level i contain all edges (a, b) such that $a - b = \pm i \pmod{n}$. Clearly, we only need to provide the color lists from level 1 to j, since level i gives the same assignment as level $2j - i$. Finally Level 0 shows the coloring of the radial edges. For example, the 4-flower on Figure 34 is colored as follows:

Level 0:
the edges $(0,1), (0,2), (0,3)(0,4), (0,5), (0,6), (0,7), (0,8)$ are colored by $RGRGRGRG$.

Level 1:
the edges $(1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,8), (8,1)$ are colored by $BGBGBGBG$;

Level 2:
the edges $(1,3), (2,4), (3,5), (4,6), (5,7), (6,8), (7,1), (8,2)$ are all colored by $BBBBBBBB$;

Level 3:
the edges $(1,4), (2,5), (3,6), (4,7), (5,8), (6,1), (7,2), (8,3)$ are colored by $RBRBRBRB$;

Level 4:
the edges $(1,5), (2,6), (3,7), (4,8), ((5,1), (6,2), (7,3), (8,4))$ are colored by $RGRG(RGRG)$.

Level 0: RGRGRGRG
Level 1: BGBGBGBG
Level 2: BBBBBBBB
Level 3: RBRBRBRB
Level 4: RGRGRGRG

8 Δs: 8 settled
20 S_2: 20 settled
8 S_3: 0 settled

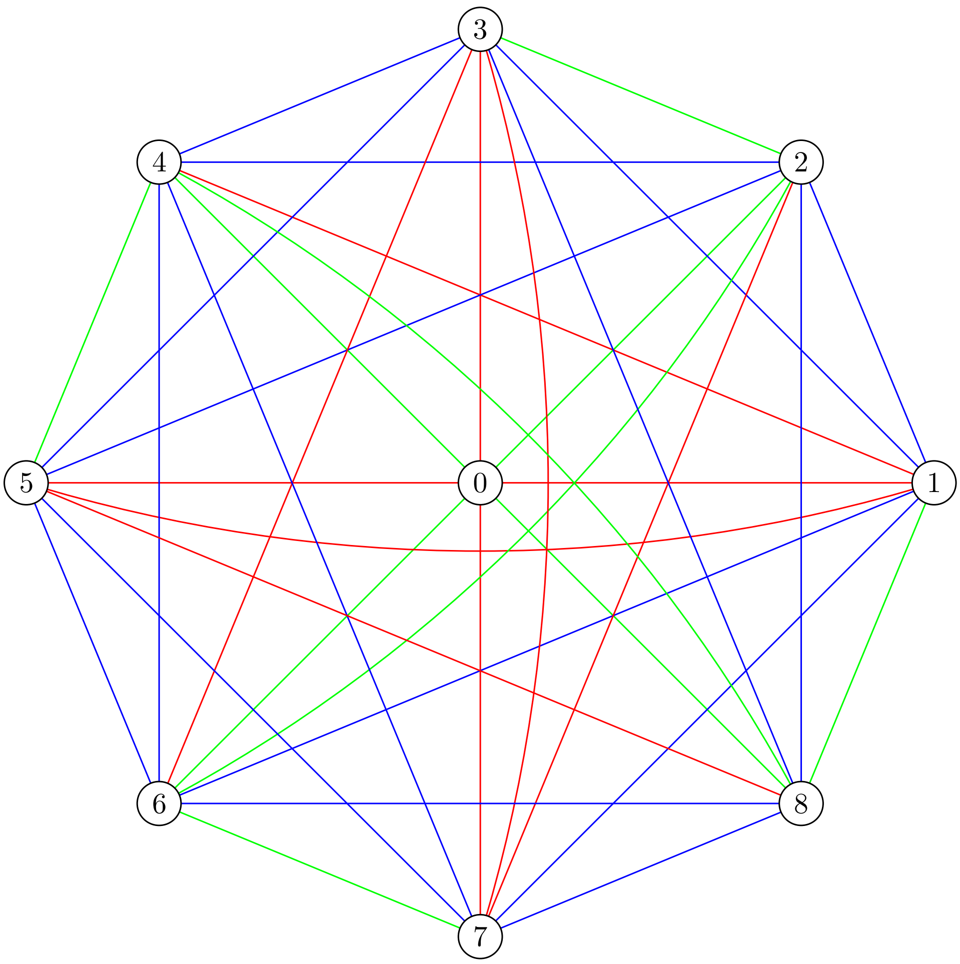


Figure 33: 4-flower example.

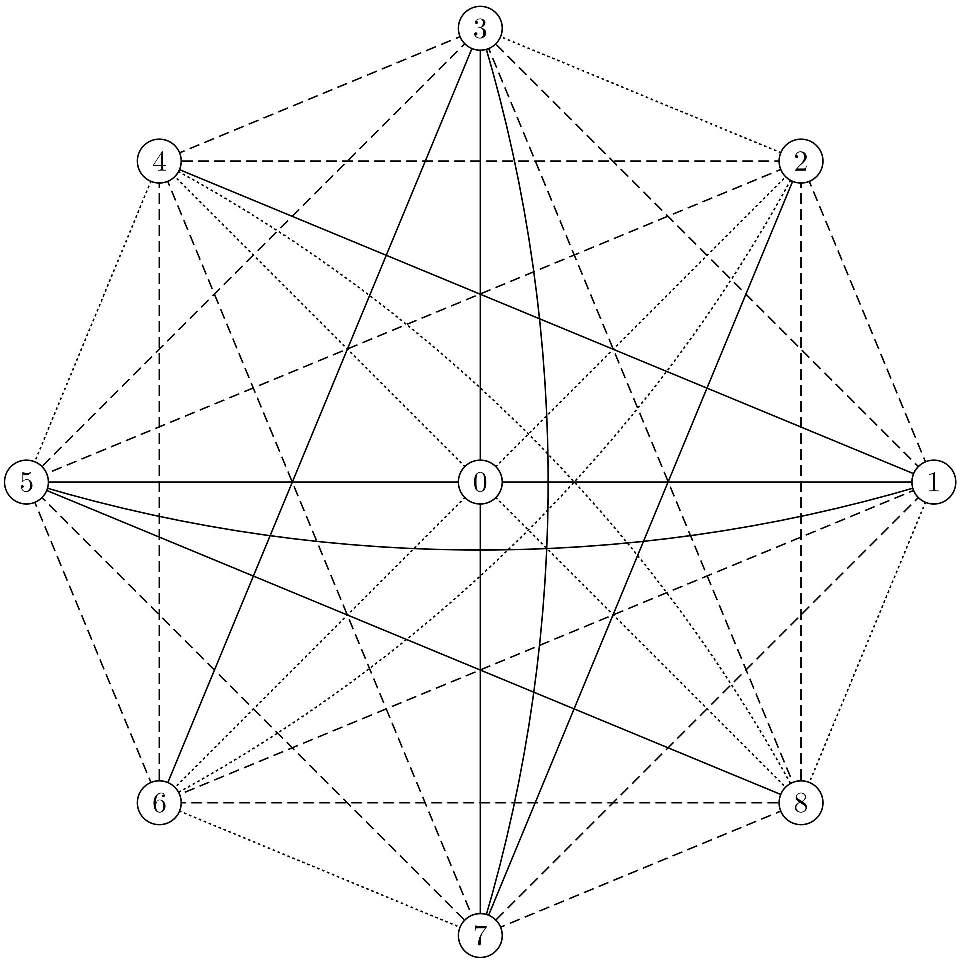


Figure 34: 4-flower example (in black and white for printing).

Level 0: RGRGRGRGRGRG
Level 1: BGBGBGBGBGBG
Level 2: BBBBBBBBBBBB
Level 3: RBRBRBRBRBRB
Level 4: RGRGRGRGRGRG
Level 5: BRBRBRBRBRBR
Level 6: BBBBBBBBBBBB

18 Δs: 18 settled
66 S_2: 66 settled
38 S_3: 20 settled
6 S_4: 0 settled

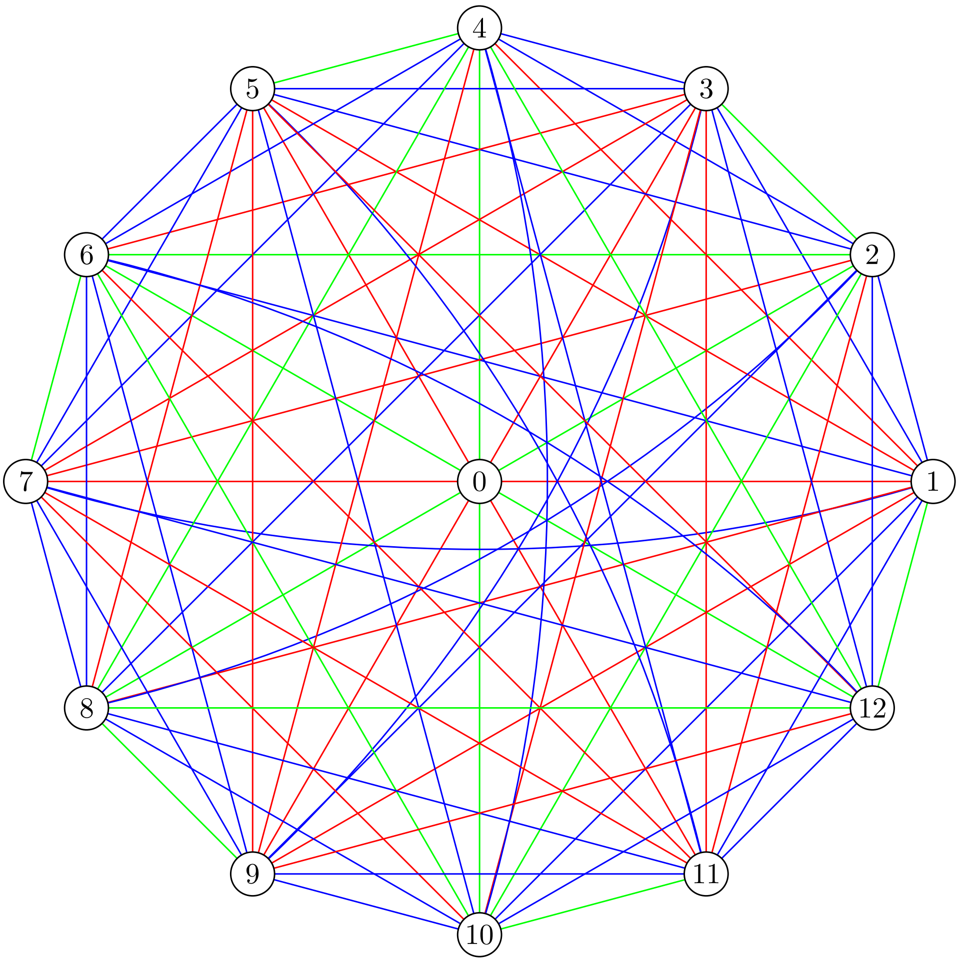


Figure 35: 6-flower example.

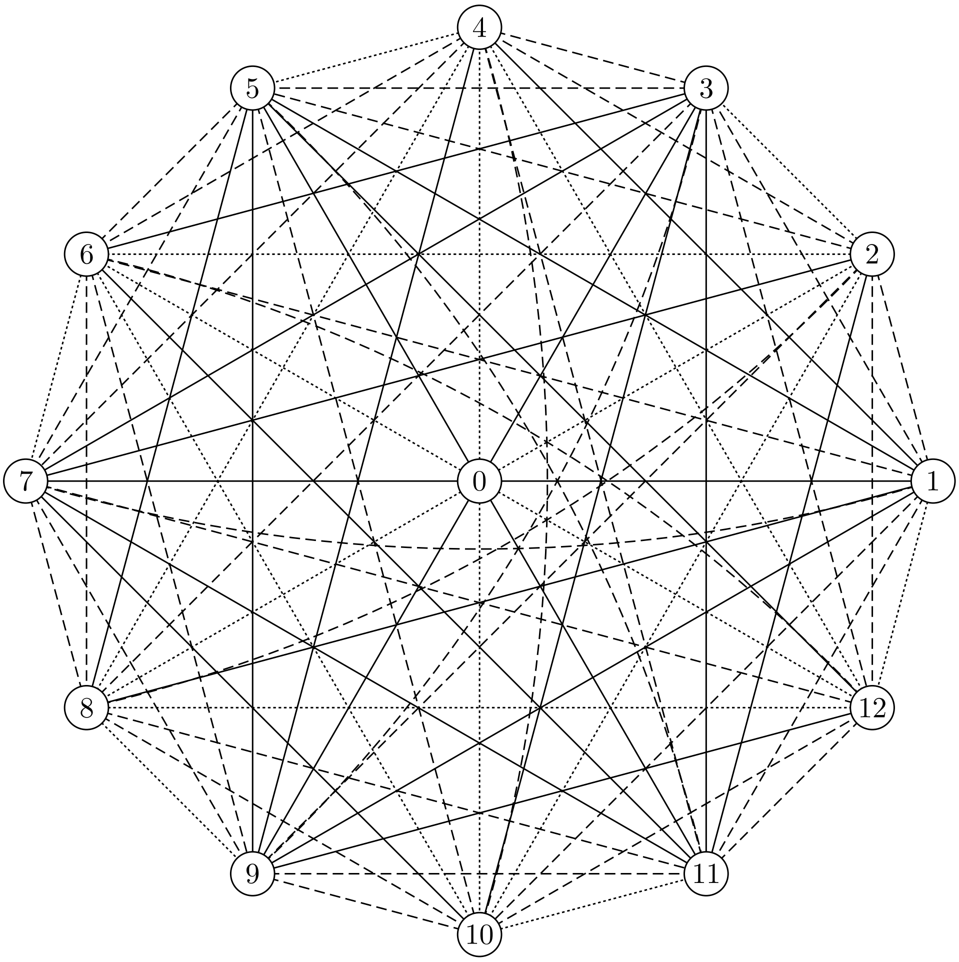


Figure 36: 6-flower example (in black and white for printing). This 3-graph was constructed by Bianca Viray in 2004.

Level 0: RGRGRGRGRGRGRGRG
Level 1: BGBGBGBGBGBGBGBG
Level 2: BBBBBBBBBBBBBBBB
Level 3: RBRBRBRBRBRBRBRB
Level 4: RGRGRGRGRGRGRGRG
Level 5: BRBRBRBRBRBRBRBR
Level 6: BBBBBBBBBBBBBBBB
Level 7: GBGBGBGBGBGBGBGB
Level 8: RRRRRRRRRRRRRRRR

32 Δs: 32 settled
192 S_2: 192 settled
256 S_3: 0 settled

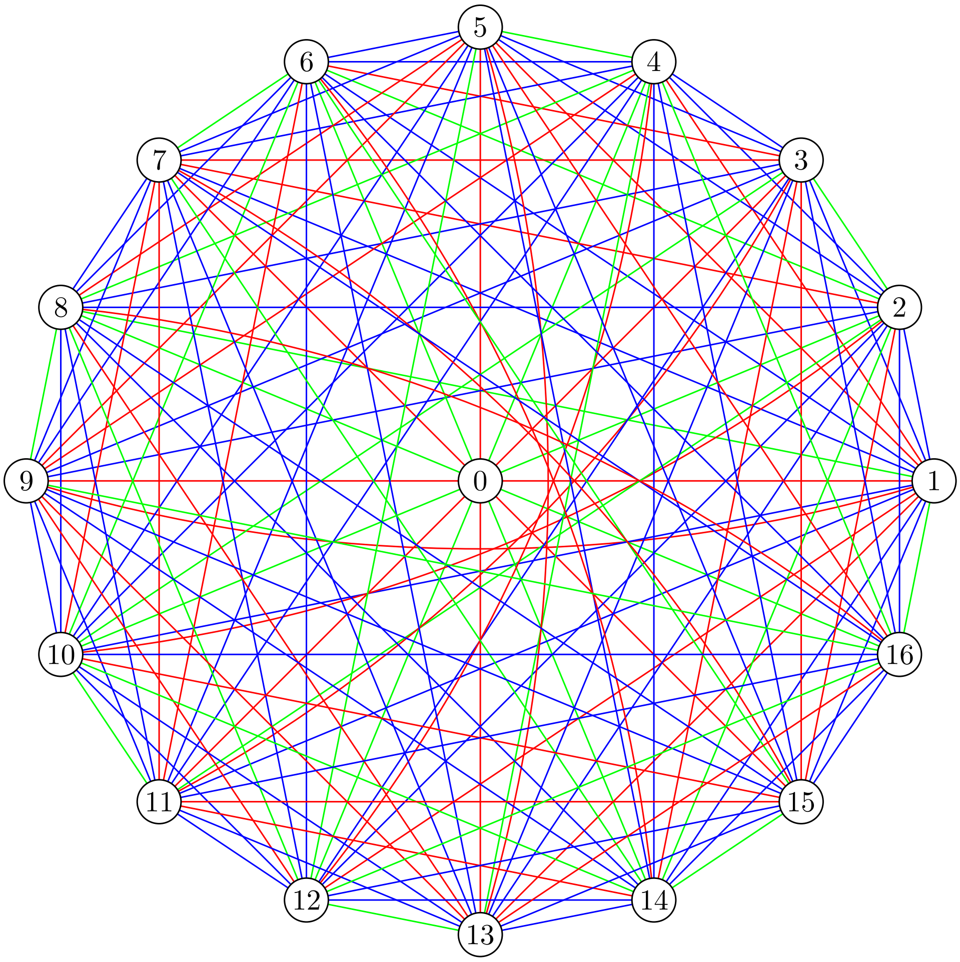


Figure 37: 8-flower example.

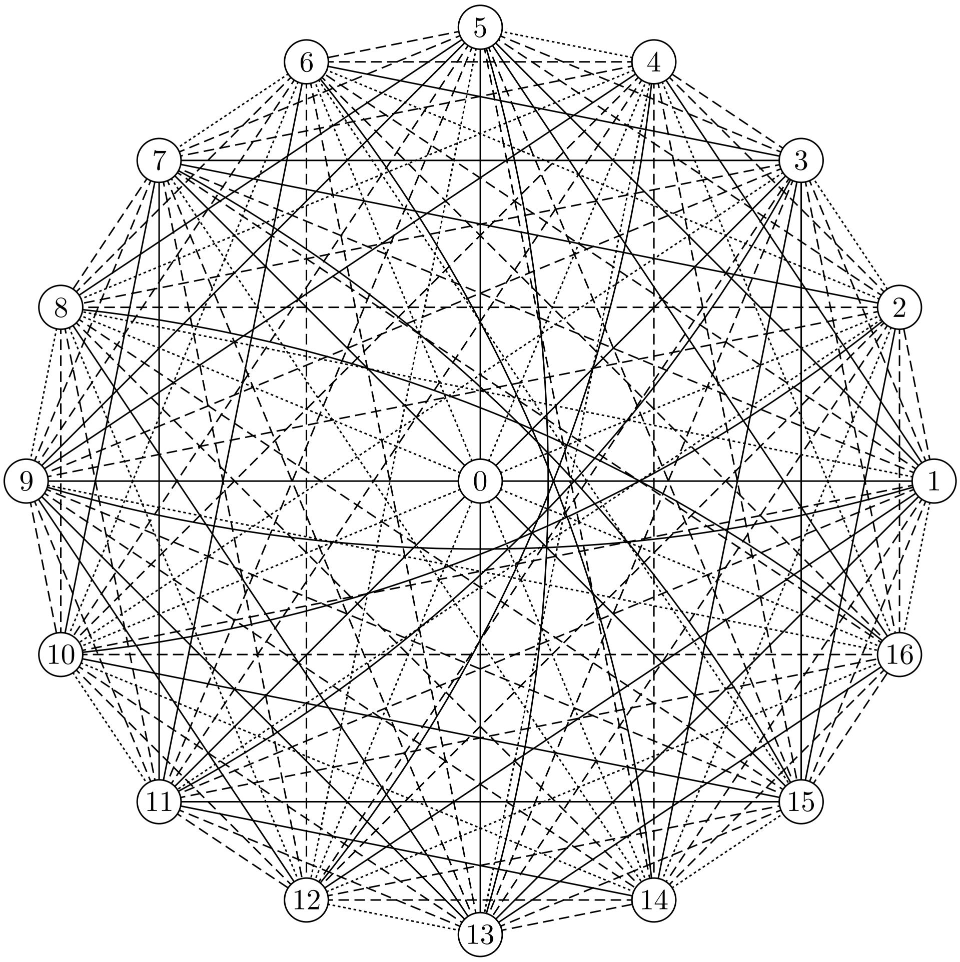


Figure 38: 8-flower example (in black and white for printing).

Level 0: RGRGRGRGRGRGRGRGRGRG
Level 1: BGBGBGBGBGBGBGBGBGBG
Level 2: BBBBBBBBBBBBBBBBBBBB
Level 3: RBRBRBRBRBRBRBRBRBRB
Level 4: RGRGRGRGRGRGRGRGRGRG
Level 5: BRBRBRBRBRBRBRBRBRBR
Level 6: BBBBBBBBBBBBBBBBBBBB
Level 7: RBRBRBRBRBRBRBRBRBRB
Level 8: RGRGRGRGRGRGRGRGRGRG
Level 9: BRBRBRBRBRBRBRBRBRBR
Level 10: BBBBBBBBBBBBBBBBBBBB

50 Δs: 50 settled
290 S_2: 290 settled
220 S_3: 120 settled
110 S_4: 0 settled

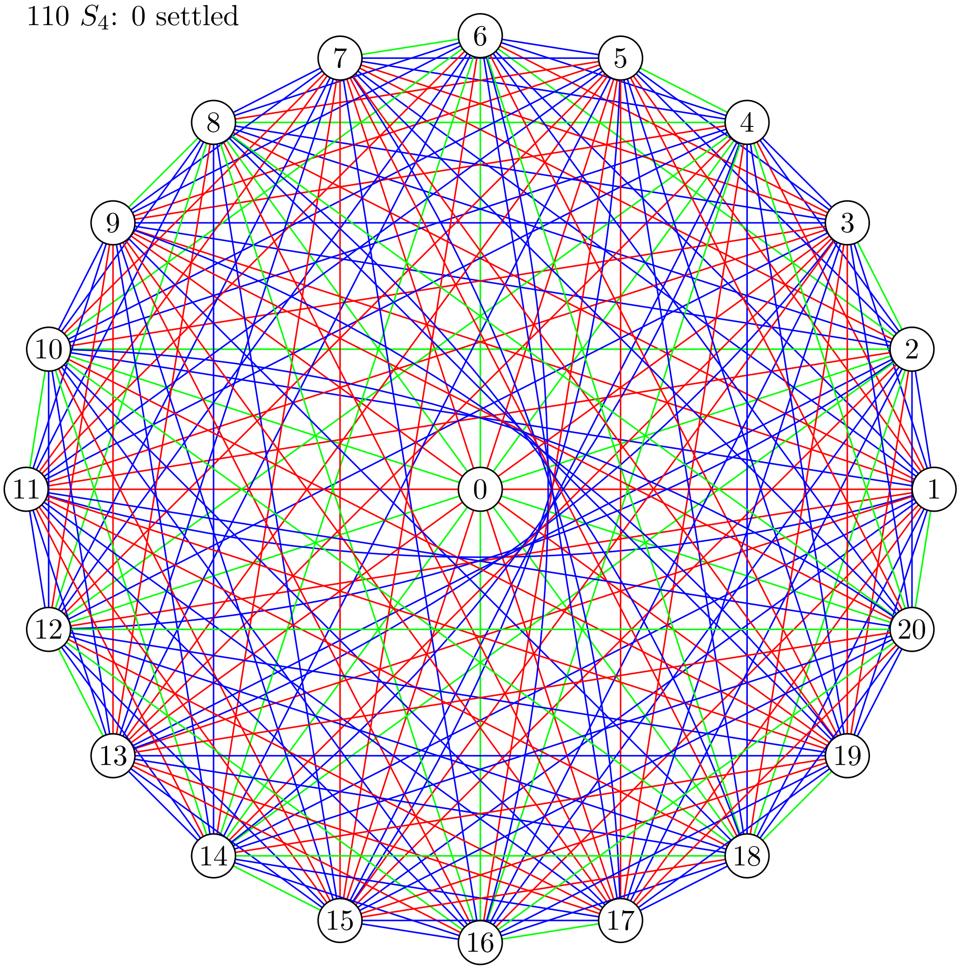


Figure 39: 10-flower example.

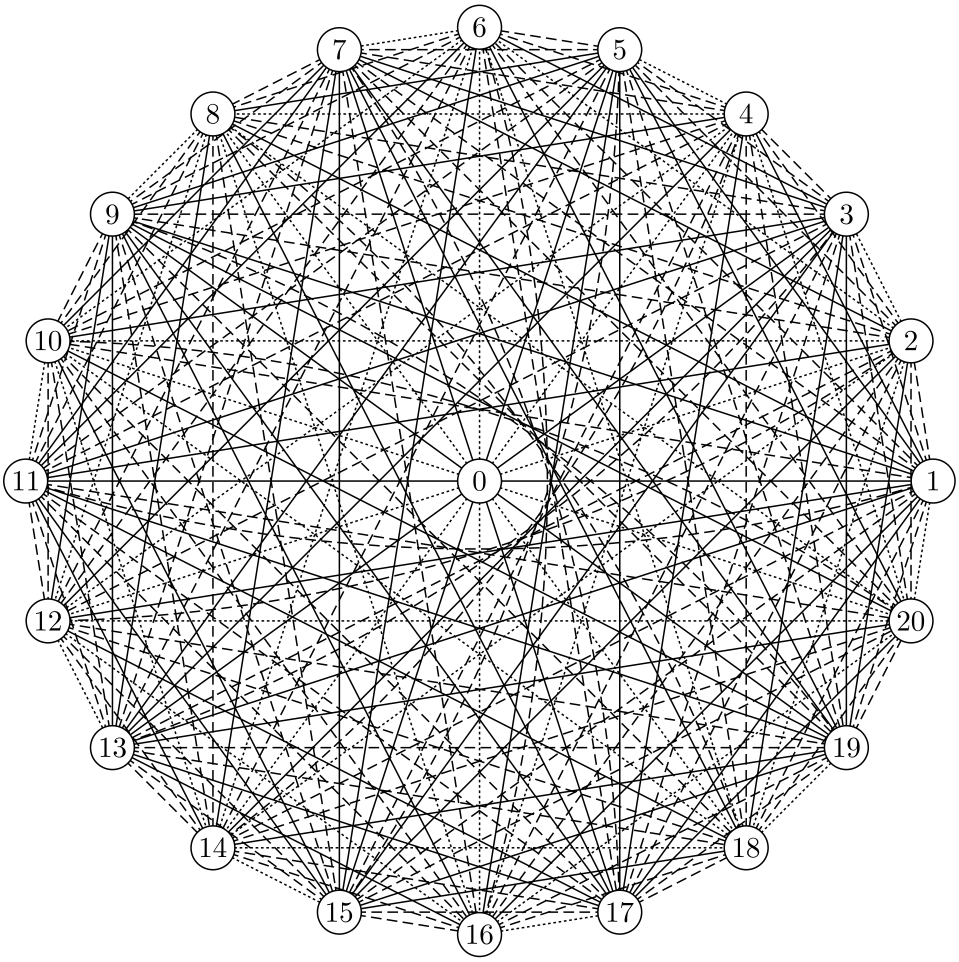


Figure 40: 10-flower example (in black and white for printing).

Level 0: RGRGRGRGRGRGRGRGRGRGRGRG
Level 1: BGBGBGBGBGBGBGBGBGBGBGBG
Level 2: BBBBBBBBBBBBBBBBBBBBBBBB
Level 3: RBRBRBRBRBRBRBRBRBRBRBRB
Level 4: RRRRRRRRRRRRRRRRRRRRRRRR
Level 5: BRBRBRBRBRBRBRBRBRBRBRBR
Level 6: BBBBBBBBBBBBBBBBBBBBBBBB
Level 7: RBRBRBRBRBRBRBRBRBRBRBRB
Level 8: RGRGRGRGRGRGRGRGRGRGRGRG
Level 9: BGBGBGBGBGBGBGBGBGBGBGBG
Level 10: BBBBBBBBBBBBBBBBBBBBBBBB
Level 11: RBRBRBRBRBRBRBRBRBRBRBRB
Level 12: RRRRRRRRRRRRRRRRRRRRRRRR

72 settled Δs
600 S_2: 600 settled
184 S_3: 76 settled
24 S_4: 0 settled

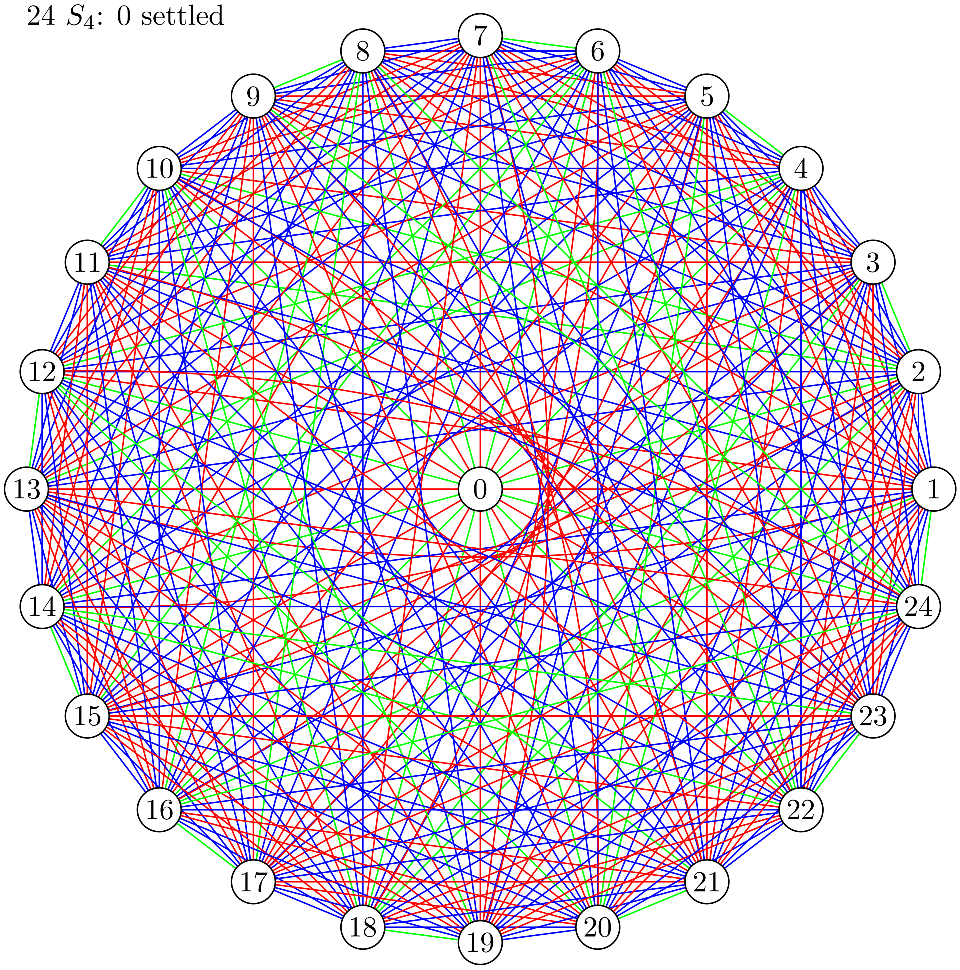


Figure 41: 12-flower example.

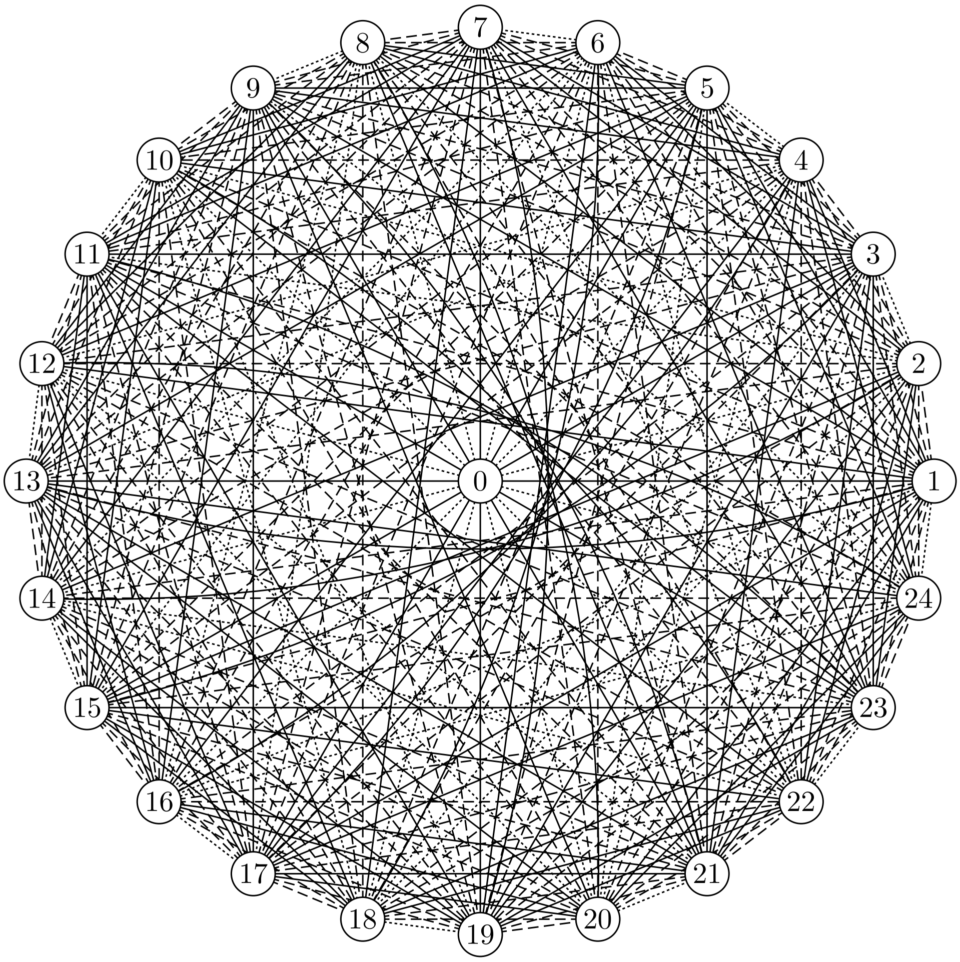


Figure 42: 12-flower example (in black and white for printing).

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