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Decomposing complete edge-chromatic graphs and hypergraphs

by

Vladimir Gurvich[1]
RUTCOR, Rutgers University
640 Bartholomew Road
Piscataway NJ 08854-8003
email: gurvich@rutcor.rutgers.edu

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ABSTRACT

A d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a complete graph whose edges are colored by d colors, or in other words, are partitioned into d subsets (some of which might be empty). We say that $\mathcal{G}$ is *complementary connected* if the complement to each chromatic component of $\mathcal{G}$ is connected on V, or in other words, if for each two vertices $u, w \in V$ and color $i \in I = \{1, \ldots, d\}$ there is a path between u and w without edges of E_i. We show that every such d-graph contains a subgraph Π or Δ , where Π has 4 vertices and 2 non-empty chromatic components each of which is a P_4, while Δ is the three-colored triangle. This simple statement implies that each Π- and Δ-free d-graph is uniquely decomposable in accordance with a tree $T = T(\mathcal{G})$ whose leaves are the vertices of V and other vertices of T are labeled by the colors of I. We can naturally interpret such a tree as a positional game (with perfect information and without moves of chance) of d players $I = \{1, \ldots, d\}$ and n outcomes $V = \{v_1, \ldots, v_n\}$. Thus, we get a one-to-one correspondence between these games and Π- and Δ-free d-graphs and, as a corollary, a characterization of the normal forms of positional games with perfect information. Another corollary of the above decomposition of d-graphs in case $d = 2$ is a characterization of the read-once Boolean functions. These results are not new; in fact, they are 25-35 years old. Yet, some important proofs did not appear in English.

Gyárfás and Simonyi recently proved a similar decomposition theorem for Δ-free d-graph. They showed that each such d-graph can be obtained from 2-graphs by substitutions. This theorem is based on results by Gallai, Cameron and Edmonds. We get some new applications of these results.

Key words: decomposition, graphs, hypergraphs, Gallai's graphs, positional games, read-once functions, substitution.

1 Complementary connected d-graphs contain Π or Δ

We consider d-graphs $\mathcal{G} = (V; E_1, \ldots, E_d)$ assuming that $d \geq 2$ is a fixed positive integral, while chromatic components E_i might be empty for some $i \in I = \{1, \ldots, d\}$. For example, we call $\mathcal{G}$ a 2- *or* 3-*graph* if $\mathcal{G}$ has only 2, respectively, 3, non-empty chromatic components.

The following 2-graph Π and 3-graph Δ given in Figure 1 will play an important role:

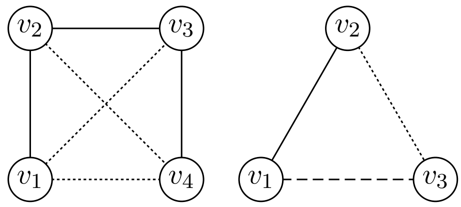


Figure 1: 2-graph Π and 3-graph Δ.

$\Pi = (V; E_1, E_2)$, where

$V = \{v_1, v_2, v_3, v_4\}$; $E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, and $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$;

$\Delta = (V; E_1, E_2, E_3)$, where

$V = \{v_1, v_2, v_3\}$, $E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, and $E_3 = \{(v_3, v_1)\}$.

The *complementary connected* (CC) d-graphs were defined in Abstract.

By convention, $\mathcal{G}$ is a CC d-graph if $|V| = 1$ and this one-vertex d-graph we will call *trivial*. Clearly, there is no CC d-graph with two vertices. It is easy to verify that Δ (respectively, Π) is a unique CC d-graph with three (respectively, four) vertices. It is also easy to see that Π and Δ are minimal CC d-graphs, that is, they do not contain non-trivial induced CC subgraphs. The next statement shows that, except Π and Δ, no other d-graph has this property.

Theorem 1. *Every non-trivial complementary connected d-graph contains* Π *or* Δ.

Proof. Given a Π- and Δ-free d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$, we will show that it is not CC, that is, the graph $G_i = (V, \overline{E_i}) = (V, \cup_{j \neq i} E_j)$ is not connected for some $i \in I$. (In the next section we will show that there is exactly one such $i \in I$.) Let us assume indirectly that $\mathcal{G}$ is CC and also Π- and Δ-free. Then $\mathcal{G}$ has the following property.

Lemma 1. *For each edge $(v', v'') \in E_i$ there exist a vertex $v \in V$ such that $(v, v'), (v, v'') \in E_j$ for some $j \neq i$.*

Proof. Since v', v'', and v cannot form a Δ, it would suffice to show that $(v, v'), (v, v'') \notin E_i$. Since $\mathcal{G}$ is complementary connected, there exists a path between v' and v'' that contains no edge from E_i. Let p be a shortest such path. Then each chord of p is of color i. Let ℓ be the length (that is, the number of edges) of p. Clearly, $\ell \neq 1$, because $(v', v'') \in E_i$. If $\ell = 2$ then $p = \{(v', v), (v, v'')\}$ and we are done. Let us show that if $\ell \geq 3$ then $\mathcal{G}$ contains a Π or Δ. Indeed, if p is monochromatic then a Π exists. Otherwise, p contains two successive edges of distinct colors, say, $(v_1, v_2) \in E_{i_1}$ and $(v_2, v_3) \in E_{i_2}$, where $i_1 \neq i_2$. Obviously, $i_1 \neq i$ and $i_2 \neq i$, since p contains no edges of color i. Thus, v_1, v_2, v_3 form a Δ. □

Now we proceed with the proof of Theorem 1 as follows.

Let $(v_{j_0}, v_{j_1}) \in E_{i_1}$. By Lemma 1, there exists $v_{j_2} \in V$ and $i_2 \in I$ such that $i_2 \neq i_1$ and $(v_{j_0}, v_{j_2}), (v_{j_1}, v_{j_2}) \in E_{i_2}$. Furthermore, since $(v_{j_1}, v_{j_2}) \in E_{i_2}$, by Lemma 1, there exists $v_{j_3} \in V$ and $i_3 \in I$ such that $(v_{j_1}, v_{j_3}), (v_{j_2}, v_{j_3}) \in E_{i_3}$ and $i_3 \neq i_2$, though $i_3 = i_1$ may hold.

Obviously, $v_{j_3} \neq v_{j_2}$ and $v_{j_3} \neq v_{j_1}$, by construction. It is also clear that $v_{j_3} \neq v_{j_0}$, because $(v_{j_0}, v_{j_2}) \in E_{i_2}$, while $(v_{j_3}, v_{j_2}) \in E_{i_3}$ and $i_3 \neq i_2$.

We will show that $(v_{j_3}, v_{j_0}) \in E_{i_3}$ too. Let us consider two cases: $i_3 = i_1$ and i_3 is distinct from both i_1 and i_2. If $i_3 = i_1$ then (v_{j_0}, v_{j_3}) must be of color i_1 too. Indeed, if $(v_{j_0}, v_{j_3}) \in E_{i_2}$ then all four vertices form a Π; if $(v_{j_0}, v_{j_3}) \in E_{i_4}$, where $i_4 \neq i_1$ and $i_4 \neq i_2$, then $(v_{j_0}, v_{j_2}, v_{j_3}$ form a Δ. If $i_3 \neq i_1$ and $i_3 \neq i_2$ then (v_{j_0}, v_{j_3}) must be in E_{i_3} too. Indeed, if $(v_{j_0}, v_{j_3}) \in E_{i_1}$ then $v_{j_0}, v_{j_2}, v_{j_3}$ form a Δ; if $v_{j_0}, v_{j_3}) \in E_{i_2}$ then $(v_{j_0}, v_{j_1}, v_{j_3}$ form a Δ; finally, if $(v_{j_0}, v_{j_3}) \in E_{i_4}$, where where $i_4 \neq i_1$ and $i_4 \neq i_2$, then both above triangles form Δs.

In general, we prove by induction that V cannot be finite. More precisely, we show that for each k there is a sequence of vertices $v_{j_0}, v_{j_1}, \ldots, v_{j_{k-1}}, v_{j_k}$ and colors $i_1, i_2, \ldots, i_{k-1}, i_k$ such that: (i) all vertices are pairwise distinct; (ii) though colors may coincide, yet, every two successive colors are distinct, that is, $i_m \neq i_{m+1}$ for every $m = 1, 2, \ldots, k-1$; and finally, (iii) $(v_{j_k}, v_{j_m}) \in E_{i_k}$ whenever $k > m$, that is, every vertex is connected by the same color to all preceding vertices.

Suppose that we already got such vertices $\{v_{j_0}, v_{j_1}, \ldots, v_{j_{k-1}}\}$ and colors $\{i_1, i_2, \ldots, i_{k-1}\}$ for $k-1$. Since $(v_{j_{k-2}}, v_{j_{k-1}}) \in E_{i_{k-1}}$, by Lemma 1, there is a vertex $v_{j_k} \in V$ such that $(v_{j_{k-2}}, v_{j_k})$, $(v_{j_{k-1}}, v_{j_k}) \in E_{i_k}$, where $i_k \neq i_{k-1}$. First, let us show that v_{j_k} is distinct from all preceding vertices, that is, $v_{j_k} = v_{j_m}$ for no $m < k$. Indeed, by the induction hypothesis, $(v_{j_{k-1}}, v_{j_m}) \in E_{i_{k-1}}$, while, by construction, $(v_{j_{k-1}}, v_{j_k}) \in E_{i_k}$ and $i_k \neq i_{k-1}$. Hence, $v_{j_k} \neq v_{j_m}$.

Now, let us prove that $(v_{j_k}, v_{j_m}) \in E_{i_k}$ for all $m < k$. Indeed, for $m = k-1$ and $m = k-2$ this holds by construction. Given $m < k-2$, let us consider four vertices $v_{j_{k-2}}, v_{j_{k-1}}, v_{j_k}$ and v_{j_m}. They are connected by six edges five of which are colored as follows: $(v_{j_{k-2}}, v_{j_k}), (v_{j_{k-1}}, v_{j_k}) \in E_{i_k}$, by construction; $(v_{j_{k-2}}, v_{j_{k-1}})$, $(v_{j_m}, v_{j_{k-1}}) \in E_{i_{k-1}}$, and $(v_{j_m}, v_{j_{k-2}}) \in E_{i_{k-2}}$, by the induction hypothesis.

Let us show that $(v_{j_m}, v_{j_k}) \in E_{i_k}$. We know that $i_k \neq i_{k-1} \neq i_{k-2}$, though i_k and i_{k-2} may coincide. If they do then $(v_{j_m}, v_{j_k}) \in E_{i_k}$. Indeed, if $(v_{j_m}, v_{j_k}) \in E_{i_{k-1}}$ then all four vertices, $v_{j_{k-2}}, v_{j_{k-1}}, v_{j_k}$, and v_{j_m}, form a Π; if $(v_{j_m}, v_{j_k}) \in E_{i_\ell}$ where $i_\ell \neq i_k$ and $i_\ell \neq i_{k-1}$ then $v_{j_{k-1}}, v_{j_k}$, and v_{j_m} form a Δ.

Now, let us suppose that $i_k \neq i_{k-2}$ and show that again $(v_{j_m}, v_{j_k}) \in E_{i_k}$. Indeed, if $(v_{j_m}, v_{j_k}) \in E_{i_{k-2}}$ then $v_{j_{k-1}}, v_{j_k}$, and v_{j_m} form a Δ; if $(a_{j_m}, a_{j_k}) \in E_{i_\ell}$ where $i_\ell \neq i_k$ and $i_\ell \neq i_{k-2}$, then $a_{j_{k-2}}, a_{j_k}$, and a_{j_m} form a Δ.

Finally, let us note that for any fixed k the d-graph induced by $V_k = \{v_{j_0}, v_{j_1}, \ldots, v_{j_{k-1}}, v_{j_k}\}$ is not complementary connected, just because v_{j_k} is an isolated vertex in $G_k = (V_k, \overline{E_{i_k}})$. Thus, V cannot be finite and we get a contradiction. □

Remark 1. *In fact, we proved a little more than Theorem 1 claims.*
Let us denote by $\mathcal{G}_\infty$ the family of infinite d-graphs satisfying all properties (i,ii,iii) mentioned above. It is easy to see that each $\mathcal{G} \in \mathcal{C}_\infty$ is complementary connected, though each finite subgraph of $\mathcal{G}$ is not. Let us mention that $\mathcal{G}_\infty$ contains only two graphs when $d = 2$, since in this case two colors must alternate.
Our arguments show that each complementary connected d-graph (finite or infinite) must contain a Π, or Δ, or an infinite subgraph from the family $\mathcal{G}_\infty$.

Remark 2. *The proof of Theorem 1 was given in [20]. The statement appears without proof in [22]. The case $d = 2$ is a little simpler than the general one, since Δ cannot exist when $d \leq 2$. This case was considered in [37, 38, 36, 19, 22]. It was also suggested as a problem for Moscow Mathematical Olympiad in 1971 (Problem 72 in [15]) and was successfully solved by five high school students.*

2 Decomposition of Π- and Δ-free d-graphs, π- and δ-free d-hypergraphs, and some applications

2.1 Decomposition tree

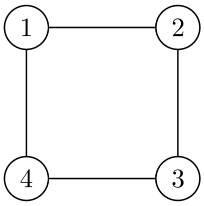
By Theorem 1, for any Π- and Δ-free d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ there exists an $i \in I$ such that the graph $G_i = (V, \overline{E_i}) = (V, \cup_{j \neq i} E_j)$ is not connected. The following lemma implies that there is exactly one such $i \in I$.

Lemma 2. *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the common vertex-set V such that both complementary graphs $\overline{G_1} = (V, \overline{E_1})$ and $\overline{G_2} = (V, \overline{E_2})$ are not connected. Then $E_1 \cap E_2 \neq \emptyset$,*

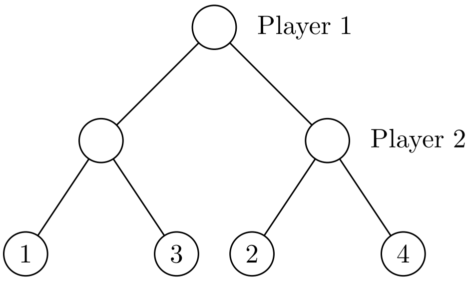
Proof. Let $V_i \subset V$ be a connected component of $\overline{G_i}$, then all edges between V_i and $V \setminus V_i$ belong to E_i, for both $i = 1$ and $i = 2$. Then $E_1 \cap E_2 \neq \emptyset$, since $V_i \neq \emptyset$ and $V_i \neq V$ for both $i = 1$ and $i = 2$. □

Given a Π- and Δ-free d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$, there exists a unique $i \in I$ such that $\overline{G_i} = (V, \overline{E_i})$ is not connected. Let us decompose it into connected components and consider the corresponding induced d-graphs (note that there are at least two of them). Each such d-graph $\mathcal{G}'$ is still Π- and Δ-free. Hence, there exists a unique $j \in I$ (note that $j \neq i$) such that ... etc. Thus, we get a decomposition tree $T = T(\mathcal{G})$ whose leaves are in one-to-one correspondence with $v_1, \ldots, v_n$, and all other vertices are labeled by $1, \ldots, d$.

Remark 3. *This decomposition was suggested in [20, 22]. Case $d = 2$ was considered before [14, 37, 38, 19, 21, 28, 27]. A more general, substitution or modular, decomposition was introduced by Gallai [14] and then studied in many papers; see [4, 5, 32, 34, 35] for a survey.*



$$\mathcal{S} = \{(1,3)(2,4)\}$$
$$\mathcal{C} = \{(1,2)(2,3)(3,4)(4,1)\}$$



	12	23	34	41
13	1	3	3	1
24	2	2	4	4

$$F = 13 \vee 24$$
$$F^d = (1 \vee 3)(2 \vee 4) = 12 \vee 13 \vee 34 \vee 41$$

Figure 2: A P_4-free graph and the corresponding positional and normal game forms.

2.2 Π- and Δ-free d-graphs and positional games

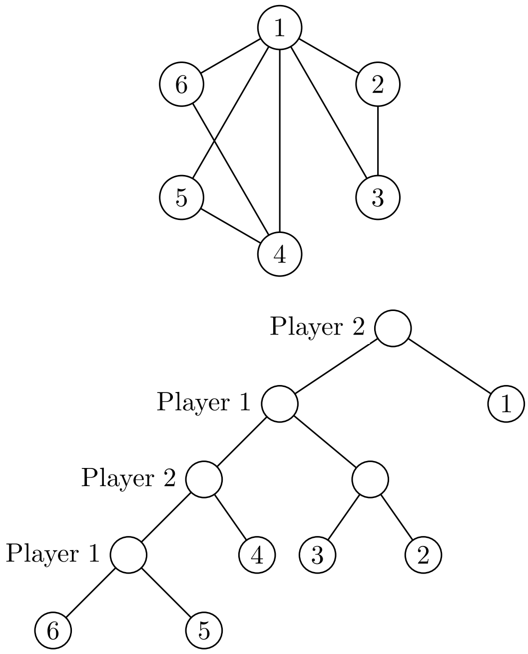
We can naturally interpret the above decomposition by $T = T(\mathcal{G})$ as a positional game (with perfect information and without moves of chance) in which $I = \{1, \ldots, d\}$ is the set of players and $V = \{v_1, \ldots, v_n\}$ is the set of outcomes.

We define this positional game P as follows. Let $T = (U \cup V, E)$ be a tree. Its vertices $U \cup V$ are *positions*; they correspond to subgraphs of $\mathcal{G}$ obtained by the decomposition. The leaves $V = L(T)$ are *final positions* or outcomes of the game; they are in one-to-one correspondence with the vertices of $\mathcal{G}$. To each non-final position $u \in U$ we assign a *player* $i = i(u) \in I$ who makes a *move* in u by choosing any successor u' of u. (This means that in the d-graph $\mathcal{G}(u)$ the complement to the chromatic component i is disconnected and one of its connected components form the d-graph $\mathcal{G}(u')$.) The game begins in the initial position s, corresponding to the original d-graph $\mathcal{G}$, and ends in a final position, which corresponds to a vertex v of $\mathcal{G}$. The unique path from s to v is called a *play*.

According to section 2.1, we must assume that there are at least two possible moves in each position and no player makes two moves in a row. Let us note, however, that these two assumptions do not reduce generality, since they can always be enforced by trivial modifications of a positional game.

Thus, to each Π- and Δ-free d-graphs $\mathcal{G}$ we assign a positional game $P = P(\mathcal{G})$.

Four examples are given in Figures 2-5. To simplify the figures we substitute j for v_j.

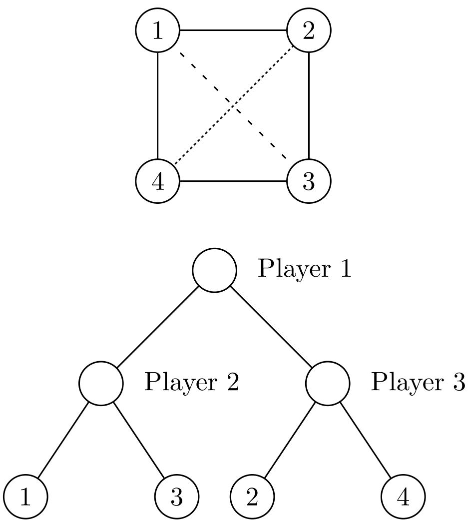


$$
\begin{aligned}
\mathcal{S} &= \{(1)(2,4)(3,4)(2,5,6)(3,5,6)\} \\
\mathcal{C} &= \{(1,2,3)(1,4,5)(1,4,6)\}
\end{aligned}
$$

	1	24	34	256	356
123	1	2	3	2	3
145	1	4	4	5	5
146	1	4	4	6	6

$$
\begin{aligned}
F &= 1(23 \vee 4(5 \vee 6)) &&= 123 \vee 145 \vee 146 \\
F^d &= 1 \vee (2 \vee 3)(4 \vee 56) &&= 1 \vee 24 \vee 34 \vee 256 \vee 356
\end{aligned}
$$

Figure 3: Another P_4-free graph and the corresponding positional and normal game forms.

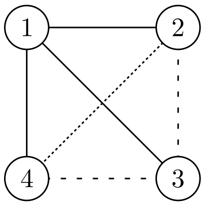


$$
\begin{aligned}
\mathcal{C}_1 &= \{(13)(24)\} \\
\mathcal{C}_2 &= \{(124)(234)\} \\
\mathcal{C}_3 &= \{(123)(134)\}
\end{aligned}
$$

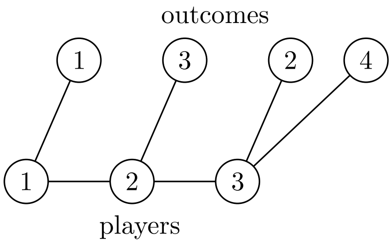
13			24	
1	1	124	2	4
3	3	234	2	4
1	1		1	1
2	3		2	3
3	4		3	4

$$
\begin{aligned}
F_1 &= 13 \vee 24 &&= 13 \vee 24 \qquad & F_{23} &= (1 \vee 3)(2 \vee 4) &&= 12 \vee 23 \vee 34 \vee 41 \\
F_2 &= (1 \vee 3)24 &&= 124 \vee 234 \qquad & F_{13} &= 13 \vee 2 \vee 4 &&= 13 \vee 2 \vee 4 \\
F_3 &= 13(2 \vee 4) &&= 123 \vee 134 \qquad & F_{12} &= 1 \vee 3 \vee 24 &&= 1 \vee 3 \vee 24
\end{aligned}
$$

Figure 4: A Π- and Δ-free 3-graph and the corresponding positional and normal game forms.



$$\begin{aligned}\mathcal{C}_1 &= \{(1)(234)\}\\ \mathcal{C}_2 &= \{(13)(124)\}\\ \mathcal{C}_3 &= \{(123)(134)\}\end{aligned}$$



1			234	
1	1	13	3	3
1	1	124	2	4
1	1		1	1
2	3		2	3
3	4		3	4

$$\begin{aligned}F_1 &= 1 \vee 324 &&= 1 \vee 234 & F_{23} &= 1(3 \vee 2 \vee 4) &&= 12 \vee 13 \vee 14\\ F_2 &= 1(3 \vee 24) &&= 13 \vee 124 & F_{13} &= 1 \vee 3(2 \vee 4) &&= 1 \vee 23 \vee 34\\ F_3 &= 13(2 \vee 4) &&= 123 \vee 134 & F_{12} &= 1 \vee 3 \vee 24 &&= 1 \vee 3 \vee 24\end{aligned}$$

Figure 5: Another Π- and Δ-free 3-graph and the corresponding positional and normal game forms.

2.3 Positional d-graphs

To show that the above mapping is bijective we construct the inverse mapping as follows.

Given a positional game P, it is not difficult to reconstruct $\mathcal{G}$ from $T = T(\mathcal{G}) = (U \cup V, E)$. For each $v_1, v_2 \in V$ let us consider the corresponding two plays in T: from s to v_1 and from s to v_2. Since T is a tree, these two plays first coincide and then separate. Let u be their last common position. We color (v_1, v_2) by the color $i = i(u)$, do so for all pairs of vertices in V, and denote the obtained d-graph by $\mathcal{G}(P)$. It is easy to see that we get exactly our original d-graph $\mathcal{G}$, that is, $\mathcal{G} = \mathcal{G}(P(\mathcal{G}))$. In particular, $\mathcal{G}(P)$ is Π- and Δ-free for any P. To see this it is sufficient to consider all positional games with 3 and 4 outcomes and verify that they do not generate Δ and Π, respectively.

We will call a d-graph $\mathcal{G}$ *positional* if it is obtained from a positional game P, that is, if $\mathcal{G} = \mathcal{G}(P)$ for some P. The arguments of the last two subsections are summarized as follows.

Theorem 2. *A d-graph $\mathcal{G}$ is positional if and only if it is Π- and Δ-free.* □

2.4 Positional d-hypergraphs

Given a positional game P, let us add to $T = (U \cup V, E)$ one extra vertex v_0 and edge (s, v_0) and denote the obtained tree by $T' = (U \cup V', E')$, where $V' = V \cup \{v_0\} = \{v_0, v_1, \ldots, v_n\}$ and $E' = E \cup \{(s, v_0)\}$. The vertex-set U and the mapping from U to $I = \{1, \ldots, d\}$ remain the same. Let us recall that $deg(u) \geq 3$ for each $u \in U$ and $i(u) \neq i(u')$ whenever u and u' are adjacent.

We get the original game P if we choose v_0 as the initial position. Yet, we can choose any

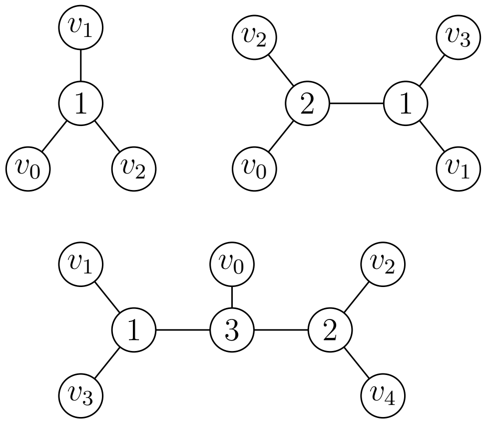


Figure 6: Three unrooted positional games with 3, 4 and 5 leaves.

$v \in V'$, as well. To distinguish positional games assigned to T and T' we call them *rooted* and *unrooted* and denote by P and P', respectively.

In sections 2.1-2.3 we assigned to the rooted games positional d-graphs and proved that they are exactly Π- and Δ-free d-graphs. In this sections we will obtained similar results for the unrooted positional games.

Let $\binom{V}{3}$ denote the set of all triplets from V'. Let us assign an arbitrary color $i \in I$ to each triplet and denote the obtained d-hypergraph by $\mathcal{H} = (V'; \mathcal{E}_1, \ldots, \mathcal{E}_d)$.

(In this paper we consider only the hypergraphs whose all hyperedges are of cardinality 3; respectively, we call them *triplets*).

Since T' is a tree, there is a unique path between any two its vertices. Furthermore, for any three distinct leaves $v_{j_1}, v_{j_2}, v_{j_3} \in V'$ there are three paths between them and there is a unique vertex $u = u(v_{j_1}, v_{j_2}, v_{j_3}) \in U$ that belongs to all three. To each triplet $v_{j_1}, v_{j_2}, v_{j_3} \in V'$ we assign the color $i(u)$, where $u = u(v_{j_1}, v_{j_2}, v_{j_3})$, and denote the obtained d-hypergraph by $\mathcal{H} = \mathcal{H}(P')$. We will call a d-hypergraph $\mathcal{H}$ *positional* if it can be obtained in this way, that is, if $\mathcal{H} = \mathcal{H}(P')$ for some unrooted positional game P'.

For example, let us consider three unrooted positional games P'_1, P'_2 and P'_3 in Figure 6. They define 1-, 2-, and 3-hypergraphs

$\mathcal{H}_1 = (V'_1; \mathcal{E}^1_1)$, $\mathcal{H}_2 = (V'_2; \mathcal{E}^2_1, \mathcal{E}^2_2)$, and $\mathcal{H}_3 = (V'_3; \mathcal{E}^3_1, \mathcal{E}^3_2, \mathcal{E}^3_3)$, where

$V'_1 = \{v_0, v_1, v_2)\}$, $V'_2 = \{v_0, v_1, v_2, v_3)\}$, $V'_3 = \{v_0, v_1, v_2, v_3, v_4)\}$;

$\mathcal{E}^1_1 = \{(v_0, v_1, v_2)\}$; $\mathcal{E}^2_1 = \{(v_1, v_3, v_0), (v_1, v_3, v_2)\}$, $\mathcal{E}^2_2 = \{(v_0, v_2, v_1), (v_0, v_2, v_3)\}$;

$\mathcal{E}^3_1 = \{(v_1, v_3, v_0), (v_1, v_3, v_2), (v_1, v_3, v_4)\}$, $\mathcal{E}^3_2 = \{(v_2, v_4, v_0), (v_2, v_4, v_1), (v_2, v_4, v_3)\}$,

$\mathcal{E}^3_3 = \{(v_0, v_1, v_2), (v_0, v_2, v_3), (v_0, v_3, v_4), (v_0, v_4, v_1)\}$.

Let us remark that merging some chromatic components of a positional d-graph or d-hypergraph results in another positional d-graph or d-hypergraph, respectively. Indeed, this operation is realized by merging the corresponding players in the corresponding game, which results in another game. For example, merging the colors 1 and 2 in $\mathcal{H}_3$ we get $\mathcal{H}'_3 = (V; \mathcal{E}^3_{1,2}, \mathcal{E}^3_3)$, where $\mathcal{E}^3_{1,2} = \mathcal{E}^3_1 \cup \mathcal{E}^3_2$.

Let us note that any induced subhypergraph of a positional d-hypergraphs is positional. In other words, like positional d-graphs, positional d-hypergraphs form a hereditary family. Hence, to characterize them it is sufficient to find all minimal non-positional d-hypergraphs. We show that, up to an isomorphism, there are only four of them.

First, there exists only one d-hypergraph with 3 vertices, $\mathcal{H}_1$, and it is positional.

There are two positional d-hypergraphs with 4 vertices: one is $\mathcal{H}_2$ and the other one is obtained from it by merging colors 1 and 2. All other d-hypergraphs with 4 vertices are not positional. There are 3 of them: δ_2, δ_3, and δ_4. They have the same vertex-set $V' = \{v_0, v_1, v_2, v_3\}$ and the same 4 triplets $\{(v_0, v_1, v_2),\ (v_0, v_1, v_3),\ (v_0, v_2, v_3),\ (v_1, v_2, v_3)\}$ that are colored in δ_ℓ by ℓ colors; where $\ell \in \{2, 3, 4\}$. In other words,

$\delta_2 = (V'; \mathcal{E}^2_1, \mathcal{E}^2_2)$, $\delta_3 = (V'; \mathcal{E}^3_1, \mathcal{E}^3_2, \mathcal{E}^3_3)$, and $\delta_4 = (V'; \mathcal{E}^4_1, \mathcal{E}^4_2, \mathcal{E}^4_3, \mathcal{E}^4_4)$, where
$\mathcal{E}^2_1 = \{(v_0, v_1, v_2)\}$, $\mathcal{E}^2_2 = \{(v_0, v_1, v_3),\ (v_0, v_2, v_3),\ (v_1, v_2, v_3)\}$,
$\mathcal{E}^3_1 = \{(v_0, v_1, v_2)\}$, $\mathcal{E}^3_2 = \{(v_0, v_1, v_3)\}$, $\mathcal{E}^3_3 = \{(v_0, v_2, v_3),\ (v_1, v_2, v_3)\}$, and
$\mathcal{E}^4_1 = \{(v_0, v_1, v_2)\}$, $\mathcal{E}^4_2 = \{(v_0, v_1, v_3)\}$, $\mathcal{E}^4_3 = \{(v_0, v_2, v_3)\}$, $\mathcal{E}^4_4 = \{(v_1, v_2, v_3)\}$.

Note that we get δ_3 (respectively, δ_2) by merging two colors of δ_4 (respectively, δ_3).

For the sake of brevity, we call a d-hypergraph δ-free if it is δ_2-,δ_3-, and δ_4-free.

We will show that, except δ_2, δ_3, and δ_4, there is only one more forbidden d-hypergraph $\pi = (V'; \mathcal{E}_1, \mathcal{E}_2)$ with 5 vertices $V' = \{v_0, v_1, v_2, v_3, v_4\}$ and 2 chromatic components

$\mathcal{E}_1 = \{(v_0, v_1, v_2), (v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_0), (v_4, v_0, v_1)$ and
$\mathcal{E}_2 = \{(v_0, v_1, v_3), (v_1, v_2, v_4), (v_2, v_3, v_0), (v_3, v_4, v_1), (v_4, v_0, v_2)$.

Theorem 3. *A d-hypergraph $\mathcal{H}$ is positional if and only if it is π- and δ-free.*

Proof. The "only if" part is easy. It is sufficient to consider all unrooted positional games with 4 and 5 outcomes and verify that between the corresponding 4- and 5-hypergraphs π and δ do not appear. All these games are either given in Figure 6 or can be obtained from them by merging players. □

To prove the "if" part we will need the following concept of projection.

Given a d-hypergraph $\mathcal{H} = (V; \mathcal{E}_1, \ldots, \mathcal{E}_d)$, and a vertex $v \in V$, let us define a d-graph $\mathcal{G} = (V \setminus \{v\}; E_1, \ldots, E_d)$ as follows: $(v', v'') \in E_i$ if and only if $(v, v', v'') \in \mathcal{E}_i$, where $v', v'' \in V \setminus \{v\}$ and $i \in I = \{1, \ldots, d\}$. We will call $\mathcal{G}$ a *projection* of $\mathcal{H}$ from v and denote it by $\mathcal{G} = p(\mathcal{H}, v)$.

By this definitions, we have $\mathcal{G} = p(\mathcal{H}, v_0)$ for $\mathcal{G} = \mathcal{G}(P)$ and $\mathcal{H} = \mathcal{H}(P')$, where the corresponding trees T and T' differ by one vertex v_0 and edge (s, v_0) added to T.

Lemma 3. *Any projection of any δ-free d-hypergraph is a Δ-free graph.*

Proof. Assume indirectly that $p(\mathcal{H}, v_0\}$ contains a Δ on v_1, v_2, v_3. Then $\{v_1, v_2, v_3\}$ and v_0 induce δ_2, δ_3, or δ_4 and we get a contradiction. □

It is also easy to verify that all 5 projections of π are isomorphic to Π.

For example, projection from v_0 results in $\mathcal{G} = p(\pi, v_0) = (V; E_1, E_2)$, where

$$\begin{tikzcd} P' \arrow[r, leftrightarrow, "v_0"] \arrow[d] & P \arrow[d, leftrightarrow] \\ \mathcal{H} \arrow[r, "\text{projection}", "\text{from } v_0"'] & \mathcal{G} \end{tikzcd}$$

Figure 7: Commutative diagram.

$V = \{v_1, v_2, v_3, v_4\}$; $E_1 = \{(v_1, v_2), (v_3, v_4), (v_4, v_1)\}$ and $E_2 = \{(v_1, v_3), (v_2, v_3), (v_2, v_4)\}$.

We can formulate an inverse claim as follows.

Lemma 4. *If a projection of a δ-free 2-hypergraph $\mathcal{H} = (V'; \mathcal{E}_1, \mathcal{E}_2)$ contains Π then $\mathcal{H}$ contains π.*

Proof. Let us assume without loss of generality that $v_0 \in V'$ and that $p(\mathcal{H}, v_0)$ contains the subgraph $\Pi = p(\pi, v_0) = (V, E_1, E_2)$ given above, where $V = V' \setminus \{v_0\}$. By the definition of projection we have

$(v_0, v_1, v_2), (v_4, v_0, v_1), (v_3, v_4, v_0) \in \mathcal{E}_1$ and $(v_0, v_1, v_3), (v_4, v_0, v_2), (v_2, v_3, v_0) \in \mathcal{E}_2$.

Furthermore, we conclude that $(v_2, v_3, v_4) \in \mathcal{E}_1$. Indeed, otherwise δ_2 appears, since (v_4, v_0, v_2), $(v_2, v_3, v_0) \in \mathcal{E}_2$ and $(v_3, v_4, v_0) \in \mathcal{E}_1$. Similarly, we conclude that $(v_1, v_2, v_3) \in \mathcal{E}_1$, (v_1, v_2, v_4), $(v_3, v_4, v_1) \in \mathcal{E}_2$. Thus, $\mathcal{H}$ contains a π. $\square$

Now the "if part" of Theorem 3 follows. Indeed, let $\mathcal{H} = (V'; \mathcal{E}_1, \mathcal{E}_2)$ be π- and δ-free, $V' = V \cup \{v_0\}$, and $\mathcal{G} = (V; E_1, E_2) = p(\mathcal{H}, v_0)$. In other words, a d-graph $\mathcal{G}$ is a projection from v_0 of a π- and δ-free d-hypergraph $\mathcal{H}$. Then $\mathcal{G}$ is Δ-free, by Lemma 3, and it is Π-free, by Lemma 4. Hence, by Theorem 2, $\mathcal{G} = \mathcal{G}(P)$, where P is a (rooted) positional game. Let s be its root. Let us add to P one new vertex v_0 and one new edge (v_0, s) and denote the obtained unrooted positional game by P'. It is easy to verify that $\mathcal{H} = \mathcal{H}(P')$. $\square$.

In fact, we proved that the Diagram in Figure 7 is commutative and all its mappings are bijective. In this diagram P and P' stand for rooted and unrooted positional games, $\mathcal{G}$ for Π- and Δ-free d-graphs, and $\mathcal{H}$ for π- and δ-free d-hypergraphs.

Let us also remark that we can generalize Lemma 4 as follows.

Proposition 1. *Any δ-free d-hypergraph is uniquely defined by any its projection.*

Proof. Let $p(\mathcal{H}, v_0) = \mathcal{G}$, where $\mathcal{G}$ is a given d-graph. By the definition of projection, for all $i \in I$ and $v', v'' \in V$ we have $(v_0, v', v'') \in \mathcal{E}_i$ in $\mathcal{H}$ if and only if $(v', v'') \in E_i$ in $\mathcal{G}$.

Now let us consider 3 vertices $v, v', v'' \in V$ distinct from v_0 and show that a color of the triplet (v, v', v'') in $\mathcal{H}$ is uniquely determined by a given coloring of the 3 edges $(v, v'), (v, v'')$ and (v', v'') in $\mathcal{G}$. Let us consider the following 3 cases.

(i) All three edges in $\mathcal{G}$ are colored by the same color , that is, (v, v'),(v, v''),$(v', v'') \in E_i$ for some $i \in I$. Then (v_0, v, v'),(v_0, v, v''),$(v_0, v', v'') \in \mathcal{E}_i$ in $\mathcal{H}$. Hence, the triplet (v, v', v'')

in $\mathcal{H}$ must be colored by the same color too, that is, $(v, v', v'') \in \mathcal{E}_i$, since otherwise the quadruple $\{v_0, v, v', v''\}$ form a δ_2 in $\mathcal{H}$.

(ii) The three edges in $\mathcal{G}$ are colored by two colors $i, j \in I$, say (v, v'),$(v, v'') \in E_i$ and $(v', v'') \in E_j$ Then (v_0, v, v'),$(v_0, v, v'') \in \mathcal{E}_i$ and $(v_0, v', v'') \in \mathcal{E}_j$ in $\mathcal{H}$. Hence, the triplet (v, v', v'') in $\mathcal{H}$ must be of color j, that is, $(v, v', v'') \in \mathcal{E}_j$, since otherwise the quadruple $\{v_0, v, v', v''\}$ form a δ_2 in $\mathcal{H}$.

(iii) The three edges in $\mathcal{G}$ are colored by the 3 distinct colors in $\mathcal{G}$, or in other words, $v, v', v'' \in V$ form a Δ. Clearly, in this case the quadruple $\{v_0, v, v', v''\}$ form δ_3 or δ_4 in $\mathcal{H}$. Hence, this case is impossible. □

This Proposition and Lemma 3 imply the following claim.

Theorem 4. *Projection $\mathcal{G} = p(\mathcal{H}, v_0)$ is a one-to-one correspondence between Δ-free d-graphs and δ-free d-hypergraphs with a fixed vertex.*

Let us note, however, that projections from different vertices may be not isomorphic. (Though, due to symmetry, all projections of π are isomorphic to Π.) For example, let us consider the unrooted game P' with 5 leaves in Figure 6. By Theorem 3, the corresponding d-hypergraph is π- and δ-free. Hence, its projection from any vertex is a Π and Δ-free d-graph. However, the projections from v_0 and from v_1 are not isomorphic. Similarly, we get two non-isomorphic rooted trees P_0 and P_1 by deleting, respectively, v_0 and v_1 from P'.

Remark 4. *The proofs of Proposition 1 and Theorems 3 and 4 were sketched in [22].*

2.5 Read-once Boolean functions

A monotone Boolean function is called *read-once* if it has a $(\vee, \wedge)$-formula in which each variable appears only once. For example, F_1 and F_2 are read-once, while F_3 and F_4 are not.

$F_1 = v_1v_2 \vee v_2v_3 \vee v_3v_4 \vee v_4v_1 = (v_1 \vee v_3)(v_2 \vee v_4)$,

$F_2 = v_1v_2v_3 \vee v_1v_4v_5 \vee v_1v_4v_6 = v_1(v_2v_3 \vee v_4(v_5 \vee v_6))$;

$F_3 = v_1v_2 \vee v_2v_3 \vee v_3v_1, \quad F_4 = v_1v_2 \vee v_2v_3 \vee v_3v_4.$

Given a function F, we define its co-occurrence graph $G(F) = (V, E)$ as follows. The vertices of $G(F)$ are all essential variables of F. Two vertices $v, v' \in V$ are connected by an edge if and only if the corresponding two variables belong to a prime implicant of F. See examples in Figures 2 and 3.

Obviously, if F is read-once then the dual function F^d is read-once too. Indeed, by de Morgan rules, $(F \vee F')^d = F^d \wedge F'^d$ and $(F \wedge F')^d = F^d \vee F'^d$, we get a read-once formula for F^d from a read-once formula for F by simply exchanging $\vee$ to $\wedge$ and vice versa.

Theorem 5. *[16, 19, 20, 24, 10, 11, 29]. The following properties of a monotone Boolean function F are equivalent:*

(i) F is read-once;

(i') F^d is read-once;

(ii) F is normal and $G(F)$ is P_4-free;

(ii') F^d is normal and $G(F^d)$ is P_4-free;

(iii) graphs $G(F)$ and $G(F^d)$ are edge-disjoint;

(iv) graphs $G(F)$ and $G(F^d)$ are edge-complementary;

(v) graphs $G(F)$ and $G(F^d)$ are edge-complementary and the obtained 2-graph is Π-free;

(vi) Any two prime implicants of F and F^d have exactly one common variable.

This Theorem is announced in [19] and proved in [20]. An improved (and simplified) version of this proof is given in [16]. It is based on Theorem 2 for $d = 2$ and on a dual subimplicant criterion [3]. This criterion, given a DNF of F, provides necessary and sufficient conditions for a set of variables to be contained by a prime implicant of F^d. Alternative proofs can be found in [10, 11, 29].

It is easy to verify that for functions F_1 and F_2 given above all claims of Theorem 5 hold (see Figures 2 and 3), while for F_3 and F_4 none of them holds. Indeed, F_3 is self-dual, that is, $F_3^d = F_3 = v_1v_2 \vee v_2v_3 \vee v_3v_1$ and $F_4^d = v_1v_3 \vee v_3v_2 \vee v_2v_4$. Hence, $G(F_3) = G(F_3^d)$, while $G(F_4)$ and $G(F_4^d)$ also have a common edge, namely, (v_2, v_3).

2.6 Normal form of positional games

Let P be a positional game, where $T = (U \cup V, E)$ is a rooted tree, s is the root, and $V = \{v_1, \dots, v_n\}$ and $I = \{1, \dots, d\}$ are the sets of outcomes and players, respectively.

A *strategy* of a player $i \in I$ is a mapping that assigns a move (u, u') to each position $u \in U$ such that $i(u) = i$. In other words, a strategy of a player i is a plan prescribing how i should play in any possible position. Let X_i be the set of all strategies of $i \in I$ and $X = \prod_{i \in I} X_i$. The n-tuples $x = (x^1, \dots, x^d) \in X$ are called *situations.* Every situation $x \in X$ uniquely defines a play that starts in the initial position s and ends in a final position $v = v(x) \in V$. The obtained mapping $g = g(P) : X \to V$ is called the normal form of P.

Four examples are given in Figures 2-5; the first two are 2-person and the last two are 3-person games. Respectively, their normal forms are 2- and 3-dimensional tables.

Let us remark that the mapping g is not injective, unless T is a star with the center s. In other words, the same outcome may occur in several situations.

Two strategies x_1^i and x_2^i of a player $i \in I$ are called *equivalent* if $g(x_1^i, x^{I \setminus \{i\}}) = g(x_2^i, x^{I \setminus \{i\}})$ for any set of strategies $x^{I \setminus \{i\}}$ of the remaining $d - 1$ players. We will merge equivalent strategies and leave only one representative of each equivalence class; see four examples in Figures 2-5.

In general, the normal form games are considered independently on the positional ones and are defined as follows. Let $I = \{1, \dots, d\}$ and $V = \{v_1, \dots, v_n\}$ be sets of players and outcomes, respectively; X_i be a set of all strategies of $i \in I$ and $X = \prod_{i \in I} X_i$ be a set of situations. We define a normal game form g as a mapping $g : X \to V$.

A game form g is called *positional* if $g = g(P)$ for a positional game P. The following simple characterization of positional game forms [20, 21] is based on Theorem 2.

A game form $g : X \to V$ is called *rectangular* if the following implication holds:

$g(x_1) = g(x_2) = v \;\Rightarrow\; g(x) = v \;\; \forall x, x_1, x_2 \in X$ such that $x^i = x^i_1$ or $x^i = x^i_2 \quad \forall i \in I$;

in other words, the implication holds for the situations $x, x_1, x_2 \in X$ whenever x is a mixture of x_1 and x_2. For example, all four game forms in Figures 2-5 are rectangular.

In general, it is easy to see that every positional game form is rectangular. Indeed, let two situations $x_1, x_2 \in X$ generate the same play p in P and let $x \in X$ be a mixture of x_1 and x_2. Then in each position u from p all three strategies $x^i_1, x^i_2, x^i \in X_i$ of the player $i = i(u)$ prescribe to stay in p. Hence, $g(x) = g(x_1) = g(x_2)$.

Subsets $K \in 2^I$ and $B \in 2^V$ are called *coalitions (of players)* and *blocks (of outcomes).*

Given a game form $g : X \to V$, we say that a (non-empty) coalition $K \subseteq I$ *is effective* for a block $B \subseteq V$ if there exists a strategy $x^K = \{x^i, i \in K\} \in X_K$ such that $g(x^K, x^{I\setminus K}) \in B$ for every strategy $x^{I\setminus K} = \{x^i, i \notin K\} \in X_{I\setminus K}$ of the complementary coalition $I \setminus K$, or in other words, if coalition K can guarantee that some outcome from B will appear whatever the rest of the players do. We will use the notation $\mathcal{E}_g(K, B) = 1$ if K is effective for B and $\mathcal{E}_g(K, B) = 0$ otherwise; $\mathcal{E}_g$ is called the *effectivity function* of a game form g.

Clearly, effectivity functions of game forms are monotone,

$\mathcal{E}_g(K, B) = 1,\; K \subseteq K' \subseteq I,\; B \subseteq B' \subseteq A \;\;\Rightarrow\;\; \mathcal{E}_g(K', B') = 1,$

superadditive,

$\mathcal{E}_g(K_1, B_1) = 1,\; \mathcal{E}_g(K_2, B_2) = 1,\; K_1 \cap K_2 = \emptyset \;\Rightarrow\;\; \mathcal{E}_g(K_1 \cup K_1, B_1 \cap B_2) = 1,$

and satisfy the following “boundary conditions”:

$\mathcal{E}_g(K, B) = 1$ if $K \neq \emptyset,\; B = V$ or $K = I,\; B \neq \emptyset$;

$\mathcal{E}_{\mathcal{G}}(K, B) = 0$ if $K = \emptyset,\; B \neq V$ or $K \neq I,\; B = \emptyset$.

By definition, $\mathcal{E}_g(I, \emptyset) = 0$ and we also assume that $\mathcal{E}_g(\emptyset, A) = 1$. Hence, by monotonicity, $\mathcal{E}_g(K, \emptyset) = 0$ and $\mathcal{E}_g(K, V) = 1$ for every $K \subseteq I$.

Remark 5. *Moulin and Peleg [33] proved that the above properties (monotonicity, superadditivity and boundary conditions) characterize the effectivity functions of the game forms.*

Obviously, the equalities $\mathcal{E}_g(K, B) = 1$ and $\mathcal{E}_g(I \setminus K, V \setminus B) = 1$ cannot hold simultaneously; in other words, two complementary (disjoint) coalitions cannot be effective for two complementary (disjoint) blocks. Indeed, if they are then, by superadditivity, we have $\mathcal{E}_g(I, \emptyset) = 1$, that is, $g(x^K, x^{I\setminus K}) \in (B \cap (V \setminus B)) = \emptyset$ for some situation $x = (x^K, x^{I\setminus K}) \in X$, in contradiction to the boundary conditions.

Yet, the opposite equalities, $\mathcal{E}_g(K, B) = 0$ and $\mathcal{E}_g(I \setminus K, V \setminus B) = 0$, can both hold. If they cannot then the game form is called *tight.* In other words, g is tight if

$$\mathcal{E}_g(K, B) = 0 \;\;\Rightarrow\;\; \mathcal{E}_g(I \setminus K, V \setminus B) = 1;\; \forall K \subseteq I, \forall B \subseteq V.$$

We will call game form g *weakly tight* if

$$\mathcal{E}_g(\{i\}, B) = 0 \;\;\Rightarrow\;\; \mathcal{E}_g(I \setminus \{i\}, V \setminus B) = 1;\;\; \forall\;\; i \in I$$

and *very weakly tight* if the above implication holds for all $i \in I$ but at most one.

By definition, for $d \leq 3$ the notions of tightness and weak tightness coincide, yet, for $n > 3$ tightness is essentially stronger. Furthermore, all three concepts (tightness, weak tightness, and very weak tightness) coincide if $d \leq 2$. It is shown in [20] that all three are also equivalent for rectangular game forms and arbitrary d.

Theorem 6. *A game form is positional if and only if it is rectangular and (very weakly) tight.*

Remark 6. *This theorem was proved in [20] and announced without proof in [21].*

It is not difficult to verify that all four game forms in Figures 2-5 are tight and rectangular. The concept of tightness can be reformulated in terms of Boolean duality as follows. Let us assign a Boolean variable to each outcome $v \in V$ and the DNF

$$F_K = \bigvee_{B \mid \mathcal{E}_g(K,B)=1} \bigwedge_{v \in B} v$$

to every coalition $K \subseteq I$. See four examples in Figures 2-5.

Then g is tight (respectively, (very) weakly tight) if F_K and $F_{I \setminus K}$ are dual for all $K \subseteq I$ (respectively, for all (but one) $K = \{i\}$; $i \in I$).

Remark 7. *A game form g is called* Nash-solvable *if for any payoff function $u : I \times V \rightarrow \mathbb{R}$ the obtained game (g, u) has at least one Nash equilibrium in pure strategies. A two-person ($d = 2$) game form g is Nash-solvable if and only if it is tight [18, 20, 23]. For zero-sum games this result was obtained earlier [9, 17]. However, for $d \geq 3$ tightness is neither necessary nor sufficient condition for Nash-solvability [23].*

3 Decomposing Δ-free d-graphs

3.1 Decomposing Gallai's d-graphs into 2-graphs by substitution

In the literature Δ-free d-graphs are known as *Gallai's graphs*, since they were introduced by Gallai in [14]. We will call them *Gallai's* d-graph which is more accurate. Gallai's d-graphs are well studied [1, 2, 6, 7, 8, 13, 26, 30, 31]. In particular, it is well-known that they are closed under substitution.

Let us substitute a d-graph $\mathcal{G}''$ for a vertex v of a d-graph $\mathcal{G}'$ and denote the obtained d-graph by $\mathcal{G} = \mathcal{G}(\mathcal{G}', v, \mathcal{G}'')$.

In is easy to see that then $\mathcal{G}$ contains both G' and G'' as induced subgraphs.

A family $\mathcal{F}$ of d-graphs is *closed* (respectively, *exactly closed*) *under substitution* if $\mathcal{G} \in F$ whenever (respectively, if and only if) $\mathcal{G}' \in F$ and $\mathcal{G}'' \in F$.

Remark 8. *Of course, we can apply these definitions to standard graphs (instead of d-graphs) as well. It is sufficient to fix $d = 2$.*

The following claim shows that Gallai's d-graphs are exactly closed under substitution.

Proposition 2. *A d-graph $\mathcal{G} = \mathcal{G}(\mathcal{G}', v, \mathcal{G}'')$ contains a Δ if and only if both $\mathcal{G}'$ and $\mathcal{G}''$ contain it.*

Proof. Let $\mathcal{G}$ contain a Δ. Clearly, this Δ cannot have exactly one edge in $\mathcal{G}''$, because then two remaining edges are of the same color. If it contains two edges in $\mathcal{G}''$ then the third one is in $\mathcal{G}''$ too and, hence, G'' contains a Δ. Finally, if all three edges are in $\mathcal{G}'$ then G' contains a Δ. Conversely, if $\mathcal{G}'$ or $\mathcal{G}''$ contains a Δ then $\mathcal{G}$ contains it too, since both $\mathcal{G}'$ and $\mathcal{G}''$ are induced subgraphs of $\mathcal{G}$. □

It is also known that each Gallai d-graph can be obtained from 2-graphs by substitutions.

Theorem 7. *(Cameron and Edmonds, [6]; Gyárfás and Simonyi, [26]).*
Let $\mathcal{G}$ be Gallai's d-graph with at least 3 non-trivial chromatic components. Then $\mathcal{G} = \mathcal{G}(\mathcal{G}', v, \mathcal{G}'')$, where $\mathcal{G}'$ and $\mathcal{G}''$ are non-trivial Gallai's d-graphs.

Clearly, we can proceed with this decomposition until there are at least 3 non-trivial chromatic components in $\mathcal{G}'$ or in $\mathcal{G}''$, since both these d-graphs are still Δ-free; see Figure 8. Thus, decomposing recursively, we will represent $\mathcal{G}$ by a binary tree $T(\mathcal{G})$ whose leaves correspond to 2-graphs.

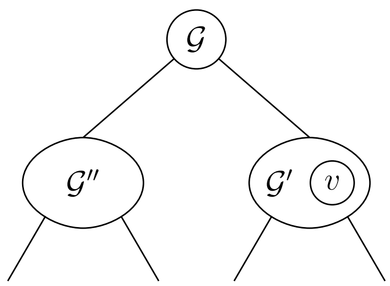


Figure 8: Decomposing $\mathcal{G}$ by the tree $T(\mathcal{G})$; substituting $\mathcal{G}''$ for v in $\mathcal{G}'$ to get $\mathcal{G}$.

3.2 Proof of Theorem 7

To make the paper self contained we give here a proof that also can be found in [26]. The following property of Gallai's d-graphs is instrumental for their decomposition.

Lemma 5. *Let $\mathcal{G} = (V; E_1, \ldots, E_n)$ be a Gallai d-graph one of whose chromatic component, say $G_1 = (V, E_1)$, is disconnected and let V_1' and V_1'' be the vertex sets of its two connected components. Then all edges between V_1' and V_1'' are homogeneously colored, that is, they all are of the same color i, where $i \neq 1$.*

Proof. Since V_1' and V_1'' are connected components of G_1, no edge between them can be of color 1. Assume indirectly that $(x', x'') \in E_2$ and $(y', y'') \in E_3$ for some $x', y' \in V_1'$ and $x'', y'' \in V_1''$. Since V_1' and V_1'' are connected, we can choose a path p' between x' and y' in

C_1' and p'' between x'' and y'' in C_1''. Then we can get a contradiction by showing that the d-graph induced by $V(p') \cup V(p'')$ contains a Δ, namely, a triangle colored by 1, 2 and 3. This is easy to show by induction on the lengths of p' and p''. $\square$

Lemma 6 ([14], [6], and [26]). *Every Gallai d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ with at least 3 non-trivial chromatic components has a color $i \in I = \{1, \ldots, d\}$ that does not span V, that is, $G_i = (V, E_i)$ is not connected.*

Proof. We copy it from [26]. Let $\mathcal{G}$ be a minimal counterexample. We may assume that for each vertex $v \in V$ and color $i \in I$ there is an edge $e \in E_i$ incident to v. Indeed, otherwise $G_i = (V, E_i)$ is not connected, since v is an isolated vertex in it. Let us fix a vertex $x \in V$ and consider the induced subgraph $\mathcal{G}_x = \mathcal{G}[V \setminus \{x\}]$. Clearly, $\mathcal{G}_x$ must have at least 3 non-trivial chromatic components. Indeed, if there are only 2 of them, say 1 and 2, then $G_i = (V, E_i)$ is disconnected for each $i = 3, \ldots, d$. Otherwise, since $\mathcal{G}$ is a minimal counterexample, $\mathcal{G}_x$ is disconnected in some color, say in color 1. Let $V_1, \ldots, V_k$ be the vertex-sets of the corresponding connected components. By Lemma 4, for each two components all edges between them are homogeneously colored, that is, they all are the same color i and, obviously, $i \neq 1$.

We will get a contradiction by showing that $G_1 = (V, E_1)$ is disconnected. Let us assume the opposite. Then there are edges of color 1 from x to $y_j \in V_j$ for each $j \in [k] = \{1, \ldots, k\}$. Let (x, u) and (x, v) be edges of colors 2 and 3.

Case 1. If u and v are in the same component, say $u, v \in V_1$, then (u, y_2) must be of color 2, since otherwise $\{x, u, y_2\}$ form a Δ, while (v, y_2) must be of color 3, since otherwise $\{x, v, y_2\}$ form a Δ. Thus, we get a contradiction with the homogeneous coloring of all edges between V_1 and V_2.

Case 2. If u and v are in different components, say $u \in V_1$ and $v \in V_2$ then (u, y_2) must be of color 2, since otherwise $\{x, u, y_2\}$ form a Δ, while (v, y_1) must be of color 3, since otherwise $\{x, v, y_1\}$ form a Δ. Again we get the same contradiction. $\square$

Gyárfás and Simonyi remark that Lemma 6 "is essentially a content of Lemma (3.2.3) in [14]". Lemmas 5 and 6 imply Theorem 7. Indeed, let $\mathcal{G} = (V; E_1, \ldots, E_d)$ be a Gallai d-graph. If it has at most 2 non-trivial chromatic components then we are done. Otherwise, by Lemma 6, there exists a non-trivial and non-connected component $G_i = (V, E_i)$. Let us decompose G_i into connected components and let $V = V_1 \cup \ldots V_k$ be the corresponding partition of V. At least one of these sets, say V_1, is of cardinality at least 2, since component i is non-trivial. By Lemma 5, for every two distinct vertex-sets $V_{j'}$ and $V_{j''}$ all edges between them are homogeneously colored, that is, there exists a color $i' \in I = \{1, \ldots, d\}$ such that $i' \neq i$ and $(v', v'') \in E_{i'}$ for every $v' \in V_{j'}, v'' \in V_{j''}$. Thus, collapsing V_1 into one vertex v we obtain a non-trivial decomposition $\mathcal{G} = \mathcal{G}(\mathcal{G}', v, \mathcal{G}'')$, where "non-trivial" means that both $\mathcal{G}'$ and $\mathcal{G}''$ are distinct from $\mathcal{G}$.

It is well-known that decomposing a graph into connected components can be executed in linear time. Hence, given a Gallai d-graph $\mathcal{G}$, its decomposition tree $T(\mathcal{G})$ can be constructed in linear time, too.

3.3 Extending Cameron-Edmonds-Lovasz' Theorem

Theorem 7 is instrumental to derive some nice properties of Gallai's colorings.

Corollary 1. *A Gallai d-graph with n vertices contains at most $n-1$ non-trivial chromatic components.*

As it was mentioned in [26], this result by Erdős, Simonovits, and Sos [13] immediately follows from Theorem 7 by induction.

Corollary 2. *If all but one chromatic components of a Gallai d-graph are perfect graphs then the remaining one is a perfect graph too.*

This claim was proved by Cameron, Edmonds, and Lovasz [7]. (Clearly, it turns into Lovasz' Perfect Graph Theorem if $d = 2$.) Later, Cameron and Edmonds [6] strengthened this claim showing that the same statement holds not only for perfect graphs but, in fact, for any family of graphs that is closed under: (i) substitution, (ii) complementation, and (iii) taking induced subgraphs. In [1] the claim is strengthened further as follows.

Theorem 8. *[1]. Let $\mathcal{F}$ be a family of graphs closed under complementation and exactly closed under substitution and let $\mathcal{G} = (V; E_1, \ldots, E_d)$ be a Gallai d-graph such that at least $d-1$ of its chromatic components, say $G_i = (V, E_i)$ for $i = 1, \ldots, d-1$, belong to $\mathcal{F}$. Then*

(a) the last component $G_d = (V, E_d)$ is in $\mathcal{F}$ too, and moreover,

(b) all 2^d projections of $\mathcal{G}$ belong to $\mathcal{F}$, that is, for each subset $J \subseteq I = \{1, \ldots, d\}$ the graph $G_J = (V, \cup_{j \in J} E_j)$ is in $\mathcal{F}$.

Proof. **Part (a).** By Theorem 7, $\mathcal{G}$ can be obtained from 2-graphs by substitutions. Such a decomposition of $\mathcal{G}$ is given by a tree $T(\mathcal{G})$ whose leaves correspond to 2-graphs. It is easy to see that by construction each chromatic component of $\mathcal{G}$ is decomposed by the same tree $T(\mathcal{G})$. Hence, all we have to prove is that both chromatic components of every 2-graph belong to $\mathcal{F}$. For colors $1, \ldots, d-1$ this holds, since $\mathcal{F}$ is exactly closed under substitution, and for the color d it holds, too, since $\mathcal{F}$ is also closed under complementation.

Part (b). It follows easily from part (a).

Given a $(d+1)$-graph $\mathcal{G} = (V; E_1, \ldots, E_d, E_{d+1})$, let us identify the last two colors d and $d+1$ and consider the d-graph $\mathcal{G}' = (V; E_1, \ldots, E_{d-1}, E_{\mathbf{d}})$, where $E_{\mathbf{d}} = E_d \cup E_{d+1}$. We assume that $\mathcal{G}$ is Δ-free and that $G_i = (V, E_i) \in \mathcal{F}$ for $i = 1, \ldots, d-1$. Then $\mathcal{G}'$ is Δ-free too and it follows from part (a) that $G_{\mathbf{d}} = (V, E_{\mathbf{d}})$ is also in $\mathcal{F}$. Hence, the union of any two colors is in $\mathcal{F}$. From this by induction we derive that the union of any set of colors is in $\mathcal{F}$. $\square$

This theorem implies Cameron-Edmonds' Theorem, as the following Lemma shows.

Lemma 7. *Let $\mathcal{F}$ be a family of graphs closed under substitution and taking induced subgraphs then $\mathcal{F}$ is exactly closed under substitution.*

Proof. Indeed, if $G = G(G', v, G'')$ then both G' and G'' are induced subgraphs of $\mathcal{G}$. □

A graph G is called a CIS-graph if each maximal clique and stable set of G intersect. By definition, CIS-graphs are closed under complementation and it is shown in [1] that they are exactly closed under substitution. However, an induced subgraph G' of a CIS graph G may be not a CIS-graph. For example, let $G = (V, E)$, where $V = \{v_0, v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2),\ (v_2, v_3), (v_3, v_4),\ (v_0, v_2), (v_0, v_3)\}$. Then G is a CIS-graph but its subgraph $G' = P_4$ induced by $V \setminus \{v_0\}$ is not.

Thus, Theorem 8 is applicable to the family $\mathcal{F}$ of the CIS-graphs, though Cameron-Edmonds' Theorem is not, because only conditions (i) and (ii) hold for $\mathcal{F}$ but not (iii).

To get more examples of families satisfying conditions of Theorem 8 let us consider the hereditary classes. Each such class is a family of graphs $\mathcal{F}$ defined by an explicitly given family (finite or infinite) of forbidden subgraphs $\mathcal{F}'$. By definition, $G \in \mathcal{F}$ if and only if G contains no induced subgraph isomorphic to a $G' \in \mathcal{F}'$.

Let us call a graph (respectively, d-graph) G *substitution-prime* if it is not decomposable by substitution, or more precisely, if $G = G(G', v, G'')$ for no G', G'' and v, except for two trivial cases: ($G = G'$ and $V(G'') = \{v\}$) or ($G = G''$ and $V(G') = \{v\}$).

Suppose that G is decomposable, $G = G(G', v, G'')$. If G' or G'' contains an induced subgraph G_0 then G also contains it, since both G' and G'' are induced subgraphs of G. However, G may contain G_0 even if G' and G'' do not. Yet, clearly, in this case G_0 is not substitution-prime. Hence, for both graphs and d-graphs, we obtain the following statement.

Proposition 3. *Family $\mathcal{F}$ is exactly closed under substitution if all graphs (respectively, d-graphs) in $\mathcal{F}'$ are substitution-prime.* □

Thus, $\mathcal{F}$ satisfies all conditions of Theorem 8 whenever $\mathcal{F}'$ is closed under complementation ($G \in \mathcal{F}'$ if and only if $\overline{G} \in \mathcal{F}'$) and $\mathcal{F}'$ contains only substitution-prime graphs. For example, these two properties hold for the family $\mathcal{F}'$ of the odd holes and anti-holes. In this case $\mathcal{F}$ is the family of Berge graphs. Thus, Theorem 8 and the Strong Perfect Graph Theorem imply Cameron-Edmonds-Lovász Theorem. Of course, it is simpler to show directly that perfect graphs are exactly closed under substitution and then apply Lovasz' perfect graph theorem in place of the strong one.

However, if $\mathcal{F}'$ contains a decomposable graph, e.g., C_4, then $\mathcal{F}$ may be not closed under substitution. For example, let $\mathcal{F}' = \{C_4, \overline{C_4}\}$ and consider the Gallai 3-graph in Figure 4. Two of its chromatic components belong to $\mathcal{F}$, while the third one, C_4, does not. As another example, let us consider $\mathcal{F}' = \{C_4, \overline{C_4}, C_5\}$. Then, by [12], $\mathcal{F}$ is the family of the split graphs. This family is not closed under substitution. Indeed, substituting a non-edge for a middle vertex of P_3 we get C_4.

The CIS-graphs form a non-hereditary family closed under complementation and exactly closed under substitution. It is not difficult to construct more examples of such families and even to characterize all families of graphs and d-graphs that are exactly closed under substitution.

Let $\mathcal{F}'$ be a family, finite or infinite, of (d-)graphs and let $\mathcal{F} = cl(\mathcal{F}')$ be its closure under substitutions. Typically, the family $\mathcal{F}$ is not hereditary.

Proposition 4. *A family $\mathcal{F}$ of (d-)graphs is exactly closed under substitution if and only if $\mathcal{F} = cl(\mathcal{F}')$, where $\mathcal{F}'$ is a family, finite or infinite, of substitution-prime (d-)graphs Furthermore, $\mathcal{F}$ is closed under complementation whenever $\mathcal{F}'$ is.*

Proof. The latter claim makes sense only for graphs and it is obvious. The former one follows from the uniqueness of canonical modular decomposition [34]. □

However, the above characterization is not constructive. For example, the substitution-prime perfect or CIS-graphs form infinite families that are difficult to describe explicitly.

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