DIMACS Technical Report 2007-05
March 2007

Game Seki[1]

by

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[1]This research was partially supported by the National Science Foundation (Grant IIS-0118635) and by DIMACS, the National Science Foundation's Center for Discrete Mathematics and Theoretical Computer Science.

ABSTRACT

Given a non-negative integral $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, the game *Seki* is defined as follows. Two players R and C take turns and it is specified who begins (this player is called the *first* and the opponent *second*). By one move a player can either reduce a strictly positive entry of A by 1 or pass. If both players pass then the game results in a draw. Player R (respectively, C) wins if a row (respectively, column) appears whose all entries are equal to 0. If after a move such a row and column appear simultaneously then the player who made this last move is the winner. (We also consider another version, Seki-I, in which this case is defined as a draw.) If neither R nor C can win in A, even being first, then A is called a *seki matrix* or simply a *seki*. Furthermore, A is called a *complete seki matrix* (CSM) if A is a seki and each player *must* pass, that is, if a player makes an active move then the opponent wins.

Seki is a difficult game. We cannot solve it and present only some partial results and conjectures mostly on the CSMs.

The game is closely related to the so-called seki (shared life) positions in GO. However, Seki is of independent interest as a combinatorial game. Those readers who do not know how to play GO can still understand the whole paper, except Appendix, where we analise (seki) positions in GO corresponding to some (seki) matrices. Already for 3×3 matrices such positions may be difficult even for advanced GO players.

Key words: combinatorial games, games with positive incentive, GO, seki, shared life, complete seki, draw, integral matrix, symmetrizable matrix, magic square.

1 Introduction

The game Seki was introduced in [3] to analyze the so-called seki (shared life) positions in GO. We refer to the website [2] that contains many examples and a systematic analysis of such positions.

Yet, we hope that Seki, as a combinatorial game, is of independent interest and present it here for mathematicians rather than for GO players. Though Seki is much easier than GO, still, it is sophisticated enough. We cannot solve it and present here only some partial results and conjectures mostly related to the complete seki matrices (CSMs). We verified these conjectures for small matrices by computer. Our experimental results are based on a computer code for repetition-free (up to permutations of rows and columns) generating all integral $m \times n$ matrices whose maximum entry (so-called *height*) is at most h. We went as far as $(m, n, h) =$ (3,3,10),(4,4,3), (5,5,2);(3,4,6),(3,5,4),(4,5,2). Having all these matrices we solve Seki by backward induction.

1.1 Main results and conjectures

The following conditions are sufficient for the second player to win. Player R can win if C begins and there is a row $i \in I$ such that $s_j^c - s_i^r \geq a_{i,j}$ for every column $j \in J$ and the inequality is strict whenever $a_{i,j} = 0$. Here s_i^r and s_j^c denote the sum of all entries of the row $i \in I$ and column $j \in J$, respectively, and $a_{i,j}$ is the corresponding entry of A. Clearly, the similar claim holds for C. It is also obvious that the first player wins if (s)he has a move such that the above conditions hold for the obtained matrix. We show that these two simple criteria are not only sufficient but also necessary for a player to win in a $2 \times n$ matrix. This observation implies that there exist only four 2×2 CSMs and there is no $2 \times n$ CSMs for $n > 2$. The same criteria also imply that an $n \times n$ matrix A is a CSM whenever A is a magic square (that is, $s_i^r = s_j^c = s$ for each $i \in I, j \in J$), whose sum $s = s(A) \geq 2$ and height $h = h(A) \leq 2$. For example, all 2×2 CSMs are of this type, since for all of them $h \leq 2$. In the opposite direction, we show that each $n \times n$ CSM of height $h \leq 1$ is a direct sum of magic squares. It seems that the same holds for $h \leq 2$ as well. Computations confirm this conjecture for $n \leq 5$.

However, there exist "non-magic" 3×3 CSMs. Furthermore, there is a symmetric 3×3 magic square (with sum $s = 6$) in which the first player, R or C, wins. This property also holds for many 4×4 magic squares. They may have other surprising properties. For example, in some of them player R (respectively, C) always wins, being the first or second. Some others are *semi-complete* seki matrices, that is, R must pass, while C can move and still draw, or vice versa. Among 3×3 matrices of height $h \leq 10$ there exist only four semi-complete seki matrices, of height 4 and 5, yet, they are not magic squares.

Let $h(n)$ be a maximum height among $n \times n$ CSMs. We show that $h(1) = h(2) = 2$. It looks like $h(3) = 3$; at least we verified that no 3×3 CSM has the height between 4 and 10. We conjecture that all seki matrices (in particular, all CSMs) are square, or equivalently, that a player, R or C, always wins in a $m \times n$ matrix with $m \neq n$. Clearly, this conjecture holds for $1 = m < n$ and we prove it for $2 = m < n$. We also verify it by computer for 3×4 (respectively, 3×5 and 4×5) matrices of height at most 6 (respectively, 4 and 2).

All $n \times n$ CSMs for $n \leq 3$ are symmetrizable, that is, they can be made symmetric by a permutation of the rows or columns. Yet, there are not symmetrizable 4×4 (respectively, 16×16) CSMs of height 2 (respectively, 1).

The game Seki is closely related to the so-called seki (shared life) positions in GO. However, Seki is of independent interest as a combinatorial game. Those readers who do not know how to play GO can still understand the whole paper, except Appendix, where we analise (seki) positions in GO corresponding to some (seki) matrices. Already for 3×3 matrices such positions may be difficult even for advanced GO players.

1.2 Seki-I

Let us recall that if after a move a row and column whose all entries are equal to 0 appear simultaneously then the player who made this last move is claimed the winner. This rule is implied by GO. Yet, it would be also logical to define each such position as a draw. We call the obtained game by *Seki-I* and study it in Section 10. It seems that Seki-I is a little simpler than Seki, although both games have similar properties.

For example, the following conditions are sufficient for the first player to win in Seki-I. Being first, R wins if there is a row $i \in I$ such that $s^c_j - s^r_i \geq a_{i,j}$ for every column $j \in J$. Obviously, the similar claim holds for C. These simple criteria imply that a magic square whose sum $s = s(A) \geq 2$ and height $h = h(A) \leq 1$ is a CSM in Seki-I. Except them, so-called 2-cycles are CSMs too; see Section 10 for the proofs and precise definitions. We conjecture that in Seki-I each CSM is a direct sum of the $(0, 1)$ magic squares and 2-cycles.

1.3 Example

Let us consider the matrix $A =$

$$\begin{matrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 3 \end{matrix}$$

If player C begins then he can win successively attacking three times the entry $a_{3,3} = 3$; while R by two moves can eliminate only entries $a_{3,1} = 1$ and $a_{3,2} = 1$. Then, after the next move of C, the third row and column disappear simultaneously, and thus C wins.

On the other hand, if R begins then she can force a draw. Of course, she should not pass, since in this case C wins, as we have just seen. Instead, R by first two moves reduces the entries $a_{1,1} = 3$ and $a_{2,2} = 3$ by 1. Each time C must respond by deleting $a_{3,1} = 1$ and $a_{3,2} = 1$, respectively, since otherwise R will delete the whole row in two moves. Thus, after two rounds they come to

$$\begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{matrix}$$

This matrix is a seki but it is still not complete, since any player can reduce $a_{3,3} = 3$ by 1. Then a CSM appears. Indeed, after a move at $a_{i,i} = 2$ for $i = 1, 2, 3$, the opponent will delete this entry and the row and column i will be eliminated simultaneously. Let us remark that R can also start by deleting $a_{3,1} = 1$ or $a_{3,2} = 1$. This will result in a draw, too.

A similar $n \times n$ example can be easily constructed: the main diagonal is 3, that is, $a_{i,i} = 3$ for $i = 1, \ldots, n$, the last row is 1, that is, $a_{n,i} = 1$ for $i = 1, \ldots, n-1$, and any other entry is 0. Again, if C begins, he can win attacking the last column 3 times; if R begins then she can force a draw attacking successively n distinct entries of the main diagonal and by this forcing C to eliminate the whole last row, except for the last entry $a_{n,n} = 3$.

1.4 $1 \times n$ and 2×2 matrices

In case $m = n = 1$, obviously, we have: if $a_{1,1} = 1$ then the first player wins; $a_{1,1} = 2$ is a CSM; $a_{1,1} > 2$ is a not complete seki matrix. In case $m = 1$ and $n > 1$ player C always wins. Similarly, R wins if $m > 1$ and $n = 1$.

Now let us consider 2×2 matrices. It is easy to verify that the first player wins in the following three:

$$\begin{matrix} 1\,2 \\ 2\,2 \end{matrix} \qquad \begin{matrix} 1\,3 \\ 3\,3 \end{matrix} \qquad \begin{matrix} 1\,2 \\ 2\,3 \end{matrix}$$

For example, in the first one, player R (respectively, C) must reduce $a_{1,2} = 2$ (respectively, $a_{2,1} = 2$) to 1. This is the only winning move. We leave to the reader finding a winning strategy in the remaining two matrices.

The following four 2×2 matrices are not complete seki:

$$\begin{matrix} B^2_1 & B^2_2 & B^2_3 & B^2_4 \\ 2\,3 & 2\,2 & 2\,3 & 3\,3 \\ 3\,2 & 2\,3 & 3\,3 & 3\,3 \end{matrix}$$

Indeed, it is easy to verify that nobody can win. Yet, a player can reduce an entry 3 of a matrix and the corresponding game is still a draw, with only two exceptions. In the third matrix player R (respectively, C) should not reduce $a_{2,1} = 3$ (respectively, $a_{1,2} = 3$), since in this case (s)he will lose. For example, the opponent wins reducing the same entry further.

We will prove that there exist exactly four 2×2 CSMs:

$$\begin{matrix} A^2_1 & A^2_2 & A^2_3 & A^2_4 \\ 1\,1 & 2\,0 & 2\,1 & 2\,2 \\ 1\,1 & 0\,2 & 1\,2 & 2\,2 \end{matrix}$$

It easy to see that in all four both players must pass. The corresponding GO-positions are given in Figure 2 of the Appendix. In section 5 we will show that there are no more 2×2 CSMs. Here and in the sequel, we identify two matrices if one can be obtained from the other by permutations of its rows and columns.

Let us note that all four 2×2 CSMs are symmetric. Moreover, they are magic squares, that is, the sum of entries in each row and in each column takes the same value s.

1.5 Magic squares

Clearly, if $s = 1$ then the first player, R or C, wins in one move in any $n \times n$ magic square. In Section 6 we show that any 2×2 magic square with $s > 1$ is a seki and that $A^2_\ell, \ell = 1, 2, 3, 4$, are only CSMs among them.

In section 8 we will show that a $n \times n$ magic square is a CSM whenever its sum $s \geq 2$ and height (the maximum entry) $h \leq 2$. One could conjecture that, in general, any $n \times n$ magic square with $s > 1$ is a seki. However, it is not true unless $n = 2$ or $h \leq 2$. For example, among 3×3 magic magic squares of height $h \leq 10$ there are two (of height 3 and 4, respectively) in which the first player, R or C, always wins; see section 1.7. Clearly, a unique magic square with $s = 1$ has this property too. Yet, all others 3×3 magic squares of height $h \leq 10$ are indeed seki.

However, there are very many interesting 4×4 magic squares of height 3: in some of them the first player, R or C, always wins, while in some others R (or C) always wins, being the first or second; see Section 1.8.

1.6 Direct sum of matrices

A trivial operation that respects CSMs is the *direct sum of matrices*, $A = A' \oplus A''$. Given an $m' \times n'$ matrix $A' : I' \times J' \to \mathbb{Z}_+$ and $m'' \times n''$ matrix $A'' : I'' \times J'' \to \mathbb{Z}_+$, let us define an $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, where $m = m' + m''$, $n = n' + n''$, and $I = I' \cup I''$, $J = J' \cup J''$; furthermore $a_{i,j} = a_{i',j'}$ if $i = i' \in I', j = j' \in J'$ and $a_{i,j} = a_{i'',j''}$ if $i = i'' \in I'', j = j'' \in J''$; otherwise, if $i \in I', j \in J''$ or $i \in I'', j \in J'$ then $a_{i,j} = 0$. The following statements are obvious.

Claim 1 *(i) If the first (respectively, the second) player wins in both A' and A'' then (s)he wins in A as well.*

(ii) If R (respectively, C) wins in both A' and A'' then (s)he wins in A.

(iii) Both A' and A'' are seki if and only if A is a seki.

(iv) Both A' and A'' are CSMs if and only if A is a CSM.

The last statement implies that a CSM may be not a magic square.

Moreover, it may be not a direct sum of magic squares. First such example was constructed in [3]; see Section 9. In the next Section we give more examples.

1.7 3×3 matrices

By a computer code, we got all 3×3 matrices of height $h \leq 10$.

1.7.1 Complete seki matrices

In particular, each 3×3 magic squares whose sum $s \geq 2$ and height $h \leq 2$ is a CSM. Thus, we obtain

B^3_1	B^3_2	B^3_3	B^3_4	B^3_5	B^3_6	B^3_7	B^3_8	B^3_9	B^3_{10}	B^3_{11}
002	011	012	012	111	022	022	112	122	200	222
020	101	111	120	111	202	211	121	212	011	222
200	110	210	201	111	220	211	211	221	011	222

Two of these CSMs, B^3_1 and B^3_{10}, are the direct sums of smaller CSMs.

The following two CSMs are direct sums too, yet, they are not magic squares:

$$\begin{matrix} 2\,0\,0 \\ 0\,2\,1 \\ 0\,1\,2 \end{matrix} \qquad \begin{matrix} 2\,0\,0 \\ 0\,2\,2 \\ 0\,2\,2 \end{matrix}$$

Finally, there are seven 3×3 CSMs of height 3:

A^3_1	A^3_2	A^3_3	A^3_4	A^3_5	A^3_6	A^3_7
033	133	301	320	320	320	033
303	313	022	212	203	213	331
330	331	121	023	032	033	312

Interestingly, the last two of them are not magic squares. In fact, A^3_6 is obtained by one move from a symmetric magic square

$$\begin{matrix} 4 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 3 \end{matrix}$$

while A^3_7 can be reduced by one move to a symmetric magic square $A =$

$$\begin{matrix} 0 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{matrix}$$

1.7.2 Magic squares in which the first player always wins

Since A^3_7 was a CSM, the first player, R or C, always wins in A. A pretty long case analysis confirms this conclusion. We leave it to the reader revealing only the first move: R (respectively, C) can win reducing $a_{1,2}$ (respectively, $a_{2,1}$) from 3 to 2. The position in the 9×9 GO-board corresponding to A^3_7 is given in Figure 3 of the Appendix.

There is another 3×3 magic square with the same property:

$$\begin{matrix} 0 & 3 & 4 \\ 3 & 1 & 3 \\ 4 & 3 & 0 \end{matrix}$$

One can verify that R (respectively, C) wins reducing $a_{2,1}$ (respectively, $a_{1,2}$) from 3 to 2. Finally, the first player always wins in the 3×3 identity matrix, since in this case $s = 1$. Interestingly, all three obtained matrices are symmetric.

Computer analysis shows that any other 3×3 magic square of height $h \leq 10$ is a seki. Moreover, both players, R and C, can start completing this seki. In section 1.8, we will see that many more interesting options can take place for 4×4 magic squares of height 3.

1.7.3 Semi-complete seki

Among 3×3 matrices of height $h \leq 10$ there are other two interesting examples:

$$\begin{array}{cc} A^3_8 & A^3_9 \\ 0\,3\,3 & 0\,3\,5 \\ 4\,0\,3 & 3\,0\,5 \\ 4\,3\,0 & 4\,4\,0 \end{array}$$

Each of the above two matrices is a not complete seki such that only C can make an active move, while R must pass. Such a matrix will be called a *semi-complete seki matrix* (semi-CSM). Let us consider, e.g., A^3_8. The corresponding position in the 9×9 GO-board is given in Figure 4 of the Appendix.

Let C reduce $a_{1,2}$ from 3 to 2. If R would pass then C can even win playing at $a_{3,2}$. Instead, R should reduce $a_{3,1}$ from 4 to 3. The obtained matrix is a (not complete) seki. Again we leave the full case analysis to the reader.

Unlike C, player R must pass in A^3_8. Indeed, due to obvious symmetry, R has only three distinct moves: at $a_{1,2}$, $a_{2,1}$, and $a_{3,2}$. If R chooses $a_{1,2}$ then C can reduce $a_{3,2}$ or, less obviously, $a_{2,1}$; if R chooses $a_{2,1}$ then C answers at $a_{1,2}$; finally, if R chooses $a_{3,2}$ then C plays at $a_{1,2}$. It is not difficult to verify that in all three cases C wins.

Now let us consider A^3_9. This example is more difficult to analyze. However, computer says that it is a semi-CSM too. In particular, C can reduce $a_{1,2}$ from 3 to 2. In return, R should reduce $a_{1,3}$ from 5 to 4. The obtained matrix is a (not complete) seki. Unlike C, player R must pass in A^3_9. Indeed, due to obvious symmetry, R has only three distinct moves: at $a_{1,2}$, $a_{1,3}$, and $a_{3,1}$. If R reduces $a_{1,2}$ from 3 to 2 then C will reduce it further to 1; if R chooses $a_{1,3}$ then C answers at $a_{2,1}$; finally, if R chooses $a_{3,1}$ then C answers at $a_{1,2}$. The program says that in all three cases C wins. Thus A^3_9 is a semi-CSM too.

Let us remark that neither A^3_8 nor A^3_9 is a magic square.

In general, 3×3 semi-CSMs are rare. Among matrices of height $h \leq 10$ there are only four of them: A^3_8, A^3_9 and two transposed matrices.

1.8 4×4 matrices of height 3

In contrast, there are many 4×4 semi-CSMs. Moreover, some of them are magic squares too. For example, there exist ten 4×4 magic squares of height 3 that are semi-CSMs in which C can move, while R cannot.

0023	0023	0033	0123	0123	0123	0133	0312	0233	1133
2300	2201	3300	2301	2310	2121	3301	2112	3302	3311
1121	2210	1122	2121	2112	2211	2122	2121	3131	2222
2111	1121	2211	2121	2121	2211	2221	2121	2222	2222

There exist thirteen 4×4 magic squares of height 3 in which R always wins, being first or second:

0023	0113	0113	0113	0113	0123	0123	0123	0123	0123	0123	0222	0223
1310	1130	3110	1220	2120	0321	0213	0213	0321	2103	1230	0222	3220
2012	2102	0221	2012	2210	3012	2220	3120	3012	2220	2202	3111	2113
2210	2210	2111	2210	1112	3210	2220	3210	3210	2220	3111	3111	2221

However, there exist only three 4×4 magic squares of height 3 in which R wins being first and draws being second.

0 1 3 3	0 2 2 3	0 2 3 3
3 1 0 3	2 2 3 0	3 3 1 1
2 3 2 0	2 1 1 3	3 1 2 2
2 2 2 1	3 2 1 1	2 2 2 2

Yet, there are sixty 4×4 magic squares of height 3 in which the first player, R or C, wins. Several examples follow:

0001	0023	0023	0023	$\cdots$	0233	1133	1133
0010	0230	0320	2300	$\cdots$	3212	1313	1322
0100	3101	3011	1130	$\cdots$	3221	3221	3212
1000	2201	2201	2102	$\cdots$	2222	3221	3221

On the other hand, there are many 4×4 CSMs that are not magic squares. For example, the following five matrices:

3 0 0 2	0 0 3 3	0 0 3 3	0 0 3 3	0 0 3 3
0 0 3 3	0 3 1 1	3 3 0 0	3 3 0 0	0 3 0 3
0 3 2 1	3 1 1 2	1 3 1 3	0 3 1 3	2 2 2 0
2 3 1 1	3 1 2 1	3 1 3 1	3 1 3 0	3 1 1 1

We leave to the reader to check that they are CSMs, indeed. In general, to verify the types of all matrices given above is a difficult task and we just refer to our computer analysis.

1.9 Two main conjectures

1.9.1 Can game Seki in a non-square matrix result in a draw?

It looks like it cannot.

Conjecture 1 *Each seki matrix (and in particular, each CSM) is a square matrix, that is,* $m = |I| = |J| = n$.
Equivalently, a player, R or C, always wins in a $m \times n$ matrix when $m \neq n$.

Obviously, C wins if $1 = m < n$ (and R wins if $1 = n < m$). In Section 8, the conjecture is proved for $2 = m < n$. It is also verified for the 3×4 (respectively, 3×5 and 4×5) matrices of height at most 6 (respectively, 4 and 2).

For example, let us consider the following two matrices

$$\begin{matrix} 2\,1\,1\,1\,1 \\ 2\,2\,2\,2\,2 \\ 2\,3\,2\,3\,3 \\ 1\,3\,6\,8\,8 \end{matrix} \qquad \begin{matrix} 2\,1\,1\,1\,1 \\ 2\,2\,2\,2\,2 \\ 2\,3\,2\,3\,2 \\ 1\,3\,6\,8\,8 \end{matrix}$$

and show that in the first one the first player, R or C, wins, while the second one R always wins, being the first or second. In any case the game does not result in a draw.

Indeed, if R begins, she can eliminate the first row and win. Yet, if C is the first, he can effectively resist by eliminating the first column. The further analysis is not difficult: R and C eliminate successively the second and third rows and columns getting matrices

$$\begin{matrix} 2\,0\,0\,0\,0 \\ 0\,2\,2\,2\,2 \\ 0\,3\,2\,3\,3 \\ 0\,3\,6\,8\,8 \end{matrix} \qquad \begin{matrix} 2\,0\,0\,0\,0 \\ 0\,2\,0\,0\,0 \\ 0\,0\,2\,3\,2 \\ 0\,0\,6\,8\,8 \end{matrix} \qquad \begin{matrix} 2\,0\,0\,0\,0 \\ 0\,2\,0\,0\,0 \\ 0\,0\,2\,0\,0 \\ 0\,0\,0\,8\,8 \end{matrix}$$

Finally, C wins. The second game will follow the same line but now R, even being second, can eliminate the third row and win.

1.9.2 How large the height of an $n \times n$ CSM can be?

Conjecture 2 *(i) For any fixed n the set of $n \times n$ CSMs is finite; or equivalently,*

(i') there is a function $h : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that the maximum entry (and hence, the height) of an $n \times n$ CSM is equal to $h(n)$.

(ii) Function h is unbounded, that is, $h(n)$ can take arbitrarily large values.

Obviously, $h(1) = 2$, since $a_{1,1} = 2$ is a unique 1×1 CSM. In Section 6 we will show that there are only four 2×2 CSMs: $A^2_\ell, \ell = 1, 2, 3, 4$. Hence, $h(2) = 2$ too. It is likely that $h(3) = 3$. At least, we verified that there is no 3×3 CSM of height h for $4 \leq h \leq 10$. Yet, we were not able to get reasonable bounds for $h(4)$, because of the memory constraints. We only know that $h(4) \geq 3$.

In general, the results of Section 9 imply that $h(n) \geq 3$ for each $n \geq 3$.

It is natural to conjecture that $h(n)$ is a non-decreasing function of n..

Perhaps, $h(n) = n$ for $n \geq 2$.

1.10 Generalized Seki and Seki-I

Let $G = (V, E)$ be a finite directed graph that has no directed cycles; its vertices $v \in V$ are called *positions* and the directed arcs $e = (v, v') \in E$ going out of a vertex v are called the *active moves* in the position v. Let $V_T \subseteq V$ be the set of all terminal positions, that is, positions in which there are no active moves.

We introduce two players R and C who take turns beginning with a specified initial position $v_0 \in V \setminus V_T$. In each position $v \in V \setminus V_T$ the player on lead can either choose an active move $e = (v, v')$ or pass. Then the opponent has to move from v' or v, respectively. A directed path from v_0 to a position $v \in V$ is called a *debut* and it is called a *play* if $v \in V_T$.

Furthermore, let $V_T^R, V_T^C \subseteq V_T$ be two sets of terminal positions and, by definition, R and C win if the play p terminates in V_T^R and V_T^C, respectively. If p terminates in a position $v \in V_T \setminus (V_T^R \cup V_T^C)$ then the result is defined as a draw. Finally, if p terminates in a position $v \in V_T^R \cap V_T^C$ then there are two options: Seki and Seki-I. In Seki-I the result is defined as a draw, while in Seki the player who made the last move is claimed the winner. The position $v \in V \setminus V_T$ is called a *complete seki position* if each player, R and C, must pass in it, otherwise the opponent wins.

2 The importance of being first or strategy thief

All above examples with symmetric matrices support the following statement.

Claim 2 *The second player never wins in a symmetric matrix, in other words, either such a matrix is a seki or the first player wins.*

Proof. The standard "strategy stealing argument" works. Assume indirectly that the second player has a winning strategy. In particular, (s)he can win if the first player pass. In this case the second player does not pass (this would be a draw), instead (s)he begins and wins. Yet, the first player can "steal" this winning strategy, since A is symmetric. We get a contradiction. □

We can generalize this claim as follows: for a player to be the first is at least as good as to be the second; this is true for any matrix. Milnor (1953) called this property of a game the *positive incentive.* For example, Seki, GO, and all games in which both players are allowed to pass have a positive incentive. More precisely, the following statement holds.

Claim 3 *If R can win (respectively, draw) being second, then she can win (respectively, draw or win) being first.*

Proof. Indeed, to become second, being first, it is enough to pass. □

Clearly, the similar claim holds for C, as well. It is also clear that Claim 3 implies Claim 2.

Claim 4 *For the following three statements:*

(i) The first player, R or C, cannot win in A;

(ii) The second player, R or C, cannot win in A;

(iii) A is a seki matrix;

we have $(i) \Rightarrow (ii)\ \&\ (iii)$ *and* $(iii) \Rightarrow (i)\ \&\ (ii)$.

Proof. Indeed, by definition, (iii) is equivalent to the conjunction of (i) and (ii). Furthermore, (i) implies (ii), since Seki is a game with positive incentive. Indeed, let us assume indirectly that (ii) does not hold. This means that for a matrix A the second player, say R, wins. Then R wins, being the first, too, since it is enough for the first player to pass to become the second one. □

3 Nine types of matrices

Given a matrix A in which R begins, let us say that A is of type:

W if R has a winning move;
L if C wins;
D if R cannot win but can draw, in other words, neither R nor C can win.

Interestingly, by Conjecture 1, only square matrices can be of type D.

Let us define the similar types assuming that C begins and consider both options. The obtained nine types of matrices are interpreted as follows:

(W, W) means the first player wins;

(W, D) that R wins if she begins and draws if plays second;

(D, W) means the same for C;

(W, L) means that R always wins;

(L, W) that C always wins;

(D, D) means seki, that is, the game is a draw, whoever begins;

(D, L) means that R wins when she plays second and only draws when begins.

(L, D) the same for C;

(L, L) means that the second player always wins.

The last three types, (D, L), (L, D), and (L, L), do not exist, since Seki is a game with a positive incentive.

We get a less obvious classification if substitute D and L by the following similar but slightly different properties:

D' means that R has no winning move but has at least one active move (not pass) that results in a draw;

L' means that R has no such a move, in other words, every move, except pass, is losing; yet, pass may result in a draw.

Now the nine types of matrices are interpreted as follows: (W,W) , (W,D'), (D',W), (W,L'), and (L',W) mean the same as before. Furthermore,

(D',D') means that each player has an active move (not pass) that results in a draw, that is, A is a (not complete) seki matrix such that each player can start completing this seki.

(D',L') is a *semi-complete* seki, that is, A is a (not complete) seki matrix such that only R, but not C, can start completing this seki.

(L',D') the same for C;

(L',L') means that A is a CSM, since the second player wins after each move of the first one, except pass.

The following property of this type is obvious but important: if A is of type (L',L') and A' is obtained by reducing an entry of A by 1 then A' is of type (W,W).

Recall that, by Conjecture 1, D or D' can appear only for square matrices.

Interestingly, all nine above types appear already among 4×4 magic squares of height 3; see Section 1.8. However, just a few types are possible for the 2×2 and 3×3 matrices (not to mention magic squares); see Sections 1.4 and 1.7.

Except for the identity matrix, there is no 2×2 magic square of type (W,W). Any other 2×2 magic square is a seki. Among them there are four of type (L',L'), CSMs, and all others are of type (D',D'). There is no 2×2 matrix (not to mention magic squares) of type (D',L'), semi-CSM.

We found out types of all 3×3 matrices of height at most 10 and got the following data. Among the magic squares there exist only three of type (W,W) (see Section 1.5), while all others are seki, $(D'D')$ or (L',L'), and no other type appears. Among CSMs of height 3 five are magic squares, $A^3_1, A^3_2, A^3_3, A^3_4, A^3_5$, and two are not A^3_6 and A^3_7; see Section 1.7.1. (As we already mentioned, each magic squares of sum s and height at most 2 is a CSM if $s \geq 2$ and is of type (W,W) if $s = 1$; see Section 7.) Furthermore, there is no 3×3 CSM of height between 4 and 10 and there exist only two matrices of type (D',L'), semi-CSMs, A^3_8 and A^3_9, yet, they are not magic squares.

4 When does the second player win?

A natural strategy in game Seki is to attack in the "bottle neck". For example, for R it looks good to move at (i,j) such that s^r_i is minimal and s^c_j is maximal. This principle provides a simple sufficient condition for the second player to win.

Claim 5 *Let $A : I \times J \to \mathbb{Z}_+$ be an $m \times n$ matrix.*
Being second, player R wins whenever there is a row $i \in I$ such that for every column $j \in J$

$$s^c_j - s^r_i \geq a_{i,j} \tag{4.1}$$

and this inequality is strict for each j such that $a_{i,j} = 0$.

Proof. Let row i satisfy (4.1). If C plays at (i', j') then R responds at (i, j) for some $j \neq j'$. If this is not possible, that is, $a_{i,j'} = 0$ for each $j' \in J$, except j), then R plays at (i, j'). It is easy to verify that conditions of Claim 5 still hold after such an exchange of moves. Indeed, even if $a_{i,j} = 1$ and it becomes 0 after R's move, still (4.1) will hold for j. It remains to note that C can pass. In this case R can choose any entry of the given row i to reduce; after this (4.1) still hold for this i and for any column j. In any case, R will remove the whole row i in s_i^r moves and win before C eliminates a column. $\square$

Of course, the similar statement holds for the player C as well.

Equation (4.1) means that R can take a burden of deleting $a_{i,j}$ and still win.

The sufficient condition of Claim 5, which we will denote by $\mathbf{S}$, has many applications. For example, $\mathbf{S}$ implies that each magic square is a CSM whenever its entries are at most 2 and its sum is at least 2; see Section 7. In Section 5.2 we will show that for $2 \times n$ matrices $\mathbf{S}$ is not only necessary but also sufficient for the player R to win, being second. As a corollary we derive that there is no $2 \times n$ seki for $n > 2$, in particular, there is no 2×3 seki; see Section 5.2. Furthermore, a 2×2 matrix is a seki matrix seki if and only if $\mathbf{S}$ fails for both players R and C. This observation implies that there exist only four 2×2 CSMs A_ℓ^2, $\ell = 1, 2, 3, 4$, given above; see Section 6.

Let us remark that the extra condition about strictness of (4.1) in $\mathbf{S}$ is essential. Indeed, if $a_{i,j} = 0$ and $s_i^r = s_j^c$ then not R but C wins. Indeed, C can begin and delete column j before R could eliminate row i or any other row. Strictness of (4.1) implies also the following strict inequality.

Claim 6 *If row $i \in I$ satisfies $\mathbf{S}$ then $s_i^r < s_j^c$ for each column $j \in J$.*

Proof. Indeed, if $a_{i,j} > 0$ then $s_i^r < s_j^c$, by (4.1); if $a_{i,j} = 0$ then (4.1) is itself strict, by Claim 5. $\square$

What would be good sufficient conditions for the first player to win? It seems natural to substitute (4.1) by a weaker inequality $s_j^c - s_i^r \geq a_{i,j} - 1$. However, the obtained condition is not sufficient. Let us consider, for example, the following three matrices:

$$\begin{matrix} 1\,1\,1 \\ 1\,1\,1 \\ 1\,1\,1 \end{matrix} \qquad \begin{matrix} 1\,1\,1 \\ 1\,2\,0 \\ 1\,0\,2 \end{matrix} \qquad \begin{matrix} 1\,1 \\ 1\,1 \end{matrix}$$

For each of them the inequality $s_j^c - s_i^r \geq a_{i,j} - 1$ holds for the first row, $i = 1$, and every column j. Moreover, all entries of the first row are strictly positive. Yet, these three matrices are CSMs, since they are magic squares of height $h \leq 2$. Hence, R cannot win even if she begins.

Of course, conditions of Claim 5, which are sufficient even for the second player to win, are also sufficient for the first player to win. However, such conditions are too strong and not necessary already for 2×2 matrices. In fact, the best what we can suggest is a triviality:

Claim 7 *Playing first, R wins whenever she can reduce the original matrix A to a matrix A' satisfying condition* **S**, *which is sufficient for R to win in A' being second.* □

Of course, the similar statement holds for the player C as well.

To illustrate Claim 7 let us consider $A =$

$$\begin{matrix} 1 & k \\ k & k \end{matrix}$$

This matrix is a CSM if $k = 1$. Yet, if $k > 1$ then R begins and wins by reducing $a_{1,2} = k$. Indeed, row 1 of the obtained matrix $A' =$

$$\begin{matrix} 1 & k-1 \\ k & k \end{matrix}$$

satisfies **S**, since $(k+1) - (1 + (k-1)) = 1 > 0$ and $(k + (k-1)) - (1 + (k-1)) = k-1 > 0$. Hence, R can win (by eliminating the first row) even if C begins. More generally, $A =$

$$\begin{matrix} \ell & k-\ell \\ k & k \end{matrix}$$

satisfies **S** whenever $0 < \ell < k$.

We will show that if $m \leq 2$ then Claims 5 and 7 provide not only sufficient but also necessary conditions for R to win, being second. On the other hand, for the matrices of size 3×3 or lager that may have entries 3 or larger these conditions are only sufficient but not necessary. Let us recall the magic square A' from Section 1.5, where R, being first, wins. However, it is easy to check that A' does not satisfy the conditions of Claim 7.

5 Cases $m \leq 2$ and $m < n$

5.1 Simple corollaries of condition S

Obviously, for each matrix $A : I \times J \to \mathbb{Z}_+$ we have

$$\sum_{i \in I} s_i^r = \sum_{j \in J} s_j^c = \sum_{i \in I, j \in J} a_{i,j} \tag{5.2}$$

Condition **S** claims that there is a row $i \in I$ such that $s_j^c \geq s_i^r + a_{i,j}$ for each column $j \in J$ and $s_j^c > s_i^r$ whenever $a_{i,j} = 0$. In particular, if $m < n$ then **S** implies that there are $i \in I$ and $j \in J$ such that $s_i^r > s_j^c$. For $m = 2$ we derive a stronger corollary. Let row 1 satisfy **S**. Then $a_{2,j} \geq s_1^r$ for each $j \in J$. Hence,

$$s_2^r = \sum_{j=1}^{n} a_{i,j} \geq n s_1^r \tag{5.3}$$

In other words, for $m = 2$, condition **S** implies that the sum of one row is at least n times the some of the other.

5.2 $1 \times n$ and $2 \times n$ matrices

As we already mentioned, first player wins in every $1 \times n$ matrix with $n > 1$. Also (s)he wins if $n = 1$ and $a_{1,1} = 1$. Otherwise, a 1×1 matrix is a seki: it is a CSM if $a_{1,1} = 2$ (see Figure 2 of the Appendix) and it is not complete if $a_{1,1} > 2$.

We will show that for $m = 2$ condition **S** is not only sufficient but also necessary for R, being second, to win. More precisely, the following claim holds.

Claim 8 *Let A be $m \times n$ matrix with $2 = m \le n$ and let C begin. Player R wins if and only if condition* **S** *holds. Moreover, A can be a seki matrix only if $m = n = 2$; if $2 = m < n$ then C wins whenever* **S** *does not hold.*

Proof. By Claim 5, condition **S** is always sufficient for R, being second, to win. To prove that it is necessary too, we assume that **S** does not hold and show that C can always maintain this situation, that is, C has a move (not pass) such that for any response of R **S** still does not hold for the obtained matrix. Clearly, **S** holds for a matrix that contains an empty row and no empty column. Hence, no empty row can appear before an empty column appears. In other words, either A is a seki matrix or C wins.

The problem that C has to solve is not too difficult, since **S** fails unless $s_2^r \ge ns_1^r$ or $s_1^r \ge ns_2^r$, and $n \ge 2$. For example, the following simple strategy works. Without loss of generality, let us assume that $s_1^r \le s_2^r$ and that row 1 does not satisfy **S**, that is, there is a column $j \in J$ such that $s_j^c - s_1^r < a_{1,j}$. If $a_{2,j} > 0$ then C reduces it by 1. It is not difficult to check that, after any move of R, condition **S** will not hold for the obtained matrix.

If $a_{2,j} = 0$ then C can reduce $a_{1,j}$. He wins in one move if $a_{1,j} = 1$. If $a_{1,j} \ge 2$ then either C can eliminate column j and win, or the matrix will be decomposed as the direct sum of 1×1 matrix $a_{1,j}$, where $a_{1,j} \ge 2$, and a $1 \times (n-1)$ matrix.

Now we should consider two cases. If $n > 2$ then C wins in in every $1 \times (n-1)$ matrix and hence, C wins in the original matrix, too. If $n = 2$ then both matrices can be seki. $\square$

Finally, let us remark that R always wins if $m = 2$ and $n = 1$.

5.3 Case $3 \le m < n$

If $m \ge 3$ then condition **S** is only sufficient but no longer necessary. Indeed, R can win, being second, even if **S** fails. Let us consider, for example, the following 3×4 matrix $A =$

$$\begin{matrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 4 & 4 \end{matrix}$$

Rows 1 and 2 have the sum $s_1^r = s_2^r = 5$ and the sum of each column is 6. Yet, **S** does not hold. Indeed, rows 1 and 2 do not satisfy **S**, since $s_1^c - s_1^r = 6 - 5 = 1 < 2 = a_{1,1}$ and $s_2^c - s_2^r = 6 - 5 = 1 < 2 = a_{2,2}$, and row 3 cannot satisfy **S**, since $s_3^r = 14$ is by far too large. Nevertheless, R wins even if C begins. Let us verify that for each move of C there is a response of R enforcing **S** for row 1 or 2. Indeed, if C plays at $(3,1)$ (respectively, at $(3,2)$) then R answers at $(2,2)$ (respectively, at $(1,1)$), and R can choose any one of these two moves if C plays elsewhere. It is easy to see that after this exchange of moves row 1 (respectively, 2) will satisfy **S** and hence, R can win.

In case $m = 2$, when R can try to enforce **S** only for row 1, player C can effectively resist this plan. Yet, if $m \geq 3$ then R may threaten to enforce **S** either for row 1 or for 2, and C cannot resist both threats.

Thus, R can win in case $m \geq 3$ even if **S** fails. Yet, the following question remains open: whether a seki matrix exists in case $m < n$? Note that if it does then a CSM exists too, since each seki can be completed. We know that for $2 \leq m < n$ the answer is negative. In this case if **S** holds then R wins, otherwise C wins and seki never happens.

It looks like no CSM can exist for $m < n$ for the following intuitive reason. “To win R must attack only rows with small sums. Hence, C can safely play in a row $i \in I$ with a large sum” ($s_i^r > s_j^c$ for some $j \in J$). Such a row always exists whenever $m < n$. However, the above arguments have a flaw. For example, let us consider the 5×6 matrix $A =$

$$\begin{matrix} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 3 \end{matrix}$$

Interestingly, R can win in five moves forcing the elimination of the row 4 whose sum is 5. This results from a combination: R does not delete row 5 herself, instead she “helps C” by eliminating first three entries of column 5. However, C must respond by eliminating the first three entries of row 5; otherwise R will win in two moves deleting the corresponding diagonal entry. Then, by the last two moves R eliminates $a_{4,4} = 2$ and wins.

However, we know that there is no $m \times n$ CSM with $m \leq 2$ and $m < n$.

6 More about 2×2 matrices

In the previous section we proved that for 2×2 matrices conditions of Claims 5 and 7 are not only sufficient but also necessary for a player to win. Hence, by negation of these conditions we can characterize 2×2 (complete) seki matrices.

6.1 Standard form

Given a 2×2 matrix $A : I \times J \to \mathbb{Z}_+$, where $I = J = \{1, 2\}$, without any loss of generality we will assume that

$$s_1^r \leq s_2^r, \;\; s_1^c \leq s_2^c, \;\; s_1^r \leq s_1^c \tag{6.4}$$

By substituting the entries of A, we can rewrite this system as follows.

$$a_{2,2} - a_{1,1} \geq a_{2,1} - a_{1,2} \geq 0. \tag{6.5}$$

6.2 2×2 complete seki matrices

By definition, in a CSM the first player cannot win, moreover, if (s)he does not pass then the opponent wins. Since for 2×2 matrices Claim 7 gives necessary and sufficient conditions for the first player to win, we get a characterization of CSMs. However, they can be characterized much simpler.

Claim 9 *There exist only four 2×2 CSMs A^2_ℓ, $\ell = 1, 2, 3, 4$, see Section 1.4.*

Proof. If the first player, say C, move at $(2, 2)$ then R can win. How?

Case 1: R has a move after which row 1 will satisfy condition **S**. However, without reducing $a_{2,2}$, the same move would win even easier, and A is a CSM. Hence this case is impossible.

Case 2: R has a move after which row 2 satisfy condition **S**. By two moves row 2 could be reduced by 2. Hence, by (5.3), $s^r_1 \geq 2(s^r_2 - 2)$ and by (6.4), $s^r_2 \geq s^r_1$. Thus, $s^r_2 \geq 2s^r_2 - 4$, that is, $s^r_2 \leq 4$. Summarizing, we obtain that $s^r_1 \leq s^r_2 \leq 4$. There are just a few matrices satisfying these inequalities and it is easy to verify that, except four matrices A^2_ℓ, $\ell = 1, 2, 3, 4$ of Section 1.4, there are no other CSMs. □

For example, it is easy to check that the following matrices are seki but not complete:

$$\begin{matrix} 3\,1 \\ 1\,3 \end{matrix} \qquad \begin{matrix} 3\,1 \\ 1\,2 \end{matrix} \qquad \begin{matrix} 3\,0 \\ 1\,3 \end{matrix}$$

6.3 2×2 seki matrices

Claims 5 and 7 provide sufficient conditions for a player to win. Claim 8 implies that for 2×2 matrices these conditions are also necessary. Thus, we obtain the following characterization of 2×2 seki matrices.

Claim 10 *A 2×2 matrix A is a seki matrix if and only if conditions of Claims 5 and 7 do not hold neither for R nor for C. In other words, for any move of a player condition* **S** *for this player does not hold in the obtained matrix A'.* □

The following four systems of two inequalities mean respectively that row 1, row 2, column 1, and column 2 satisfy condition **S**

$$a_{2,1} - a_{1,2} \geq a_{1,1}$$
$$a_{2,2} - a_{1,1} \geq a_{1,2}$$

$$a_{1,1} - a_{2,2} \geq a_{2,1}$$

$a_{1,2} - a_{2,1} \geq a_{2,2}$

$a_{1,2} - a_{2,1} \geq a_{1,1}$
$a_{2,2} - a_{1,1} \geq a_{2,1}$

$a_{1,1} - a_{2,2} \geq a_{1,2}$
$a_{2,1} - a_{1,2} \geq a_{2,2}$

Recall that each inequality must be strict whenever right-hand-side is 0.
It is convenient to rewrite the above four systems as follows.

$a_{2,1} \geq a_{1,1} + a_{1,2} = s_1^r$
$a_{2,2} \geq a_{1,1} + a_{1,2} = s_1^r$

$a_{1,1} \geq a_{2,1} + a_{2,2} = s_2^r$
$a_{1,2} \geq a_{2,1} + a_{2,2} = s_2^r$

$a_{1,2} \geq a_{1,1} + a_{2,1} = s_1^c$
$a_{2,2} \geq a_{1,1} + a_{2,1} = s_1^c$

$a_{1,1} \geq a_{1,2} + a_{2,2} = s_2^c$
$a_{2,1} \geq a_{1,2} + a_{2,2} = s_1^c$

These systems allow us to simplify a bit our characterization of 2×2 seki matrices. To do so, except the standard form (6.5), let us also assume for simplicity that each entry is at least 2:

$$a_{1,1} \geq 2 \;\; a_{1,2} \geq 2 \;\; a_{2,1} \geq 2 \;\; a_{2,2} \geq 2 \tag{6.6}$$

Otherwise, after reduction by 1 an entry can become 0 and we have to consider strict inequalities in **S** too, which doubles the number of cases. So, we leave to the reader characterizing the 2×2 seki matrices that may have entries 0 or 1. Under the above assumptions Claim 10 can be reformulated as follows.

Claim 11 *A 2×2 matrix A satisfying (6.4), (6.5), and (6.6) is a seki matrix if and only if it also satisfies to at least one of the following three linear systems:*

$a_{1,1} + a_{1,2} - a_{2,1} \geq 2$
$a_{1,1} + a_{2,1} - a_{2,2} \geq 2$

$a_{1,1} + a_{1,2} - a_{2,1} \geq 2$
$a_{1,1} - a_{1,2} + a_{2,1} \geq 2$

$a_{1,1} + a_{1,2} - a_{2,2} \geq 2$ $\square$

6.4 2×2 matrices in which the first player wins

Also Claim 7 provides a simple characterization of the 2×2 of type (W, W). There are three types of them:

$$\begin{array}{ccc} 0\,1 & 0\,a & 1\,a \\ a\,0 & a\,b & a\,c \end{array}$$

where $1 \le a \le b+1$ and $1 \le a \le c \ge 2$. In other words, the following characterization holds.

Claim 12 *A 2×2 matrix A is of type (W, W) if and only if one of the following three systems holds:*

(i) $a_{1,1} = a_{2,2} = 0,\ a_{1,2} = 1,\ a_{2,2} \ge 1;$

(ii) $a_{1,1} = 0,\ 1 \le a_{1,2} = a_{2,1} \le a_{2,2} + 1;$

(iii) $1 \le a_{1,1} \le a_{1,2} = a_{2,1} \le a_{2,2} \ge 2.$

Proof. . It follows easily from **S**. We leave details to the reader. □

7 CSMs and magic squares of height 1 and 2

Let us recall that a matrix $A : I \times J \to \mathbb{Z}_+$ is called a *magic square* with the sum $s = s(A)$ if $s^r_i = s^c_j = s$ for every $i \in I$ and $j \in J$. Obviously, in this case $|I| = |J|$. It is also clear that the first player wins in one move when $s = 1$.

In Section 1.8 we saw that a 4×4 magic square of height 3 may be of any type. Yet, if height is at most 2 then only CSMs, (L', L'), remain.

A matrix of height 1 (respectively, 2) will be called a $(0, 1)$ (respectively, $(0, 1, 2)$) matrix. We conjecture that every $(0, 1, 2)$ CSM is a direct sum of magic squares with $s > 1$, and vice versa. However, we can prove only one direction of this conjecture, and the inverse one only for $(0, 1)$ CSMs.

Claim 13 *Let A be a $(0, 1, 2)$ magic square with $s(A) > 1$ then A is a CSM.*

Proof. It follows from Claims 5 and 7. Let $A : I \times J \to \mathbb{Z}_+$ be a $(0, 1, 2)$ magic square and let a player, say C, make a move (not pass) reducing a positive entry a_{i_0,j_0} by 1. Clearly, for the obtained matrix A' we have $s'_{i_0} = s'_{j_0} = s - 1$, while $s'_i = s'_j = s$ for all $i \in I \setminus \{i_0\}$ and $j \in J \setminus \{j_0\}$. We show that R wins in A', unless $s = 1$, in which case the game is already over and C won. Indeed, let R play at (i_0, j_1) for some $j_1 \ne j_0$, if possible, and at (i_0, j_0) otherwise. Clearly, in the latter case a_{i_0,j_0} becomes 0, since it was reduced twice and each entry of A was at most 2. It is also clear that both, row i_0 and column j_0, are removed by the last move of R (and hence, she wins), since otherwise R would not move at (i_0, j_0). If the former case, when R made a move at (i_0, j_1), for the remaining matrix A'' we have: $s''_{i_0} = s - 2$, $s''_{j_0} = s''_{j_1} = s - 1$, while $s''_i = s''_j = s$ for all $i \in I \setminus \{i_0\}$ and $j \in J \setminus \{j_0, j_1\}$. It is easy to check that conditions of Claim 5 hold, since i_0 satisfies (4.1) for any j. Indeed, for

$j = j_0$ or $j = j_1$ (4.1) holds, since entries a_{i_0,j_0} and a_{i_0,j_1} being at most 2 were reduced by 1 and $s''_{j_0} - s''_{i_0} = s''_{j_1} - s''_{i_0} = (s-1)-(s-2) = 1$. Hence, (4.1) holds for columns j_0 and j_1 and it is strict whenever $a''_{i_0,j_0} = 0$ or $a''_{i_0,j_1} = 0$, respectively. Also (4.1) holds and it is strict for any other column j, since $s''_j - s''_{i_0} = s-(s-2) = 2$ and $a_{i_0,j_0} \le 2$. □

Clearly, the above claim and Claim 1 imply also that A is a CSM whenever it is a direct sum $A = \oplus_{i=1}^k A_i$ of k $(0,1,2)$ magic squares A_i and $s(A_i) > 1$ for $i = 1,\dots,k$. It is likely that the inverse statement holds, too.

Conjecture 3 *Every* $(0,1,2)$ *CSM* A *is a direct sum of magic squares, that is,* $A = \oplus_{i=1}^k A_i$, *with* $s(A_i) > 1$ *for* $i = 1,\dots,k$. □

By a computer code we verified this conjecture for $n \times n$ matrices with $n \le 5$.

One could start a proof of the conjecture as follows. Given A, let $s = min\{s^r_i,\ i \in I;\ s^c_j,\ j \in J\}$ be the minimum sum in all rows and columns of A. Furthermore, let $A_{i,j}$ denote the matrix obtained from A by deleting its row i and column j. We would like to show that there is no entry $a_{i,j} > 0$ such that $s^r_i = s, s^c_j > s$ or $s^r_i > s, s^c_j = s$. This would imply that A is a direct sum, $A = A' \oplus A''$, where A' is a magic square with sum $s(A') = s$. Then Conjecture 3 follows by induction. The next claim is a step in this direction.

Claim 14 *Let* $A : I \times J \to \{0,1,2\}$ *be a* $(0,1,2)$ $m \times n$ *matrix with minimum sum* s. *If there are* $i \in I$ *and* $j \in J$ *such that* $s^r_i = s$, $s^c_j > s$, *and* $a_{i,j} > 0$ *then player* R *can begin at* (i,j) *and win, unless there are also* $i' \in I, j' \in J$ *such that* $s^r_{i'} > s$, $s^c_{j'} = s$, $a_{i',j'} > 0$, $a_{i,j'} = 2$, *and* R *cannot win in* $A_{i,j'}$ *even if she begins. Yet, even in this case, a move at* (i,j) *guarantees at least a draw to* R, *unless in* $A_{i,j'}$ *player* C *wins even if* R *begins.*

Proof. If $s^r_i = s$, $s^c_j > s$, and $a_{i,j} > 0$ then R can start at (i,j) and play in row i all the time. Clearly, she wins, unless there is a column $j' \in J$ such that $s^r_{i'} > s$, $s^c_{j'} = s$, $a_{i',j'} > 0$, $a_{i,j'} = 2$. In this, and only in this, case C has a defense strategy. He can start at $a_{i',j'}$ and then play in column j' all the time. If both players leave the entry $a_{i,j'} = 2$ untouched then in $s-2$ moves A will be reduced to a direct sum $A' \oplus A_{i,j'}$, where A' is a 1×1 CSM $a_{i,j'} = 2$. Such a strategy of R fails only if C wins in $A_{i,j'}$ even if R begins. Yet, if R can begin and make at least a draw in $A_{i,j'}$ then she can still move at (i,j) and guarantee at least a draw in A too. □

This claim implies Conjecture 3 for $(0,1)$ CSMs, though we cannot finish the proof in general.

Claim 15 *Every* $(0,1)$ *CSM* A *is a direct sum of magic squares;* $A = \oplus_{i=1}^k A_i$ *with* $s(A_i) > 1$ *for* $i = 1,\dots,k$.

Proof. By Claim 14, for each $(0,1)$ CSM $A : I \times J \to \{0,1\}$ there are subsets $I' \subseteq I$ and $J' \subseteq J$ and an integral $s > 1$ such that $s^r_i = s^c_j = s$ for every $i \in I'$, $j \in J'$ and $s^r_i > s$, $s^c_j > s$

for every $i \notin I'$, $j \notin J'$; moreover, $a_{i,j} = 0$ for every $i \in I'$ and $j \notin J'$ and $j \in J'$ and $i \notin I'$. Otherwise, A would not be a CSM, by Claim 14. (Note that this claim does not imply the same for $(0,1,2)$ matrices.) Furthermore, $|I'| = |J'|$, and A is a direct sum of two matrices: $A = A_s \oplus A_{s^+}$. They both are CSMs, since A is a CSM. Now we can decompose the second matrix in a similar way, etc. Finally, we get decomposition $A = \oplus_{i=1}^{k} A_i$, where A_i are $(0,1)$ magic squares whose sum $s(A_i) > 1$ for $i = 1, \ldots, k$. $\square$

8 Non-symmetrizable complete seki matrices

By Claim 2, the second player cannot win in a symmetric matrix, that is, either it is a seki matrix, or the first player wins. The latter option can take place, indeed; see for example Section 1.5.

Our computations show that, for $n \leq 3$, all $n \times n$ CSMs are symmetric. More precisely, they are *symmetrizable*, that is, can be made symmetric by a permutation of rows or columns. However, the following 4×4 CSM A is not symmetrizable:

$$
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 2 & 0 & 2 \\
1 & 1 & 2 & 0 \\
2 & 0 & 1 & 1
\end{array}
$$

Indeed, A has row $\mathbf{e}$ that consists of all ones but no such column. Yet, A is a CSM, since it is a $(0,1,2)$ magic square with sum 4. Moreover, there are even non-symmetrizable $(0,1)$ CSMs. However, their size is much larger. For example, let us consider the following 16×16 binary matrix:

$$
\begin{array}{cccccccccccccccc}
0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&1&0&1&0&0&0&1&0&0&0&1&0&0\\
1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&1\\
0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0\\
0&0&0&1&0&1&0&1&0&0&0&1&0&1&0&0\\
1&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&1&0&1&0&1&0&0&0&1\\
0&0&1&0&1&0&0&0&1&0&1&0&1&0&0&0\\
0&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0\\
1&0&0&0&0&0&1&0&1&0&1&0&0&0&1&0\\
0&1&0&0&1&0&0&0&0&1&0&0&0&1&0&1\\
0&0&1&0&0&0&1&0&0&0&1&0&1&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&1\\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0
\end{array}
$$

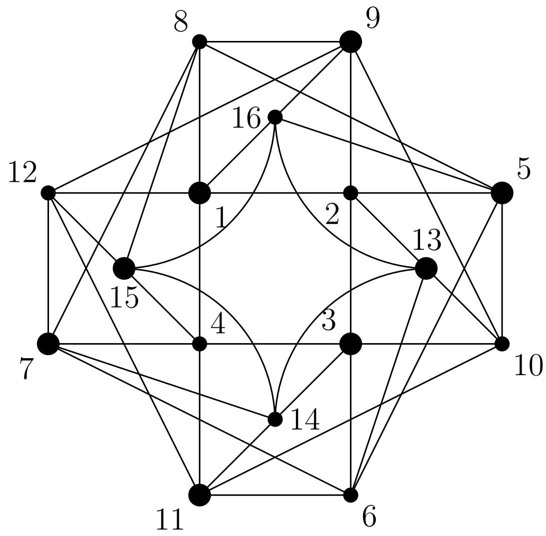


Figure 1: The adjacency matrix of this graph is a non-symmetrizable CSM.

Again, A is a CSM, since it is a magic square with sum 5. Yet, A is non-symmetrizable; see, e,g., [1].

Remark 1 *In fact, this example is just a slight modification (resulting in a magic square) of the non-symmetrizable* 16×16 *matrix from [1]. The graph in Figure 5 of [1] has vertices of degree 2,3, and 5. We add several edges to make it a regular (bipartite) graph of degree 5; see Figure 1. However, the automorphism group,* $\mathbb{Z}_4$*, remains the same, and the obtained* 16×16 *adjacency matrix remains non-symmetrizable.*

9 $(3,1,1,-1)$- and $(3,2,2,-2)$-extensions

Let us recall four 3×3 CSMs $A_i^3; i = 3,4,5,6$ and three 2×2 seki matrices B_1^2, B_4^2, and A_4^2 (the last one is a CSM too):

$$\begin{array}{ccccccc}
A_4^2 & A_3^3 & B_1^2 & A_5^3 & A_4^3 & B_4^2 & A_6^3 \\
 & 3\ 0\ 1 & & 3\ 2\ 0 & 3\ 2\ 0 & & 3\ 2\ 0 \\
2\ 2 & 0\ 2\ 2 & 2\ 3 & 2\ 0\ 3 & 2\ 1\ 2 & 3\ 3 & 2\ 1\ 3 \\
2\ 2 & 3\ 2\ 1 & 3\ 2 & 0\ 3\ 2 & 0\ 2\ 3 & 3\ 3 & 0\ 3\ 3
\end{array}$$

It is easy to notice that the 3×3 matrices A_3^3, A_4^3, A_5^3 and A_6^3 can be viewed as extensions of the 2×2 matrices A_4^2, B_1^2, B_1^2, and B_4^2, respectively.

More precisely, given an $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, we define two new $(m+1) \times (n+1)$ matrices A' and A'' as follows. Let us add one new row i_0 to I and one new column j_0 to J and set $a'_{i_0,j_0} = 3$. Furthermore, let us choose an entry a_{i^*,j^*} in A such that $a_{i^*,j^*} \geq 1$ (respectively, $a_{i^*,j^*} \geq 2$) and reduce it by 1, that is, set $a'_{i^*,j^*} = a_{i^*,j^*} - 1$ (respectively, by 2, that is, $a''_{i^*,j^*} = a_{i^*,j^*} - 2$). All other entries of A remain the same, that is, $a_{i,j} = a'_{i,j} = a''_{i,j}$ whenever $i \in I \setminus \{i^*\}$ or $j \in J \setminus \{j^*\}$. Finally, let us define $a'_{i^*,j_0} = a'_{i_0,j^*} = 1$ (respectively,

$a''_{i^*,j_0} = a''_{i_0,j^*} = 2$) and $a'_{i,j_0} = a'_{i_0,j} = a''_{i,j_0} = a''_{i_0,j} = 0$ for all other i and j, that is, for $i \in I \setminus \{i^*\}$ and $j \in J \setminus \{j^*\}$. We will call two obtained matrices, A' and A'', respectively, the $(3,1,1,-1)$- and $(3,2,2,-2)$-*extension* of A at (i^*,j^*).

For example, $A'' = A^3_5$ is the $(3,2,2,-2)$-extension of B^2_1 at $(1,1)$ and $A' = A^3_3$ is the $(3,1,1,-1)$-extension of A^2_4 at $(2,2)$. Let us notice that both last matrices are CSMs.

Moreover, our computations show that *all* $(3,1,1,-1)$-extension of ten 3×3 CSMs, A^3_i, $i = 1,2,3,4,5$ and B^3_j, $j = 6,7,8,9,11$, are CSMs too.

A^3_1	A^3_2	A^3_3	A^3_4	A^3_5	B^3_6	B^3_7	B^3_8	B^3_9	B^3_{11}
0 3 3	1 3 3	3 0 1	3 2 0	3 2 0	0 2 2	0 2 2	1 1 2	1 2 2	2 2 2
3 0 3	3 1 3	0 2 2	2 1 2	2 0 3	2 0 2	2 1 1	1 2 1	2 1 2	2 2 2
3 3 0	3 3 1	1 2 1	0 2 3	0 3 2	2 2 0	2 1 1	2 1 1	2 2 1	2 2 2

Furthermore, the $(3,1,1,-1)$-extensions of A^3_6 at (i^*,j^*) are CSMs whenever $(i^*+j^* \geq 4$, that is, at $(1,3)$, $(1,3)$, $(2,3)$, $(3,1)$, $(3,2)$, and $(2,2)$. However, at $(1,1)$ it is of type (W,W) (the first player wins) and it is not defined at $(1,2)$ and at $(2,1)$, since $a_{1,2} = a_{2,1} = 0$.

Similarly, the $(3,1,1,-1)$-extensions of A^3_7 at (i^*,j^*) are CSMs whenever $\min(i^*,j^*) \geq 2$, that is, at $(2,2)$, $(2,3)$, $(3,2)$, and $(3,3)$. However, at $(1,2)$ and at $(2,1)$ it is a semi-CSM, of type (D',L') and (L',D'), respectively; while at $(1,3)$ and $(3,1)$ it is of type (L',W) and (W,L'), that is, C and R win, respectively; finally, it is not defined at $(1,1)$, since $a_{1,1} = 0$.

In [3] it was erroneously announced that $(3,1,1,-1)$-extensions of CSMs are CSMs; more precisely, if A' is a $(3,1,1,-1)$-extension of A at (i^*,j^*) then A' is a CSM whenever $s^r_{i^*} \geq 4$, $s^c_{j^*} \geq 4$ and A is a CSM. The above analysis of the CSMs A^3_6 and A^3_7 shows that this claim is an overstatement. However, among all 2×2 and 3×3 CSMs there are only above five counterexamples: $(3,1,1,-1)$-extension of A^3_6 at $(1,1)$ and of A^3_7 at $(1,2)$, $(1,3)$, $(2,1)$, and $(3,1)$ are not CSMs.

Let us also notice that the conditions $s^r_{i^*} \geq 4$, $s^c_{j^*} \geq 4$ above are absolutely necessary. Indeed, if A is a CSM and $s^r_{i^*} \leq 3$ (respectively, $s^c_{j^*} \leq 3$) then, being first, R (respectively C) wins in A' in at most 3 moves.

Now, let us proceed with $(3,2,2,-2)$-extensions. We already mentioned that all $(3,2,2,-2)$-extensions of (not complete 2×2 seki matrices) B^2_1 and B^2_4 are CSMs. The same property holds for the following seven 3×3 matrices:

A^3_1	A^3_2	A^3_4	A^3_5	B^3_9	B^3_{11}	B^3_{12}
0 3 3	1 3 3	3 2 0	3 2 0	1 2 2	2 2 2	3 1 1
3 0 3	3 1 3	2 1 2	2 0 3	2 1 2	2 2 2	1 2 2
3 3 0	3 3 1	0 2 3	0 3 2	2 2 1	2 2 2	1 2 2

Furthermore, let us consider the following nine 3×3 matrices:

C^3_1	C^3_2	C^3_3	C^3_4	C^3_5	C^3_6	C^3_7	C^3_8	C^3_9
2 0 0	2 0 0	3 0 1	3 0 1	3 0 1	3 0 2	3 0 1	3 0 2	3 1 2
0 2 3	0 3 3	0 3 2	0 3 3	0 2 3	0 3 3	0 3 1	0 3 2	1 2 3
0 3 2	0 3 3	1 2 2	1 3 2	1 3 1	2 3 1	1 1 3	2 2 2	2 3 1

For the first four of them, C_i^3, $i = 1, 2, 3, 4$, the $(3, 2, 2, -2)$-extensions at (i^*, j^*) are CSMs whenever $\min(i^*, j^*) \geq 2$; it is of type (W, W) at $(1, 1)$ and not defined at $(1, 2)$,$(1, 3)$,$(2, 1)$, and $(3, 1)$. The same for C_5^3, except, it is not defined at $(3, 3)$ either. The same for C_6^3, except, it is (W, W) not only at $(1, 1)$ but at $(1, 3)$ and $(3, 1)$ too. For C_7^3 the $(3, 2, 2, -2)$-extension is (L', L') (a CSM) at $(3, 3)$, it is (W, W) at $(1, 1)$ and $(2, 2)$ and not defined otherwise. For C_8^3 it is (L', L') at $(3, 3)$, it is (W, W) at $(1, 1)$ and $(2, 2)$, it is (D', D') (a not complete seki) at $(1, 3)$, $(2, 3)$, $(3, 1)$, and $(3, 2)$, and it is not defined at $(1, 2)$ and $(2, 1)$. Finally, for C_9^3 it is (L', L') at $(2, 3)$, $(3, 2)$, and $(1, 1)$; it is (D', D') at $(1, 3)$, $(3, 1)$, and $(2, 2)$, and it is not defined at $(1, 2)$, $(2, 1)$, and $(3, 3)$.

Let us apply the $(3, 1, 1, -1)$- and $(3, 2, 2, -2))$-extensions recursively beginning with the 2×2 matrices A_4^2 and, respectively, B_1^2. We obtain the following two infinite sequences:

$$A_4^2 \qquad \begin{matrix} 22 \\ 22 \end{matrix} \quad \begin{matrix} 310 \\ 112 \\ 022 \end{matrix} \quad \begin{matrix} 3100 \\ 1210 \\ 0112 \\ 0022 \end{matrix} \quad \begin{matrix} 31000 \\ 12100 \\ 01210 \\ 00112 \\ 00022 \end{matrix} \quad \begin{matrix} 310000 \\ 121000 \\ 012100 \\ 001210 \\ 000112 \\ 000022 \end{matrix} \quad \ldots$$

$$B_1^2 \qquad \begin{matrix} 32 \\ 23 \end{matrix} \quad \begin{matrix} 320 \\ 212 \\ 023 \end{matrix} \quad \begin{matrix} 3200 \\ 2120 \\ 0212 \\ 0023 \end{matrix} \quad \begin{matrix} 32000 \\ 21200 \\ 02120 \\ 00212 \\ 00023 \end{matrix} \quad \ldots$$

Claim 16 *Except for B_1^2, both sequences contains only CSMs.*

This implies that for CSMs $h(n) \geq 3$ whenever $n \geq 3$; see Section 1.9.2. Indeed, each sequence contains a (unique) $n \times n$ matrix for each $n \geq 2$ and it is a CSM with $a_{1,1} = 3$ whenever $n \geq 3$.

Proof. Let us note that both sequences contain only magic squares with sum 4 and 5, respectively. Moreover, except for $a_{1,1} = 3$ (respectively, $a_{1,1} = a_{n,n} = 3$), all entries take only values $0, 1$, and 2. Thus, conditions of Claim 13 "almost" hold and by the same arguments we can show that after "almost" any active move of a player the opponent wins in 3 (respectively, in 4) moves. However, there are two important exceptions when the opponent cannot win immediately; instead (s)he can reduce $n \times n$ to the $(n - 1) \times (n - 1)$ matrix from the same sequence in which one entry is also reduced by 1. Thus, we can finish the proof by induction on n, since in both sequences the 3×3 matrices are CSMs, A_3^3 and A_4^3, respectively.

Case 1. If a player, say R, reduces $a_{1,1} = 3$ then C answers at $a_{2,1}$. It is easy to see that R must play at $a_{1,2}$. (In case of the second sequence, C will repeat by playing at $a_{2,1}$ once more and again R must answer at $a_{1,2}$). As a result of this exchange of moves, the the original $n \times n$ matrix is reduced to the direct sum of two matrices: a 1×1 CSM $a_{1,1} = 2$ and the $(n - 1) \times (n - 1)$ matrix from the same sequence in which the first entry is 2 instead of 3. Since C has to move in the obtained game, he can enforce the same reduction again, etc.

Case 2. If a player, say R, reduces $a_{2,1}$ then C answers by reducing $a_{1,1} = 3$ to 2. By this he threatens to eliminate the first column by the next 2 (respectively, 3) next moves.

The only defense of R is to play at $a_{1,2}$. (In case of the second sequence, C will proceed by eliminating of $a_{2,1}$ and R must eliminate $a_{1,2}$) in return). The above exchange of moves results in exactly the same reduction as in Case 1. Thus, C wins again.

Let us notice that the matrices of the second sequence are double-symmetric, that is, $a_{i,j} = a_{j,i}$ and $a_{i,j} = a_{n-i+1,n-j+1}$ for any i and j. Hence, we can substitute $(1,2)$,$(2,1)$, and $(1,1)$ by $(n-1,n)$,$(n,n-1)$, and (n,n), respectively.

It remains to verify that for any other move of any player the opponent can win in 3 (respectively, in 4) moves by applying the strategy suggested in the proof of Claim 13. □

10 Game Seki-I

The winner in Seki was defined as follows. Player R (respectively, C) wins if a row (respectively, column) appears whose all entries are equal to 0. If after a move such a row and column appear simultaneously then the player who made this last move is the winner.

The last rule is motivated by GO. Let us consider here a slightly different game, *Seki-I*, which, by definition, is a draw when zero row and column appear simultaneously; all other rules of Seki-I are the same as in Seki. The new game is no longer related to GO. However, it is also logical and it seems that Seki-I is simpler than Seki. Both games have similar properties but, by definition, some winning positions in Seki become draw positions in Seki-I. Hence, the CSMs in Seki-I form a proper subset of the CSMs in Seki. Indeed, let us recall that A is a CSM (that is, of type $(L'L')$) if and only if each matrix A' obtained by reducing a strictly positive entry of A by 1 is of type (W,W).

Clearly, Claim 1 holds for Seki-I too.

Let us show that Clam 5 can be simplified for Seki-I as follows.

Claim 17 *Let $A : I \times J \to \mathbb{Z}_+$ be an $m \times n$ matrix whose every row and column contain a strictly positive entry. Being first, player R wins whenever there is a row $i \in I$ such that for every column $j \in J$*

$$s^c_j - s^r_i \geq a_{i,j}. \tag{10.7}$$

Proof. Let row i satisfy (10.7). We will show that R can win just reducing its entries $a_{i,j}$ in any order. Indeed, let R play at (i,j) and C answer at (i',j'). We prove that (10.7) still holds for the same row i in the obtained matrix A'.

Case 1: $i' = i$. It is obvious. Indeed, s^r_i is reduced by 2, while s^c_j by at most 1 for each j unless $j' = j$. In the last case s^c_j is reduced by 2 but $a_{i,j}$ too.

Case 2: $i' \neq i$ but $j' = j$. Then s^c_j is reduced by 2 but s^r_i and $a_{i,j}$ are reduced by 1 each. Hence, (10.7) still holds for $j' = j$. Obviously, it holds for all other $j'' \in J$ too.

Case 3: $i' \neq i$ and $j' \neq j$. It is again obvious. Indeed, s^c_j, $s^c_{j'}$, and s^r_i are reduced by 1 each, while for any other column $j'' \in J$ the sum $s^c_{j''}$ does not change. We still have to consider separately the case $s^r_i = a_{i,j} = 1$. Then $s^c_j \geq 1$ for all $j \in J$ and, moreover, $s^c_i \geq 2$. Thus, no zero column appears after eliminating $a_{i,j}$ and R wins in one move. □

Of course, the similar statement holds for the player C as well.

Let us note that 1×1 CSM $a_{1,1} = 2$ is not a CSM in Seki-I. Indeed, each player, R or C can reduce 2 to 1 and Seki-I is still a draw. Let us recall four 2×2 CSMs

$$\begin{array}{cccc} A_1^2 & A_2^2 & A_3^2 & A_4^2 \\ 1\,1 & 2\,0 & 2\,1 & 2\,2 \\ 1\,1 & 0\,2 & 1\,2 & 2\,2 \end{array}$$

It is easy to verify that in Seki-I only A_1^2 remains a CSM, while the other three matrices become not complete seki.

By Claim 13, each $(0,1,2)$ magic square with sum $s(A) > 1$ is a CSM in Seki. Yet, for Seki-I only the following weaker claim holds.

Claim 18 *If A is a $(0,1)$ magic square with sum $s(A) > 1$ then A is a CSM.*

Let us remark that if $s(A) = 1$ then Seki-I is finished in one move and the result is a draw. However, A is not a CSM in this case. Claim 18 follows easily from Claims 17.

Proof. Let $A : I \times J \to \mathbb{Z}_+$ be a $(0,1)$ magic square with $s(A) > 1$ and let a player, say C, make an active move eliminating a_{i_0,j_0}. The game is not yet over, since $s(A) > 1$ and it is easy to see that in the obtained matrix A' the row i satisfies (10.7). Indeed, for $j = j_0$ we have $s^c_j(A') - s^r_i(A') = s^c_j(A) - s^r_i(A) = 0$, since A is a magic square. Still (10.7) holds for A', since $a'_{i_0,j_0} = a_{i_0,j_0} - 1 = 0$ too. For $j \neq j_0$ we have $s^c_j(A') - s^r_i(A') \geq s^c_j(A) - s^r_i(A) + 1 \geq 1 \geq a_{i,j}$. Hence, by Claim 18, R wins in A' after any active move of C. By symmetry, C wins in A' after any active move of R. Hence, A is a CSM. $\square$

Clearly, the above claim and Claim 1 imply also that in Seki-I A is a CSM whenever it is a direct sum $A = \oplus_{i=1}^k A_i$ of k $(0,1)$ magic squares A_i and $s(A_i) > 1$ for $i = 1, \ldots, k$. The inverse statement holds, too.

Claim 19 *In Seki-I every $(0,1)$ CSM A is a direct sum of magic squares, that is, $A = \oplus_{i=1}^k A_i$, with $s(A_i) > 1$ for $i = 1, \ldots, k$.*

The same statement holds for Seki as well, see Claim 3, and its proof for Seki-I is similar.

Yet, there are also CSMs of height two in Seki-I, for example, the 2-cycles:

$$\begin{array}{ccc} & & 0\,0\,0\,2\,2 \\ & 0\,0\,2\,2 & 0\,0\,2\,0\,2 \\ 0\,2\,2 & 0\,2\,0\,2 & 0\,2\,0\,2\,0 \\ 2\,0\,2 & 2\,0\,2\,0 & 2\,0\,2\,0\,0 \\ 2\,2\,0 & 2\,2\,0\,0 & 2\,2\,0\,0\,0 \end{array}$$

For any $n \geq 3$ we define the $n \times n$ 2-cycle by equations:

$a_{1,n} = a_{1,n-1} = a_{2,n} = a_{2,n-2} = a_{3,n-1} = a_{3,n-3} = \ldots$

$\ldots = a_{n-1,3} = a_{n-1,1} = a_{n,2} = a_{n,1} = 2;$

and all remaining entries are equal to 0.

Claim 20 *Every 2-cycle is a CSM.*

Proof. We have to show that for each move of the first player there is a winning reply of the second one. Due to the symmetry of 2-cycles, all first moves are equivalent. Let us consider, for example, the 3×3 matrix and assume that C makes a move, say at $(1, 2)$. We have to show that now R can win. Indeed, R replies at $(1, 3)$ threatening to eliminate the first row. Still, C has a (unique) defense , at $(3, 2)$. Now, if R plays at $(1, 3)$ again then she will lose and if at $(1, 2)$ then C can force a draw answering at $(2, 3)$. Yet, R has a stronger move, at $(2, 1)$. Now, C cannot play at $(1, 2)$,$(1, 3)$,$(2, 1)$, or $(1, 3)$, since then R wins in one move. If instead C plays at $(3, 1)$ or $(3, 2)$ then still R wins replying at $(1, 2)$ or $(2, 1)$, respectively; and both these moves are winning for R in case C would pass after $(2, 1)$. The same analysis works for longer 2-cycles as well. $\square$

By Claim 1, we can strengthen Claim 18 as follows.

Claim 21 *Every direct sum of* $(0, 1)$ *magic squares with sum greater than* 1 *and* 2*-cycles is a CSM in Seki-I.* $\square$

We conjecture that the inverse is also true.

Conjecture 4 *Every CSM in Seki-I is a direct sum of* $(0, 1)$ *magic squares with sum greater than* 1 *and* 2*-cycles. In particular, each CSM is a square matrix.*

11 Appendix

11.1 Game Seki and seki in GO

Here we explain how game Seki is related to seki positions in GO. We assume that the rules of GO and standard notation are known.

Given an $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, let us consider a position P in GO with m White and n Black groups that are indexed by I and J, respectively, and let $a_{i,j}$ be the number of common free points (so-called liberties or dame) between the White group $i \in I$ and Black group $j \in J$. Thus, players R and C in game Seki correspond to Black and White in GO. Indeed, R (respectively, C) wants to delete all positive entries of a row (respectively, of a column); accordingly, Black (respectively, White) wants to surround completely a White (respectively, Black) group.

Of course, only very special, "seki-type", positions in GO get an adequate description in this way. We assume that numbers m and n remain unchanged, that is, no two groups of the same color can ever be connected. Furthermore, we assume that each of the $m + n$ involved groups has no eyes and, moreover, no group can make an eye even if the opponent will always pass.

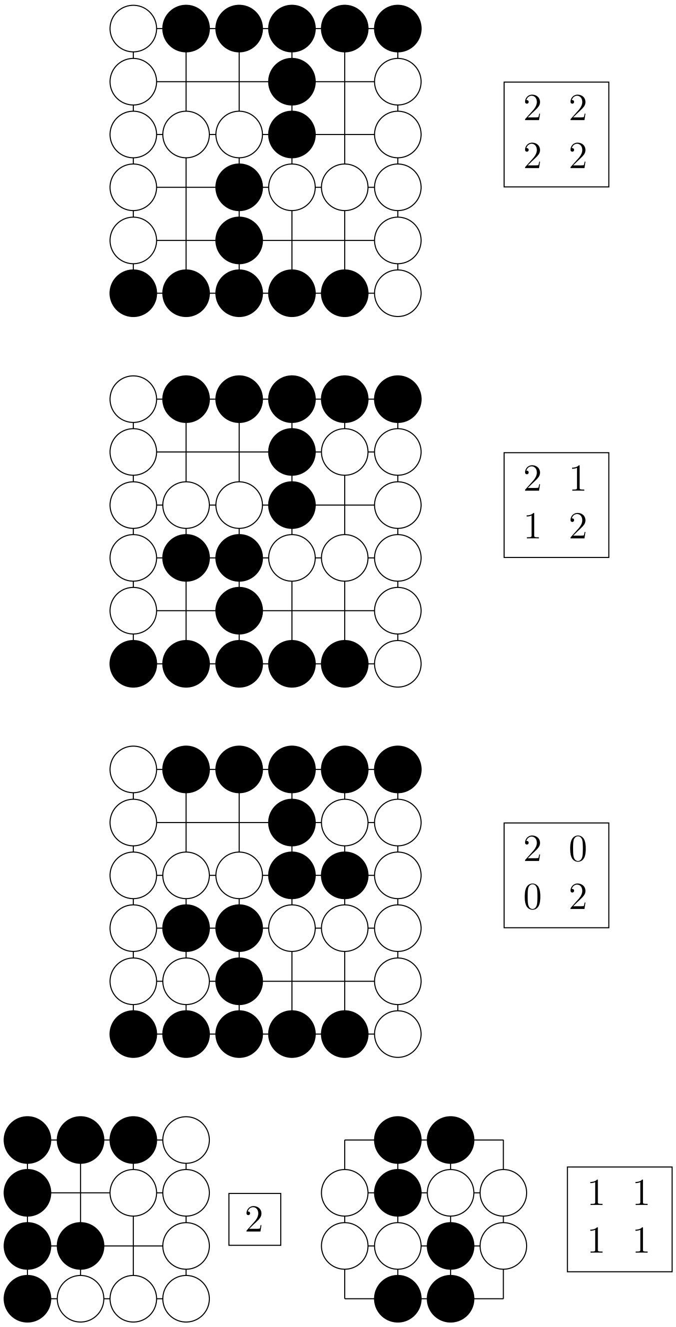
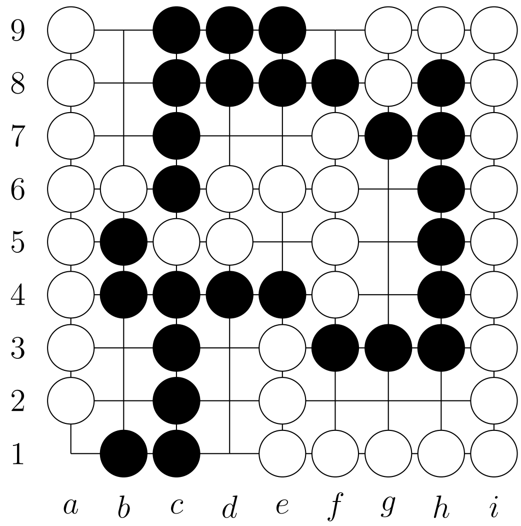


Figure 2: Five standard complete seki positions in GO and the corresponding 1×1 and four 2×2 complete seki matrices.

Standard examples of complete seki positions are given in Figure 2. The fourth position involves two groups, one White and one Black, with two dame between them. The corresponding 1×1 CSM is $a_{1,1} = 2$. The remaining four positions involve four groups each, two White and two Black. They correspond exactly to four CSMs A^2_ℓ, $\ell = 1, 2, 3, 4$, from Section 1.4.

In terms of GO, Conjecture 2 means that in each (complete) seki position the numbers of involved White and Black groups must be equal.

11.2 More sophisticated complete seki positions



0	3	3
3	3	1
3	1	2

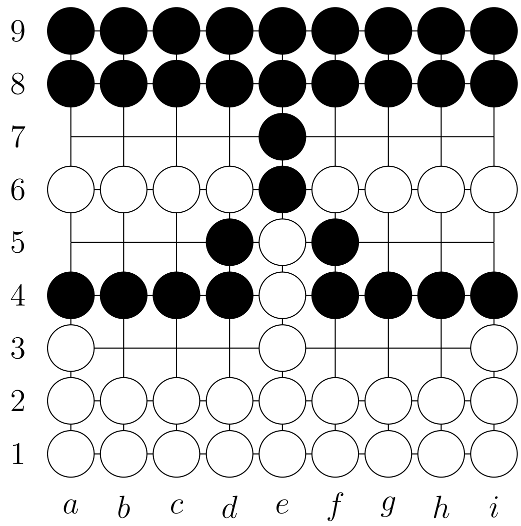
Figure 3: A complete seki position corrersponding to the matrix A^3_7.

Let us consider position in Figure 3. Is it a complete seki? This question may be difficult even for advanced players, because such a complicated seki would hardly appear in practice. To answer the question one should be smart in playing game Seki with matrices rather than in GO. In fact, the answer is positive, since the corresponding matrix is the CSM A^3_7.

Indeed, the left, right, and middle White groups correspond to the rows i_1, i_2 and i_3, while the right, lower, and upper Black groups correspond to the columns j_1, j_2 and j_3, respectively. Since $a_{1,1} = 0$, the corresponding White and Black groups have no dame in common. (Let us remark that if all nine entries of a 3×3 matrix are strictly positive then the corresponding GO position cannot exist on a plane board, by Kuratowski's Theorem.) The remaining eight positive entries $a_{1,2} = 3$, $a_{1,3} = 3$, $a_{2,1} = 3$, $a_{2,2} = 3$, $a_{2,3} = 1$, $a_{3,1} = 3$, $a_{3,2} = 1$, $a_{3,3} = 2$ correspond to the following eight sets of dame: $\{a1, b2, b3\}$, $\{b7, b8, b9\}$, $\{f2, g2, h2\}$, $\{d1, d2, d3\}$, $\{f9\}$, $\{g4, g5, g6\}$, $\{e5\}$, $\{d7, e7\}$.

Now the analysis of Section 1.5 can be translated as follows. If a player, Black or White, occupies d1,d2, or d3 then in the obtained position all six groups have the same number of dame, 6. However, this position cannot be a seki, since the previous one was a complete seki. Now the first player must win. Indeed, Black (respectively, White) wins playing at a1, b2, or b3 (respectively, at f2,g2, or h2). We leave complete case analysis to the reader.

11.3 Semi-complete seki positions



$$\boxed{\begin{array}{ccc} 0 & 3 & 3 \\ 4 & 0 & 3 \\ 4 & 3 & 0 \end{array}}$$

Figure 4: A semi-complete seki position corresponding to A_8^3.

Let us consider a GO position corresponding to the semi-complete seki matrix A_8^3 from Section 1.7.3; see Figure 4. The lower, left, and right White groups correspond to the rows i_1, i_2 and i_3, while the upper, right, and left Black groups correspond to the columns j_1, j_2 and j_3, respectively. The six positive entries of A_8^3: $a_{1,2} = 3$, $a_{1,3} = 3$, $a_{2,1} = 4$, $a_{2,3} = 3$, $a_{3,1} = 4$, $a_{3,2} = 3$, correspond to the following six sets of dame: $\{f3, g3, h3\}$, $\{b3, c3, d3\}$, $\{a7, b7, c7, d7\}$, $\{a5, b5, c5\}$, $\{f7, g7, h7, i7\}$, $\{g5, h5, i5\}$.

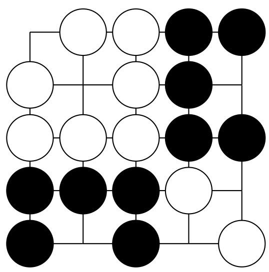
Now the analysis of Section 1.7.3 can be translated as follows. Since A_8^3 is a semi-CSM, the obtained GO position is a semi-complete seki, that is, only White can make an active move such that the obtained position is still a seki, while Black must pass. Indeed, White can occupy f3,g3, or h3. If after this Black will pass then White can even win playing at g5, h5, or i5. Instead, Black should occupy f7,g7,h7, or i7 that results in a (not complete) seki. We leave the full case analysis to the reader. Unlike White, Black must pass. Indeed, due to the obvious symmetry, he has only three distinct options: to play at f3,g3, or h3, at a7,b7,c7, or d7, or finally, at g5,h5, or i5. In the first case White can occupy g5,h5, or i5, or, less obviously, a7,b7,c7, or d7; in the second case, he answers at f3,g3, or h3; finally, in the third case he plays at f3,g3, or h3. Again, we leave it to the reader to verify that in all three cases White wins.

11.4 Seki with eyes

Let us now generalize game Seki in order to include into consideration groups with one eye. Given an $m \times n$ seki-matrix $A : I \times J \to \mathbb{Z}_+$, let us add one new special row i_0 to I and one new special column j_0 to J, and introduce arbitrary integral non-negative entries $a_{i_0,j}$ and a_{i,j_0} for all $j \in J$ and $i \in I$. The entry a_{i_0,j_0} is irrelevant and can be defined arbitrarily. Let us denote the obtained extended matrix by $A' : I' \times J' \to \mathbb{Z}_+$, where $I' = I \cup \{i_0\}$ and $J' = J \cup \{j_0\}$.

The entries $a_{i_0,j}$ and a_{i,j_0} are interpreted as the numbers of eyes of the Black group $j \in J$ and White group $i \in I$, respectively. In fact, keeping only GO in mind, we could restrict ourselves by the case when $a_{i_0,j}$ and a_{i,j_0} take only values 0 or 1. Yet, for the sake of generality we may allow arbitrary non-negative integral entries.

The rules of game Seki are changed as follows: after column j (row i)) is eliminated in A player C (respectively, R) needs $a_{i_0,j}$ (respectively, a_{i,j_0}) extra moves to eliminate it in A', and (s)he cannot reduce $a_{i_0,j}$ (respectively, a_{i,j_0}) before column j (row i)) is completely eliminated in A. We return to the standard game Seki if all entries of i_0 and j_0 equal 0. The definitions of the winner, loser, seki, complete seki, etc., are the same for the generalized and standard versions.



$$\begin{array}{cc|c} 1 & 1 & 0 \\ \hline 1 & 1 & \end{array}$$

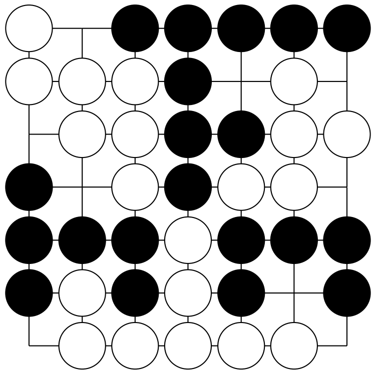
Figure 5: Complete seki with eyes.

Two examples of the generalized game Seki and the corresponding positions in GO are given on Figures 5 and 6. The position in Figure 5 is a complete seki and the corresponding 1×2 matrix is a CSM. Let us note that in this case $m \neq n$: two Black groups and only one White group are involved in this seki. The second White group is alive, since it has two eyes. In the next section we will show that the position in Figure 6 is a complete seki too.

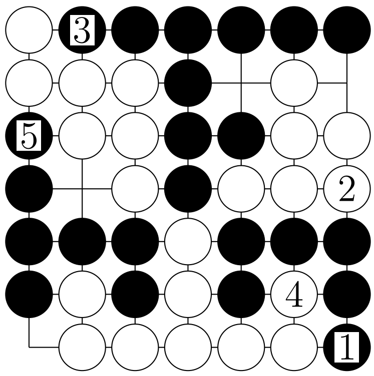
Thus, Conjecture 2, does not hold for seki with eyes, that is, such a seki may have m White and n Black groups such that $m \neq n$ and there is at most one eye in each group. Let us recall that, by Conjecture 2, $m = n$ whenever all groups have no eyes. Is it true that $m = n$ if each group has exactly one eye?

11.5 Complications

Even if we restrict ourselves by the seki-type positions satisfying all the above conditions, still GO is much more complicated game than Seki.



0	1	2
1	2	0
2	0	1



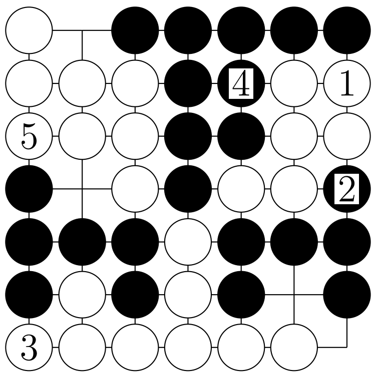


Figure 6: A non-seki position corresponding to a CSM.

Let P be a position in GO and A be the corresponding matrix. It may happen that P is a complete seki, while A is not a seki matrix, and vice versa, A may be a CSM, while P is not a seki position at all.

It can be explained as follows. In game Seki if R eliminates a row or C a column then (s)he is immediately claimed the winner and the game is over.

In GO the situation is more sophisticated, since groups can be sacrificed. If White takes a large Black's group in one (or several) moves after Black has taken a smaller White's group then White may win in GO, though R certainly wins in the corresponding game Seki.

Let us recall an example from [3]; see Figure 6. Here A is a CSM. However, P is not a seki, since both Black and White can sacrifice a group and take a larger opponent's group a bit later. If White begins he can force a draw; if Black begins she wins by 2 points; see Figure 6.

Nevertheless, let us confirm that our example in Figure 3 is a real complete seki. Indeed, in this position there are two Black and two White groups that cannot be sacrificed, since otherwise all three opponent's groups will become connected. Only the left White group (i_1) could be exchanged for the right Black group (j_1), and vice versa. However, these two groups are of equal size, each has nine stones. To sacrifice one of them a player must add to it one extra stone that will result in loss of two points.

Let us also note that to each CSM we can assign a complete seki position defined in a graph G (though we cannot guarantee that G is the 19×19 planar grid). Indeed, given a CSM $A : I \times J \to \mathbb{Z}_+$, we can easily choose G such that each White group $i \in I$ and each Black group $j \in J$ are the neighbors in G and, moreover, they have sufficiently many stones adjacent in G (though they may have no common liberties). In this case no group can be sacrificed, since all groups of a player become safe whenever an opponent's group is killed.

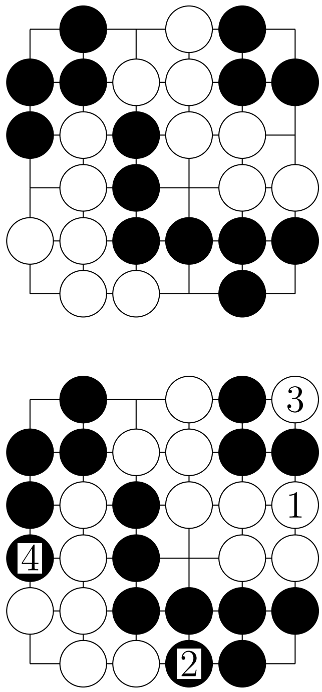
Now let us recall another example from [3]; see Figure 7. Here A is not a seki matrix. Clearly, C can eliminate the second column in two moves. Doing so he wins in game Seki. However, in GO the corresponding strategy is bad for White. Indeed, though White takes first the small group in the upper right corner, yet, Black will retaliate in two moves by taking the larger group in the opposite corner. It is not difficult to check that in fact this position in GO is a complete seki.

Thus, analyzing a position P in GO, one must take into account not only its seki matrix A but also many other factors, such as sizes of groups, exchange options, etc.

In [2] there are many other interesting examples of seki positions in GO in which each player can take an opponent's group, yet, it would result in a loss.

Thus, CSMs may correspond to complete seki positions in GO. However, some CSMs do not and, vice versa, a complete seki position may correspond to a non-seki matrix; see Sections 11.5. In general, CSMs describe only some very special seki positions in GO. There are many other types, of totally different nature; see, for example, Haneseki and Nakadeseki in [2].

What do we call "a group"? It is not that easy to be accurate in this issue. Typically, a Black (respectively, White) group is a set of Black (respectively, White) stones that form a



$$\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & \end{array}$$

Figure 7: A complete seki position whose matrix is not a seki.

connected component in the GO-graph. Standardly, such a graph is the 19×19 (or 13×13, or 9×9) plane grid. Yet, sometimes several connected components with common neighbors (*dame or liberties*, in terms of GO) form just one group. For example, in Figure 5 two White stones in the lower right corner are not connected, yet, they form a single group. More examples can be found in [2].

Finally, let us remark that not every seki-matrix A can be realized on a standard GO-board. For example, the sum of all entries of A may be too large.

Furthermore, if A contains a 3×3 submatrix whose all entries are strictly positive then A cannot be realized either. Moreover, it cannot be realized on a planar graph (not to mention a planar grid), since the graph $K_{3,3}$ is not planar, by Kuratowski's Theorem. Yet, a 3×3 CSM A_7^3 has only one 0 entry and the corresponding position is realized on the board 9×9, see Figure 3. In general, all seki matrices could be realized if instead of the standard GO board we allow to play GO on arbitrary graphs.

11.6 Bridge and Single-Suit two-person game

The relation between GO and Seki is somewhat similar to the relation between Bridge and the Single-suit two-person game [4]. This game was introduced in 1929 by Emanuel Lasker who called it Whistette. It is defined as follows: $2n$ cards with the numbers $1, \ldots, 2n$ are shuffled and divided into two hands A and B of n cards each, held by players R and C. Let us say, C begins. He chooses one of his cards and put in on the table. Then R, after seeing this card, selects one of her own. The player with the higher card wins the trick and gets the lead. Two cards of the trick are removed and play continues until there are no more cards. The goal of each player is to win as many tricks as possible.

The game is not trivial already for $n = 3$. For example, let C and R get $6, 4, 2$ and $5, 3, 1$ (or in Bridge terms, A,Q,10 and K,J,9), respectively. If C leads with 6 then he will win only the first trick, yet, if C leads with 2 or 4 then he will get two tricks. For another example, let C and R get $5, 4, 2$ and $6, 3, 1$ (or in Bridge terms, K,Q,10 and A,J,9), respectively. If C leads with 5 or 4 and R wins with 6 then she will get no more tricks, yet, if R discards 1 in the first trick then she will win the remaining two tricks. For larger n, say $n = 20$, Whistette becomes complicated. Many interesting results are obtained in [4]. Recently, Johan Wästlund [6] got a polynomial algorithm solving Whistette.

In some situations, Bridge is reduced to Whistette. Let us consider, for example, the six-card end-play in which North and West have only trumps, say, $3, 6, 10, J, Q, A$ and $4, 5, 7, 8, 9, K$, respectively. However, such situations are very rare. Furthermore, a positions with $n > 6$ cannot appear in Bridge at all, since there are only 13 cards in each suit. Thus, it is not necessary for a Bridge expert to be good at Whistette. Similarly, games GO and Seki demand very different skills. Unlike Bridge and GO, Whistette and Seki are pretty boring games. Yet, they are complicated too. At least, there are many positions that are difficult to analyze even for advanced Bridge and GO players.

Acknowledgments The authors are thankful to Endre Boros, Jeff Kahn, and Vladimir Oudalov for helpful discussions and also to Klaus Heine and Harry Fearnley who saved the old manuscript [3] that otherwise would be lost.

The third author was partially supported by the National Science Foundation (Grant IIS-0118635) and by DIMACS, Center for Discrete Mathematics and Theoretical Computer Science.

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