DIMACS Technical Report 2008-15
November 2008

Neighborhood hypergraphs of digraphs and some matrix permutation problems

by

Vladimir A. Gurvich[1]
RUTCOR, Rutgers Center for Operations Research
Rutgers, The State University of New Jersey
640 Bartholomew Road, Piscataway, NJ 08854-8003, USA
gurvich@rutcor.rutgers.edu

Igor E. Zverovich
RUTCOR, Rutgers Center for Operations Research
Rutgers, The State University of New Jersey
640 Bartholomew Road, Piscataway, NJ 08854-8003, USA
igor@rutcor.rutgers.edu

[1]This research was partially supported by DIMACS, Center for Discrete Mathematics and Theoretical Computer Science, Rutgers University.

ABSTRACT

Given a digraph D, the set of all pairs $(N^-(v), N^+(v))$ constitutes the neighborhood dihypergraph $\mathcal{N}(D)$ of D. The Digraph Realization Problem asks whether a given dihypergraph H coincides with $\mathcal{N}(D)$ for some digraph D. This problem was introduced by Aigner and Triesch [2] as a natural generalization of the Open Neighborhood Realization Problem for undirected graphs, which is known to be NP-complete.

We show that the Digraph Realization Problem remains NP-complete for orgraphs (orientations of undirected graphs). As a corollary, we show that the Matrix Skew-Symmetrization Problem for square $\{0, 1, -1\}$ matrices ($a_{ij} = -a_{ji}$) is NP-complete. This result can be compared with the known fact that the Matrix Symmetrization Problem for square $0-1$ matrices ($a_{ij} = a_{ji}$) is NP-complete.

Extending a negative result of Fomin, Kratochvíl, Lokshtanov, Mancini, and Telle [15] we show that the Digraph Realization Problem remains NP-complete for almost all hereditary classes of digraphs defined by a unique minimal forbidden subdigraph.

Finally, we consider the Matrix Complementation Problem for rectangular $0-1$ matrices, and prove that it is polynomial-time equivalent to graph isomorphism. A related known result is that the Matrix Transposability Problem is polynomial-time equivalent to graph isomorphism.

2000 Mathematics Subject Classification:
68Q25 (Analysis of algorithms and problem complexity),
68R10 (Graph theory in relation to computer science),
05C62 (Graph representations),
05C20 (Directed graphs (digraphs), tournaments),
05C65 (Hypergraphs),
05C70 (Factorization, matching, covering and packing),
05C60 (Isomorphism problems),
05B20 ($(0, 1)$-matrices (combinatorics)),
05C85 (Graph algorithms).
Neighborhood hypergraphs of digraphs and orgraphs, Graph Isomorphism Problem, matrix symmetrization, matrix complementation, symmetrizability, skew-symmetrizability, involutory automorphisms

1 Introduction

Let $D = (V, A)$ be a digraph without loops and multiple arcs. For a vertex $v \in V$, we denote

$$N^-(v) = \{u \in V : (u, v) \in A\},$$

the *in-neighborhood* of v, and

$$N^+(v) = \{w \in V : (v, w) \in A\},$$

the *out-neighborhood* of v. Suppose that we know all pairs $(N^-(v), N^+(v))$, is it possible to restore the digraph? To formalize the problem, let us define a *directed hypergraph*, or shortly *dihypergraph*, as an ordered pair $(V, A) = H$ consisting of a finite set V, the *vertex-set* of H, and a finite multi-set of hyperarcs, a *hyperarcs* $a \in A$ being an ordered pair $(a^-, a^+) = a$ of some subsets a^- and a^+ of V. It is possible that $a^- = \emptyset$ or $a^+ = \emptyset$ or $a^- = a^+$. Also note that a^- and a^+ are not necessarily disjoint.

Definition 1. *The* neighborhood dihypergraph *of a digraph $D = (V, A)$, $\mathcal{N}(D)$, has V as its vertex-set, and $A(\mathcal{N}(D)) = \{(N^-(v), N^+(v)) : v \in V\}$.*

An obvious property of $\mathcal{N}(D)$ is that the number of vertices is the same as the number of hyperarcs. The following problem was proposed by Aigner and Triesch [2].

Decision Problem 1 (Digraph Realization Problem)**.**
Instance: *A directed hypergraph H.*
Question: *Does $H = \mathcal{N}(D)$ hold for some digraph D?*

This problem generalizes the *Open Neighborhood Realization Problem* for undirected graphs: given a hypergraph H (with possible multiple hyperedges), the problem is asking to find a graph G for which H is the hypergraph of open neighborhoods $\mathcal{N}^{\mathrm{op}}(G)$, of vertices of G, that is $V(H) = V(G)$ and $E(H) = \{N(v) : v \in V(G)\}$. Here $N(v) = \{w \in V(G) : vw \in E(G)\}$ is the *neighborhood* of a vertex v of G. The Open Neighborhood Realization Problem was proposed by Sós [21] under the name the Star System Problem, and it is also attributed to G. Sabidussi by Babai [4]. Also, Babai [4] noticed that the problem is at least as hard as graph isomorphism. The *Graph Isomorphism Problem* is well-known: Are two given graphs isomorphic? Boros, Gurvich, and Zverovich [8] survey different equivalent formulations of the problem.

The *Closed Neighborhood Realization Problem* is defined in a similar way, using the *closed neighborhoods* $N[v] = \{v\} \cup N(v)$ of vertices. Also, one can consider a hypergraph $\mathcal{N}(G)$ of open and closed neighborhoods of G, that is, for each vertex v either $N(v)$ or $N[v]$ is a hyperedge of $\mathcal{N}(G)$. The *Neighborhood Realization Problem* is to decide whether a given hypergraph H is $\mathcal{N}(G)$ for some graph G.

Theorem 1 (Lalonde [16, 17])**.** *The* Open Neighborhood Realization Problem*, the* Closed Neighborhood Realization Problem*, and the* Neighborhood Realization Problem *are NP-complete.*

An undirected graph G can be viewed as a digraph on $V(G)$ if we replace every edge $uv \in E(G)$ by the corresponding pair (u, v), (v, u) of opposite arcs.

Corollary 1 (Aigner and Triesch [2])**.** *The* Digraph Realization Problem *is NP-complete.*

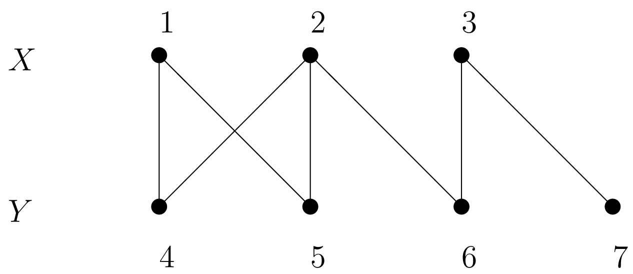
Theorem 1 has an interesting interpretation. A square matrix $A = (a_{ij})$ is *symmetric* if $a_{ij} = a_{ji}$ for all i and j. A square matrix A is *symmetrizable* if it is possible to permute rows of A in such a way that the resulting matrix is symmetric. The Neighborhood Realization Problem is equivalent to the *Matrix Symmetrization Problem*: Is a given square $0-1$ matrix is symmetrizable? If we additionally require that all entries in the main diagonal are 0s (respectively, 1s), then we obtain a problem which is equivalent to the Open (respectively, Closed) Neighborhood Realization Problem. The three symmetrization problems are NP-complete.

We show that the Digraph Realization Problem remains NP-complete for orgraphs (orientations of undirected graphs) and for almost all hereditary classes of digraphs defined by a unique minimal forbidden subdigraph. As a corollary, we show that the Matrix Skew-Symmetrization Problem for square $\{0, 1, -1\}$ matrices is NP-complete. The problem is to bring a matrix to skew form ($a_{ij} = -a_{ji}$) using permutations of rows. Then we consider the Matrix Complementation Problem for rectangular $0 - 1$ matrices: to construct the complementary matrix (defined by $\overline{a}_{ij} = 1 - a_{ji}$) using row and column permutations. We prove that it is polynomial-time equivalent to graph isomorphism.

2 Representations

It is convenient to represent hypergraphs as bipartite graphs. and as their adjacency matrices. A *bigraph* $B = (X, Y, E)$ is defined as a bipartite graph on vertex-set $V = X \cup Y$ with a fixed order (X, Y) of its parts. Here $X \cap Y = \emptyset$ and $E \subseteq X \times Y$. To a bigraph $B = (X, Y, E)$ we can associate its *X-Y-adjacency* matrix $A(B) = (a_{ij}) \in \{0, 1\}^{X \times Y}$ defined by $a_{ij} = 1$ if and only if $(i, j) \in E$. Conversely, any $0 - 1$ matrix $A = (a_{ij})$ can be viewed as the X-Y adjacency matrix $A = A(B)$ of a corresponding bigraph $B = (X, Y, E)$, where X is the set of row indices of A, Y is the set of column indices of A, and $(i, j) \in E$ if and only if $a_{ij} = 1$, see an example in Figure 1.

Now we consider similar representations of a dihypergraph H. Let us define a *directed bigraph* $B = (X, Y, A)$ as a bipartite digraph on vertex-set $X \cup Y$ with a fixed order (X, Y) of its parts, i.e., where $X \cap Y = \emptyset$ and $A \subseteq (X \times Y) \cup (Y \times X)$.



$B = (X, Y, E)$

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$A(B)$

Figure 1: A bigraph $B = (X, Y, E)$ and its adjacency matrix $A(B)$.

Definition 2. *Given a dihypergraph H, we construct a directed bigraph B_H as follows. For every vertex v of H, we introduce a vertex in X, which is also called v. For every hyperarc $a = (a^-, a^+)$, we introduce a vertex $a \in Y$. Whenever $v \in a^-$, there is the arc (v, a) in B_H. Whenever $v \in a^+$, there is the arc (a, v) in B_H.*

As an example, consider the neighborhood dihypergraph $H = (V, A)$ of the digraph D shown in Figure 2: $V = \{u, v, w, x\}$, $A = \{a_u, a_v, a_w, a_x\}$, where
$a_u = (\{v\}, \emptyset)$,
$a_v = (\{w\}, \{u, w\})$,
$a_w = (\{v, x\}, \{v\})$, and
$a_x = (\emptyset, \{w\})$.

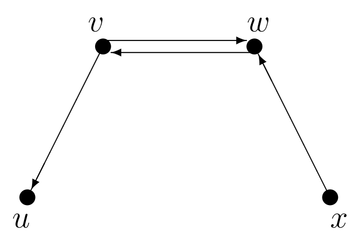


Figure 2: A digraph D.

The directed bigraph B_H of H is shown in Figure 3.

Consider a directed bigraph $B = (X, Y, A)$ and an automorphism $\alpha : (X \cup Y) \to (X \cup Y)$ of the underlying bipartite digraph B, that is for which $(i, j) \in A$ if and only if $(\alpha(i), \alpha(j)) \in A$. The automorphism α *involutory* if $\alpha(i) = j$ implies $\alpha(j) = i$, that is α^2 is identity, and it

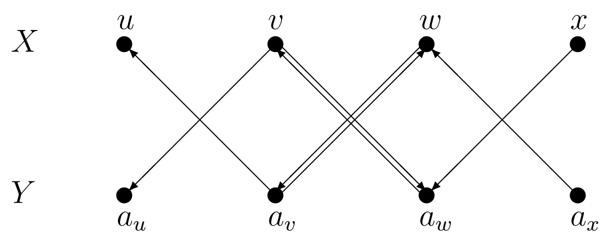


Figure 3: The directed bigraph B_H of H.

is called *switching* if $\alpha(X) = Y$ and $\alpha(Y) = X$. The Digraph Realization Problem for a directed hypergraph H can be equivalently formulated in terms of B_H: Does B_H admit an involutory switching automorphism α such that x and $\alpha(x)$ are non-adjacent for all $x \in X$?

To a directed bigraph $B = (X, Y, A)$ we can associate its X-Y-*adjacency* matrix $A(B) = (a_{ij}) \in \{0, 1, -1, \pm 1\}^{X \times Y}$ defined by

- $a_{ij} = 0$ if and only if $i \in X$, $j \in Y$, $(i, j) \notin A$ and $(j, i) \notin A$,
- $a_{ij} = 1$ if and only if $i \in X$, $j \in Y$, $(i, j) \in A$ and $(j, i) \notin A$,
- $a_{ij} = -1$ if and only if $i \in X$, $j \in Y$, $(j, i) \in A$ and $(i, j) \notin A$,
- $a_{ij} = \pm 1$ if and only if $i \in X$, $j \in Y$, $(i, j) \in A$ and $(j, i) \in A$.

We have

$$A(B_H) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & \pm 1 & 0 \\ 0 & \pm 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for the directed bigraph B_H of Figure 3.

3 Orgraph realizations and skew symmetrization

An *orgraph* is an orientation of an undirected graph. In other words, an orgraph is a digraph having no pairs of opposite arcs. Here we consider Decision Problem 1 for orgraphs – the Orgraph Realization Problem.

Theorem 2. *The* Orgraph Realization Problem *is NP-complete.*

Proof. We construct a polynomial-time reduction from the Neighborhood Realization Problem, which is NP-complete by Theorem 1. Let H be an instance to the problem represented as a bigraph $B = (X, Y, E)$. In terms of B, the problem is to recognize whether B has an involutory automorphism α (that is α^2 is identical) which switches the parts ($\alpha(X) = Y$). Without loss of generality, we may assume that all vertex degrees in B are at least three. To satisfy this assumption we can add $i \leq 3$ new vertices into each part, making them adjacent to all vertices in the opposite part.

Now we transform B into a directed bigraph $B' = (X', Y', A)$ by replacing every edge $e = xy \in E$, where $x \in X$ and $y \in Y$, by a directed 6-cycle

$$C^e = (x = x_1^e, y_1^e, x_2^e, y = y_2^e, x_3^e, y_3^e), \tag{1}$$

and put the vertices x_i^e and y_i^e into the parts X' and Y' of B', respectively, see Figure 4 for an illustration.

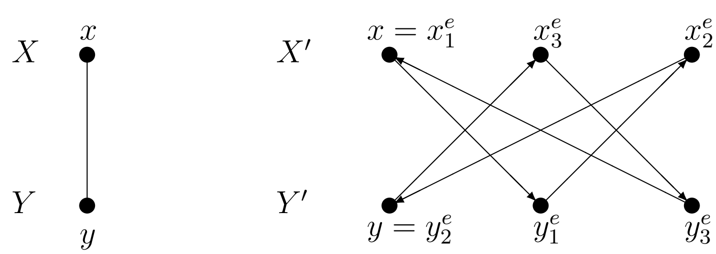


Figure 4: The construction of a directed bigraph $B' = (X', Y', A)$.

The directed bigraph B' represents a dihypergraph H' which is considered as an instance to the Orgraph Realization Problem. In terms of B', the problem is to recognize whether B' has an involutory automorphism α' which switches the parts X and Y', and such that x' and $\alpha'(x')$ are always non-adjacent, where $x' \in X'$.

Suppose that B admits an involutory automorphism α that switches the parts X and Y. If some vertices $x \in X$ and $y = \alpha(x) \in Y$ are adjacent, let $a = xy$, then we define $\alpha'(x) = y$,

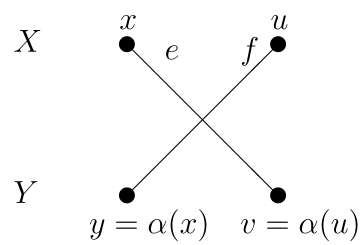


Figure 5: The edges $e = xv$ and $f = uy$ of B.

$\alpha'(x_2^a) = y_3^a$ and $\alpha'(x_3^a) = y_1^a$, see the correspondence in Figure 4. Now consider two edges $e = xv$ and $f = uy$ of B such that $y = \alpha(x) \neq v = \alpha(u)$, as it is shown in Figure 5.

The vertices

$$x = x_1^e, y_1^e, x_2^e, v = y_2^e, x_3^e, y_3^e$$

of the directed cycle C^e will be mapped by α' to the vertices

$$y = y_2^f, x_3^f, y_3^f, u = x_1^f, y_1^f, x_2^f$$

of the directed cycle C^f, respectively, as it is shown in Figure 6. It is easy to see that α' is an involutory automorphism of B' that switches X' and Y'. Also, x' and $\alpha'(x')$ are non-adjacent for all $x' \in X'$.

Conversely, let α' be an involutory automorphism of B' switching X' and Y', and such that x' and $\alpha'(x')$ are non-adjacent for all $x' \in X'$. The degree assumption implies that α' pairs the vertices of X with the vertices of Y. Thus, α' induces an involutory bijection α on B that switches X and Y. Finally,α is an automorphism of B. Indeed, let $y = \alpha(x)$ and $v = \alpha(u)$ for some distinct vertices $x, u \in X$. Suppose that $e = xv$ is an edge of B. It is easy to see that the directed 6-cycle C^e can be mapped by α' to another directed 6-cycle as in Figure 6 only. It shows that u and y must be adjacent. □

A square matrix $A = (a_{ij})$ is called *skew* if $a_{ij} = -a_{ji}$ for all i and j. In other words, $A = -A^T$, where A^T is the transpose of A. Clearly, all entries on the main diagonal must be zeroes. A square matrix A is *skew-symmetrizable* if it is possible to obtains a skew matrix permuting rows of A.

Decision Problem 2 (Skew-Symmetrization Problem)**.**
Instance: *A square* $\{0, 1, -1\}$ *matrix* A.
Question: *Is* A *a skew-symmetrizable matrix?*

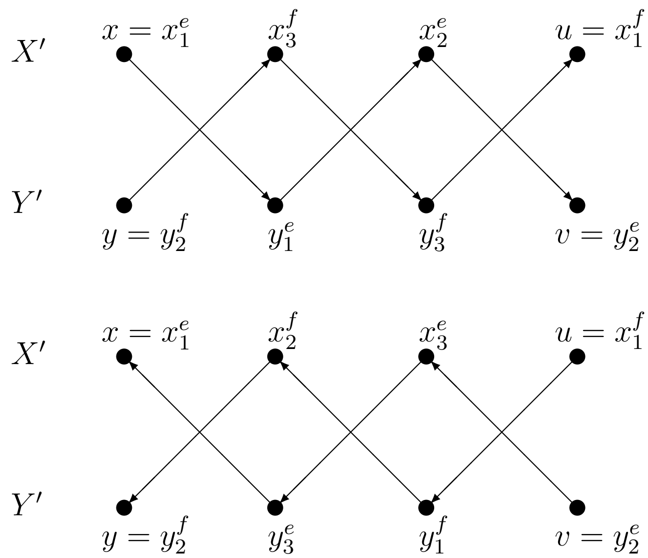


Figure 6: The automorphism α'.

The Orgraph Realization Problem is essentially the same as the Skew-Symmetrization Problem. Let a dihypergraph H be an instance to the Orgraph Realization Problem. We may assume that $|V(H)| = |A(H)|$. The directed bigraph B of H does not have pairs of opposite arcs (otherwise H has no orgraph realizations). The $\{0, 1, -1\}$ adjacency matrix of B is skew-symmetrizable if and only if $H = \mathcal{N}(D)$ for some orgraph D.

Corollary 2. *The* Matrix Skew-Symmetrization Problem *is NP-complete.*

It is interesting to study the Matrix Skew-Symmetrization Problem within hereditary classes of orgraphs, in particular for D-free orgraphs.

4 Skew transposability

We write $A \to B$ if a matrix A can be transformed to a matrix B with row and column permutations. Here we consider the following problem which is related to skew symmetrizability. A square matrix A is *skew-transposable* if $A \to -A^T$, where A^T is the transpose of A.

Decision Problem 3 (Skew Transposability Problem)**.**
Instance: *A square* $\{0, 1, -1\}$ *matrix* A*.*
Question: *Is* A *a skew-transposable matrix?*

Here is a relation between the two problems.

Proposition 1. *Every skew-symmetrizable matrix A is skew-transposable.*

Proof. By the definition of skew-symmetrizability, there exists a permutation matrix P such that PA is skew-symmetric, that is $PA = -(PA)^T = -A^T P^T$. To show that $A \to -A^T$, we apply P to the columns of PA: $PAP = -A^T P^T P = -A^T$, meaning that A skew-transposable. □

If we represent a square $\{0, 1, -1\}$ matrix A as a directed bigraph $B = (X, Y, A)$, then the matrix $-A^T$ produces the *reversed* directed bigraph $B' = (Y, X, A)$. For example, let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

We have

$$-A^T = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

The corresponding directed bigraphs B and B' are shown in Figure 7.

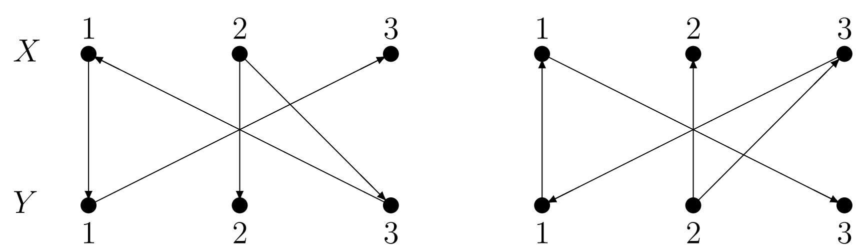


Figure 7: The directed bigraphs B and B'.

Now we clarify the complexity of Decision Problem 3.

Proposition 2. *The* Skew Transposability Problem *is polynomial-time equivalent to graph isomorphism.*

Proof. The Skew Transposability Problem is equivalent to checking whether B and B' are isomorphic, which a particular case of graph isomorphism. Conversely, suppose we want to check isomorphism of graphs G and H. We represent G as a a directed bigraph $B_G = (X_G, Y_G, A_G)$, where $X_G = V(G)$, $Y_G = E(G)$, and every edge $e = uv \in E(G)$ produces two arcs (u, e) and (v, e) in B. A similar bigraph $B_H = (X_H, Y_H, A_H)$ is defined for H, and

$B'_H = (Y'_H, X'_H, A'_H)$ is obtained by reversing of B_H. Let B is disjoint union of B_G and B'_H. Accordingly, B' is disjoint union of B'_G and B_H. Assuming that both G and H do not have isolated vertices, G and H are isomorphic if and only if B and B' are. □

5 Digraph realizations within hereditary classes

Fomin, Kratochvíl, Lokshtanov, Mancini, and Telle [15] studied the Open Neighborhood Realization Problem within hereditary classes.

Definition 3. *Let $\mathcal{P}$ be hereditary class of graphs. A* $\mathcal{P}$-realization *of a hypergraph H is a graph $G \in \mathcal{P}$ such that $\mathcal{N}(G) = H$. If $\mathcal{P}$ is defined by a unique minimal forbidden induced subgraph H, then a $\mathcal{P}$-realization is called an* H-free realization *of H.*

Definition 3 is extended to digraphs in a straightforward way.

A *star-like graph* consists of $k \geq 1$ paths $Q_i = (u_0, u_{i1}, u_{i2}, \ldots, u_{id_i})$, $i = 1, 2, \ldots, k$, having a common vertex u_0. Here $d_i \geq 0$ for $i = 1, 2, \ldots, k$. An example of a star-like graph with $k = 3$, $d_1 = 3$, $d_2 = 4$, and $d_3 = 2$ is shown in Figure 8.

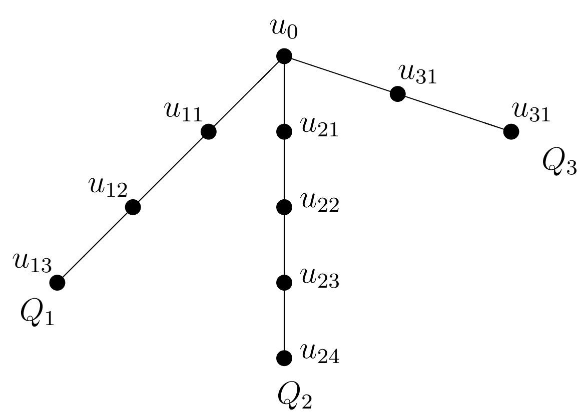


Figure 8: An example of a star-like graph.

If every connected component of a graph G is star-like, then G is called an *S-graph*. Fomin, Kratochvíl, Lokshtanov, Mancini, and Telle [15] proved the following result in the complementary form (for closed neighborhood hypergraphs).

Theorem 3. *If H is not an S-graph, then it is NP-hard to decide whether a given hypergraph has an H-free realization.*

Theorem 3 can be easily extended to $\mathcal{P}$-realizations, where $\mathcal{P}$ is a hereditary class with a finite set $Z(\mathcal{P})$ of minimal forbidden induced subgraphs.

Theorem 4. *If $Z(\mathcal{P})$ is a finite set and it does not contain an S-graph, then it is NP-hard to decide whether a given hypergraph has a $\mathcal{P}$-realization.*

If H is an S-graph, then complexity of the H-free realization problem is unknown, except the following polynomial-time solvable cases: $H \in \{\overline{P}_1, \overline{P}_2, \overline{P}_3, \overline{P}_4, \overline{C}_3, \overline{C}_4\}$, where P_k and C_k are the path and the cycle with k vertices, and $\overline{G}$ is the complement of G, see Fomin, Kratochvíl, Lokshtanov, Mancini, and Telle [15].

We are going to extend Theorem 3 and Theorem 4 to digraphs.

A *star-like digraph of type* 1 is obtained from a star-like graph G if we replace every edge $uv \in E(G)$ by the corresponding pair (u, v), (v, u) of opposite arcs. A *star-like digraph of type* 2 consists of $k \geq 1$ directed paths

$$Q_i = (u_0, u_{i1}, u_{i2}, \ldots, u_{id_i}),$$

$i = 1, 2, \ldots, k$, having a common vertex u_0, and of $l \geq 0$ directed paths

$$R_j = (v_{j1}, v_{j2}, \ldots, v_{je_j}, u_0),$$

$j = 1, 2, \ldots, l$, having a common vertex u_0. Here $d_i \geq 0$ and $e_j \geq 0$ for all i and j. An example of a star-like graph with $k = 3$, $d_1 = 3$, $d_2 = 4$, $d_3 = 2$, $l = 2$, $e_1 = 3$ and $e_2 = 2$ is shown in Figure 9.

If every weakly connected component of a digraph D is a star-like digraph of type i, then D is called an *S_i-digraph*, $i = 1, 2$.

Theorem 5. *If a digraph D has at least one arc, then it is NP-hard to decide whether a given dihypergraph H has a D-free realization.*

Proof. First we apply Theorem 3 to a *symmetric* dihypergraph H, that is $a^- = a^+$ for every hyperarc (a^-, a^+) of H.

Property 1. *If D is not an S_1-digraph, then it is NP-hard to decide whether a symmetric dihypergraph H has a D-free realization.*

Proof. A digraph is *symmetric* if (u, v) is an arc if and only if (v, u) is an arc. Essentially, a symmetric digraph is an undirected graph. Clearly, every realization of a symmetric dihypergraph is a symmetric digraph, and Theorem 3 implies the result, since D is not an S_1-digraph. $\square$

Now we consider S_2-digraphs.

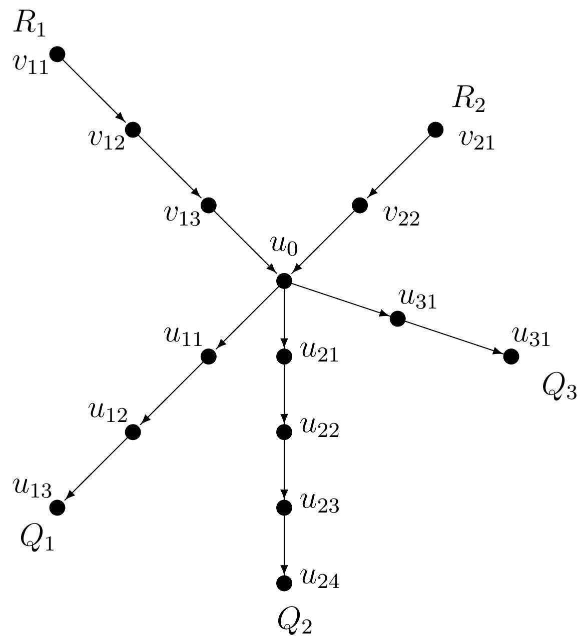


Figure 9: An example of a star-like digraph of type 2.

Property 2. *If D is not an S_2-digraph, then it is NP-hard to decide whether a given directed hypergraph has a D-free realization.*

Proof. We modify the proof of Theorem 2 in the following way. Instead of a directed 6-cycle C^e for an edge $e = xy$ as in (1), we introduce a $(4t+2)$-cycle C^e

$$C^e = (x = x_1^e, y_1^e, x_2^e, y_2^e, \dots, x_t^e, y = y_t^e, \dots, x_{2t+1}^e, y_{2t+1}^e) \tag{2}$$

for a fixed $t \geq 1$. The resulting dihypergraph and directed bigraph are denoted by H' and B', respectively. We shall specify t so that every realization of H' does not contain the forbidden induced subdigraph D. Let t_1 be the minimum length of a cycle (not necessarily directed) in D. If D is acyclic then $t_1 = \infty$. A *knot vertex* of D is a vertex u such that either

- $|N^-(u)| + |N^+(u)| \geq 3$, or
- $|N^-(u)| = 2$, or
- $|N^+(u)| = 2$.

Let t_2 be the minimum length of a path (not necessarily directed) in D that connects two knot vertices in D. If D does not have such paths, then $t_2 = \infty$. At least one of t_1 and t_2 is finite, since D is not an S_2-digraph. It is sufficient to take $t = \min\{t_1, t_2\}$. $\square$

Property 1 and Property 2 show that the problem is NP-hard unless D is both an S_1-digraph and an S_2-digraph. But it is possible only if D does not have arcs. □

Let O_n be an arcless digraph of order n.

Open Problem 1. *How hard is to decide whether a given directed hypergraph has an O_n-free realization, $n \geq 3$?*

For $n \leq 2$, the problem is trivially polynomial-time solvable.

6 Matrix complementation

Here we consider another interesting problem related to $0-1$ matrices. Let $A = (a_{ij})$ be an $m \times n$ matrix with $a_{ij} \in \{0, 1\}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The *complement* of A is the matrix $\overline{A} = (\overline{a}_{ij})$ defined by: $\overline{a}_{ij} = -a_{ij}$ for all i and j. Recall that $A \rightarrow B$ means that a matrix A can be transformed to a matrix B with row and column permutations.

Decision Problem 4 (Matrix Complementation Problem)**.**
Instance: *A $0-1$ matrix A.*
Question: *Does $A \rightarrow \overline{A}$ hold?*

As an example, consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Permuting row 1 and row 2, we obtain

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Now, permutation of column 2 and column 3 gives

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \overline{A},$$

therefore $A \rightarrow \overline{A}$.

We show that the Matrix Complementation Problem is polynomial-time equivalent to graph isomorphism. One can mention a related result of McCarthy and McKay [20] which says that the problem $A \rightarrow A^T$, where A is a square $0-1$ matrix A and A^T is the transpose of A, is also polynomial-time equivalent to graph isomorphism.

An obvious necessary condition for $A \to \overline{A}$ is that $A_0 = A_1$, where A_k denotes the total number of entries $a_{ij} = k$ in A. However, this condition is not sufficient. For example, it is impossible to get $\overline{A}$ from the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

where $A_0 = A_1 = 6$. Indeed, permuting columns of A, one can obtain the following six matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and, unlike $\overline{A}$, no one of them has two rows (011). Thus, $A \to \overline{A}$ does not hold.

Theorem 6. *The* Matrix Complementation Problem *and the* Graph Isomorphism Problem *are polynomial-time equivalent.*

Proof. First we represent A and $\overline{A}$ as bigraphs $B = (X, Y, E)$ and $B' = (X', Y', E')$, respectively. The bigraphs B and B' are *isomorphic* if there are bijections $\alpha : X \leftrightarrow X'$ and $\beta : Y \leftrightarrow Y'$ such that $(i, j) \in E$ if and only if $(\alpha(i), \beta(j)) \in E'$. The corresponding recognition problem is called *Bigraph Isomorphism*.

Fact 1. $A \to \overline{A}$ *holds if and only if the bigraphs* B *and* B' *are isomorphic.*

Proof. Indeed, a permutation α of rows and a permutation β of columns is nothing but an isomorphism of corresponding bigraphs. □

The *bi-complement* of B is the bigraph $\overline{B} = (X, Y, \overline{E})$, where

$$\overline{E} = \{xy : x \in X, y \in Y, xy \notin E\}.$$

Clearly, B' is isomorphic to $\overline{B}$. A bigraph is *self-bi-complementary* if B and $\overline{B}$ are isomorphic, see Bhave and Raghunathan [6]. In this terminology, Fact 1 says that $A \to \overline{A}$ holds if and only if B is a self-bi-complementary bigraph. Recognition of self-bi-complementary bigraphs is a particular case of the Bigraph Isomorphism Problem, therefore the Matrix Complementation Problem is not harder than graph isomorphism.

Fact 2. *The* Graph Isomorphism Problem *is polynomial-time reducible to recognition of self-bi-complementary bigraphs.*

Proof. Let G and H be an instance to the Graph Isomorphism Problem. Without loss of generality, we may assume that $|V(G)| = |V(H)| = n$, $|E(G)| = |E(H)| = m$ (otherwise G and H are not isomorphic) and both G and H do not have isolated vertices (otherwise we add a dominating vertex to each of them obtaining an equivalent instance).

We subdivide every edge of G and H with a new vertex, and denote the resulting graphs by G' and H', respectively. G' can be considered as a bigraph having $V(G)$ as its X-part (old vertices) and the set of $|E(G)|$ new vertices as its X-part. Similar situation takes place for H'. Now we use the graphs G' and H' to construct a bigraph $B = (X, Y, E)$ such that $G \cong H$ if and only if B is self-bi-complementary. For that, we take disjoint copies of G' and $\overline{H'}$ [the bi-complement of H], and introduce all edges between the X-part of G' and and the Y-part of $\overline{H'}$. Figure 10 illustrates the construction.

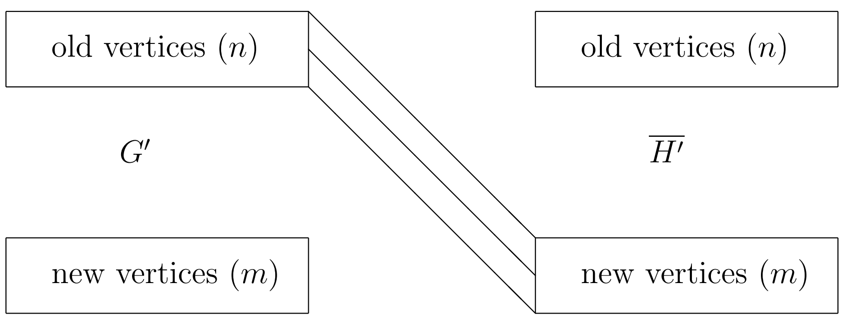


Figure 10: The construction of B.

The bi-complement $\overline{B}$ of B is shown in Figure 11, where $\overline{G'}$ and H' are the bi-complements of G' and H', respectively, and all edges between the X-part H' of and the Y-part of $\overline{G'}$ are included.

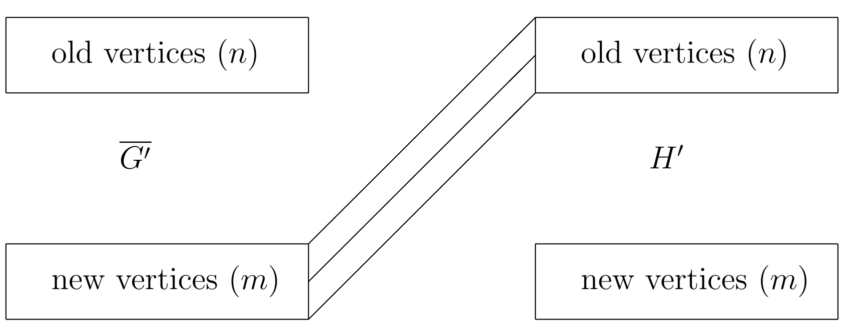


Figure 11: The bi-complement $\overline{B}$ of B.

If we have an isomorphism $\phi : V(G) \to V(H)$, then we can obviously extend ϕ to isomorphisms of G' and H', and $\overline{H'}$ and $\overline{G'}$. In turn, they induce an isomorphism of the bigraphs B and $\overline{B}$.

Conversely, let α, β be an isomorphism of B and $\overline{B}$. The assumptions imply that $\deg_B u \geq m + 1 > \deg_{\overline{B}} v$ for all old vertices u, v of G'. It shows that α transforms the old vertices of G' to the old vertices of H'. Similarly, $\deg_B u = 2 < n + 2 \leq \deg_{\overline{B}} v$ for all new vertices u, v of G'. Hence β transforms the new vertices of G' to the new vertices of H'. As a result, we obtain an isomorphism of G' and H' which induces an isomorphism of G and H. □

Now the result follows from Fact 1 and Fact 2. □

Fact 2 is similar to a known result of Colbourn and Colbourn [14, 12] that recognizing whether a graph is self-complementary is polynomially equivalent to the graph isomorphism problem. The Matrix Complementation Problem can be viewed as a particular case of the following *Matrix Negation Problem* (if we replace 0 by -1): Given a matrix A over a set of integers, whether $A \to -A$. It is not hard to show that the Matrix Negation Problem is polynomial-time equivalent to graph isomorphism.

7 Tournament realizations and anti-symmetrization

A *tournament* is an orientation of a complete undirected graph. Decision Problem 1 for tournaments is trivial. However, Aigner and Triesch [2] proposed an interesting variant of the problem. Given a digraph $D = (V, A)$, define the (+)-*neighborhood hypergraph*, $H = \mathcal{N}^+(D)$, by $V(H) = V$ and $E(H) = \{N^+(u) : u \in V\}$.

Decision Problem 5 (Digraph (+)-Realization Problem)**.**
Instance: *A hypergraph H.*
Question: *Does $H = \mathcal{N}^+(D)$ hold for some digraph D?*

This problem is simple in general: Aigner and Triesch [2] noted that it is equivalent to finding a perfect matching in a bipartite graph. But they were unable to solve Decision Problem 5 for tournaments.

We represent a hypergraph H as an (undirected) bigraph $B = (X, Y, E)$. The problem is to find an involutory switching automorphism α such that x and $\alpha(x)$ are always non-adjacent, and $x \in X$ is adjacent to $\alpha(x') \in Y$ if and only if the vertices $x' \in X$ and $\alpha(x) \in Y$ are non-adjacent. Illustrations for the oriented triple and the transitive triple are given in Figure 12 and Figure 13, respectively.

To a bigraph $B = (X, Y, E)$ we can associate its X-Y-*adjacency* matrix $A(B) = (a_{ij}) \in \{0, 1\}^{X \times Y}$ defined by $a_{ij} = 1$ if and only if $(i, j) \in E$. Conversely, any $0-1$ matrix $A = (a_{ij})$ can be viewed as the X-Y adjacency matrix $A = A(B)$ of a corresponding bigraph $B =$

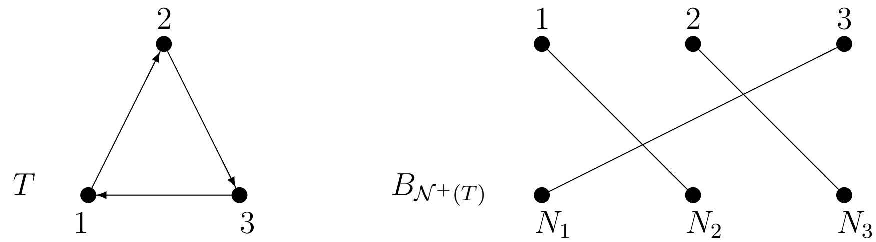


Figure 12: An illustration for the oriented triple.

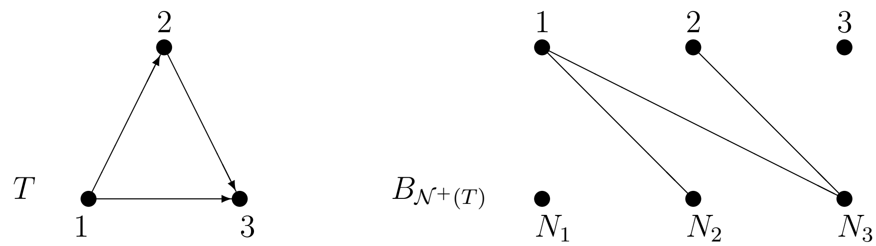


Figure 13: An illustration for the transitive triple.

(X, Y, E), where X is the set of row indices of A, Y is the set of column indices of A, and $(i, j) \in E$ if and only if $a_{ij} = 1$. Here are the adjacency matrices of the bigraphs of Figure 12 and Figure 13, respectively:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we reformulate the problem in terms of square $0-1$ matrices as follows. Does a given $0-1$ square matrix A admits a permutation of rows such that the resulting matrix B has the properties:

(all-0 diagonal) $b_{ii} = 0$ for all i, and

(anti-symmetry) $b_{ij} \neq b_{ji}$ for all $i \neq j$?

It is called the *Matrix Anti-Symmetrization Problem.*

Conjecture 1. *The* Matrix Anti-Symmetrization Problem *is NP-hard.*

Acknowledgments: the first author was partially supported by DIMACS, Center for Discrete Mathematics and Theoretical Computer Science, Rutgers University; Graduate

School of Science and Technology, University of Tokyo; Center for Algorithmic Game Theory and Research Foundation, University of Aarhus.

References

[1] M. Aigner and E. Triesch, Reconstructing a graph from its neighborhood lists, in: *Combinatorics, geometry and probability (Cambridge, 1993)*, (Cambridge Univ. Press, Cambridge, 1997) 51–61

[2] M. Aigner and E. Triesch, Reconstructing a graph from its neighborhood lists, Combin. Probab. Comput. **2** (2) (1993) 103–113

[3] M. Aigner and E. Triesch, Realizability and uniqueness in graphs, Trends in discrete mathematics. Discrete Math. **136** (1-3) (1994) 3–20

[4] L. Babai, The star system problem is at least as hard as graph isomorphism problem, in: *Combinatorics. Vol. II, Hungary, 1976*, A. Hajnal and V. T. Sós, Eds., Coll. Math Soc. J. Bolyai **18** (North-Holland, Amsterdam, 1978) 1214

[5] K. V. S. Bhat, An algorithm for matrix symmetrization, J. Franklin Inst. **312** (1) (1981) 89–93

[6] N. S. Bhave and T. T. Raghunathan, Enumeration of bipartite self-complementary graphs, Indian J. Pure Appl. Math. **28** (11) (1997) 1451–1459

[7] A. Bondy, G. Durán, M. C. Lin, and J. L. Szwarcfiter, Self-clique graphs and matrix permutations, J. Graph Theory **44** (3) (2003) 178–192

[8] E. Boros, V. Gurvich, and I. Zverovich, Neighborhood hypergraphs of bipartite graphs, J. Graph Theory, **58** (1) (2008) 69–95

[9] K. Cameron, Induced matchings, in: *First Montreal Conference on Combinatorics and Computer Science, 1987*, Discrete Appl. Math. **24** (1-3) (1989) 97–102

[10] P. A. Catlin, Nonisomorphic graphs having the same vertex neighborhood family, in: *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978)*, Congr. Numer. **XXI** (Utilitas Math., Winnipeg, Man., 1978) 189–193

[11] C. J. Colbourn, The complexity of symmetrizing matrices, Inform. Process. Lett. **9** (3) (1979) 108–109

[12] C. J. Colbourn and M. J. Colbourn, Isomorphism problems involving self-complementary graphs and tournaments, in: *Proceedings of the Eighth Manitoba Conference on Numerical Mathematics and Computing (Univ. Manitoba, Winnipeg, Man., 1978)*, Congr. Numer. **XXII** (Utilitas Math., Winnipeg, Man., 1979) 153–164

[13] C. J. Colbourn and B. D. McKay, A correction to Colbourn's paper on the complexity of matrix symmetrizability: "The complexity of symmetrizing matrices" [Inform. Process. Lett. **9** (3) (1979) 108–109], Inform. Process. Lett. **11** (2) (1980) 96–97

[14] M. J. Colbourn and C. J. Colbourn, Graph isomorphism and self-complementary graphs, S.I.G.A.C.T. News **10** (1) (1978) 25–29

[15] F. V. Fomin, J. Kratochvíl, D. Lokshtanov, F. Mancini, and J. A. Telle, On the complexity of reconstructing H-free graphs from their star systems, Manuscript (2007) 11 pp.

[16] F. Lalonde, Le problème d'étoiles pour graphes bouclés est NP-complet, Reserch Report No. 79-2 (Dept. of Mathematics and Statistics, University of Montreal, 1979) (in French)

[17] F. Lalonde, Le problème d'étoiles pour graphes bouclés est NP-complet, Discrete Math. **33** (3) (1981) 271–280 (in French)

[18] A. Lubiw, Some NP-complete problems similar to graph isomorphism, SIAM J. Comput. **10** (1) (1981) 11–21

[19] D. McCarthy and B. D. McKay, On symmetrizing matrices, in: *Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1979)*, Congr. Numer. **XXIII–XXIV** (Utilitas Math., Winnipeg, Man., 1979)

[20] D. McCarthy and B. D. McKay, Transposable and symmetrizable matrices, J. Austral. Math. Soc. Ser. A **29** (4) (1980) 469–474

[21] V. T. Sós, Problem with a comment by László Babai, in: *Combinatorics. Vol. II, Hungary, 1976*, A. Hajnal and V. T. Sós, Eds., Coll. Math Soc. J. Bolyai **18** (North-Holland, Amsterdam, 1978) 1214