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Assignability of 3-dimensional totally tight matrices

by

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ABSTRACT

A 3-dimensional *totally tight matrix* $A = (a_{ijk})$ has the property that every 2×2 submatrix has a constant line [a row or a column]. We prove that all such matrices are *assignable*, that is it is possible to assign a label to each of the axial planes so that every a_{ijk} is equal to at least one of the corresponding labels. The result can be easily extended to the case of multi-dimensional matrices.

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Assignable 3-dimensional matrices, totally tight matrices

A 3-dimensional $l \times m \times n$ matrix $A = (a_{ijk})$ has three sets of "axial" planes, $P_1, P_2, \ldots, P_l, Q_1, Q_2, \ldots, Q_m$, and $R_1, R_2, \ldots, R_n$. Such a matrix is called *assignable* if it is possible to assign labels p_i, q_j and r_k to the axial planes P_i, Q_j and R_k so that every a_{ijk} is equal to at least one of p_i, q_j or r_k. A 3-dimensional *totally tight matrix* has the following *TT property*: every 2×2 submatrix has a constant line [a row or a column]. Here a 2×2 submatrix is obtained by taking two distinct axial planes from the same set, say P_i and P_j, choosing a pair of distinct elements in P_i, and the corresponding pair of elements in P_j.

Here is our main result.

Theorem 1. *Every 3-dimensional totally tight matrix $A = (a_{ijk})$ is assignable.*

Proof. We say that a plane P_r *dominates* a plane P_s *by* x (notation $P_r \rightarrow_x P_s$) if, whenever $a_{rjk} \neq a_{sjk}$, we have $a_{rjk} = x$. Here x is the *domination parameter*. Similar definitions are applied to the planes Q_j and R_k.

Claim 1. *A 3-dimensional matrix $A = (a_{ijk})$ is totally tight matrix if and only if, for every distinct planes P_r and P_s, either $P_r \rightarrow_x P_s$ or $P_s \rightarrow_x P_r$ for some x, and similarly for the planes Q_j and R_k.*

Proof. Straightforward. □

The three binary relations $\rightarrow_x$ on the sets P_i, Q_j and R_k determine three digraphs, denoted by D_P, D_Q and D_R, on the same sets. A *sink* in a digraph is a vertex v such that, for every other vertex u, there is an arc (u, v). Note that the definition allows arcs out-coming from a sink. We shall distinguish two cases.

Case 1. At least one of the three digraphs D_P, D_Q or D_R has a sink.

Without loss of generality, let P_1 be a sink in the digraph D_P. The 2-dimensional plane P_1 is assignable, see Boros, Gurvich, Makino, and Papp [1]. We assign labels to all rows and columns of P_1, and then consider them as labels of all planes Q_j and R_k. Now, for every plane $P_i \neq P_1$, we have $P_i \rightarrow_{x_i} P_1$, since P_1 is a sink. We assign label x_i to P_i, thus obtaining an assignment for the matrix A. Note that the plane P_1 remains unlabeled.

Case 2. No one of the three digraphs D_P, D_Q or D_R has a sink.

The domination relation $P_r \rightarrow_x P_s$ is called *strict* if $P_s \rightarrow_y P_r$ does not hold for any y. We choose labels p_i and q_j for all planes P_i and Q_j according to the strict domination relation, that is we choose the domination parameters as labels.

Claim 2. *For every plane R_k, all entries that are not satisfied by the labels p_i and q_j are the same.*

Proof. Suppose that there exists R_k which contains distinct entries u and v that are not satisfied by the labels p_i and q_j. We may assume that u and v are in the same plane P_i or Q_j. Indeed, otherwise the entries u and v are opposite corners of a rectangle in R_k. By the TT property, at least one of the two other corners must be either u or v. Thus, we always can choose u and v in the same plane P_i or Q_j. Let $u \in P_1 \cup Q_1$ and $v \in P_2 \cup Q_1$.

Since $p_1 \neq u$, $p_2 \neq v$ and $u \neq v$, P_1 non-strictly dominates P_2 by u, and P_2 non-strictly dominates P_1 by v. The plane P_1 strictly dominates some plane P_3 by $p_1 \neq u$, therefore

P_1	u	p_1	α
P_2	v	p_1	α
P_3	u	$\beta \neq p_1$	α

Here α and β are some strings of entries, β does not contain p_1, but it contains at least two distinct entries. We may choose an entry $x \in \beta$ distinct from u, and obtain the following submatrix

$$\begin{pmatrix} v & p_1 \\ u & x \neq u, p_1 \end{pmatrix}.$$

If $p_1 \neq v$, we have a contradiction to the TT property. Thus, $p_1 = v$:

P_1	u	v	α
P_2	v	v	α
P_3	u	$\beta \neq v$	α

Now we see that P_2 strictly dominates P_3 by v, a contradiction to the fact that $p_2 \neq v$. □

Finally, we state an algorithm that produces an assignment for an arbitrary matrix of Case 2.

Step 1. Assign labels p_i and q_j to all P_i and Q_j according to the strict domination relation.

Step 2. Based on Claim 2, assign the non-satisfied constant to every plane R_k. □

Finally note that our method is easily extended to n-dimensional totally tight matrices for all $n > 3$.

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References

[1] E. Boros, V. Gurvich, K. Makino, and D. Papp, On acyclyc, or totally tight, two-person game forms, RUTCOR Research Report 3-2008 (Rutgers University, 2008)