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Not complementary connected and not CIS d-graphs form weakly monotone families

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ABSTRACT

A d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a complete graph whose edges are arbitrarily partitioned into d subsets (colored with d colors); $\mathcal{G}$ is a Gallai d-graph if it contains no three-colored triangle Δ; furthermore, $\mathcal{G}$ is a CIS d-graph if $\bigcap_{i=1}^{d} S_i \neq \emptyset$ for every set-family $\mathcal{S} = \{S_i \mid i \in [d]\}$, where $S_i \subseteq V$ is a maximal independent set of $G_i = (V, E_i)$, the ith chromatic component of $\mathcal{G}$, for all $i \in [d] = \{1, \ldots, d\}$. A conjecture suggested in 1978 by the third author says that every CIS d-graph is a Gallai d-graph. In this paper we obtain a partial result. Let Π be the two-colored d-graph on four vertices whose two non-empty chromatic components are isomorphic to P_4. It is easily seen that Π and Δ are not CIS d-graphs but become CIS after eliminating any vertex. We prove that no other d-graph has this property, that is, every non-CIS d-graph $\mathcal{G}$ distinct from Π and Δ contains a vertex $v \in V$ such that the sub-d-graph $\mathcal{G}[V \setminus \{v\}]$ is still non-CIS. This result easily follows if the above Δ-conjecture is true, yet, we prove it independently.

A d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is complementary connected (CC) if the complement $\overline{G}_i = (V, \overline{E}_i) = (V, \bigcup_{j \in [d] \setminus \{i\}} E_j)$ to its ith chromatic component is connected for every $i \in [d]$. It is known that every CC d-graph $\mathcal{G}$, distinct from Π, Δ, and a single vertex, contains a vertex $v \in V$ such that the reduced sub-d-graph $\mathcal{G}[V \setminus \{v\}]$ is still CC.

It is not difficult to show that every non-CC d-graph with contains a vertex $v \in V$ such that the sub-d-graph $\mathcal{G}[V \setminus \{v\}]$ is not CC.

Keywords: d-graph, complementary connected, Gallai, weakly monotone, minimal and locally minimal.

1 Introduction: Gallai and CIS d-graphs

A d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a complete graph whose edges are arbitrarily partitioned into d subsets (colored with d colors). Graph $G_i = (V, E_i)$ is called the ith chromatic component of $\mathcal{G}$, where $i \in [d] = \{1, \ldots, d\}$. Some of these components might be empty $E_i = \emptyset$. A d-graph is called k-colored if k is the number of its non-empty chromatic components. Obviously, $0 \leq k \leq d$; moreover, $k = 0$ if and only if $\mathcal{G}$ consists of a single vertex. Let us note that in case $d = 2$ a d-graph is just a graph, or more precisely, a pair: a graph and its complement. Thus, d-graphs can be viewed as a generalization of graphs.

Let us choose a maximal independent set $S_i \subseteq V$ in every graph G_i and denote by $\mathcal{S} = \{S_i \mid i \in [d]\}$ the obtained set-family; furthermore, let $S = \bigcap_{i=1}^{d} S_i$. Obviously, $|S| \leq 1$ for every $\mathcal{S}$; indeed, if $v, v' \in S$ then $(v, v') \notin E_i$ for all $i \in [d]$, that is, this edge has no color. We say that $\mathcal{G}$ has the CIS property and call $\mathcal{G}$ a CIS d-graph [1] if $S \neq \emptyset$ for every $\mathcal{S}$.

Two d-graphs Π and Δ given in Figure 1 will play an important role:

Π is defined for any $d \geq 2$ by $V = \{v_1, v_2, v_3, v_4\}$;
$E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$, and $E_i = \emptyset$ whenever $i > 2$;

Δ is defined for any $d \geq 3$ by $V = \{v_1, v_2, v_3\}$,
$E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, $E_3 = \{(v_3, v_1)\}$, and $E_i = \emptyset$ whenever $i > 3$.

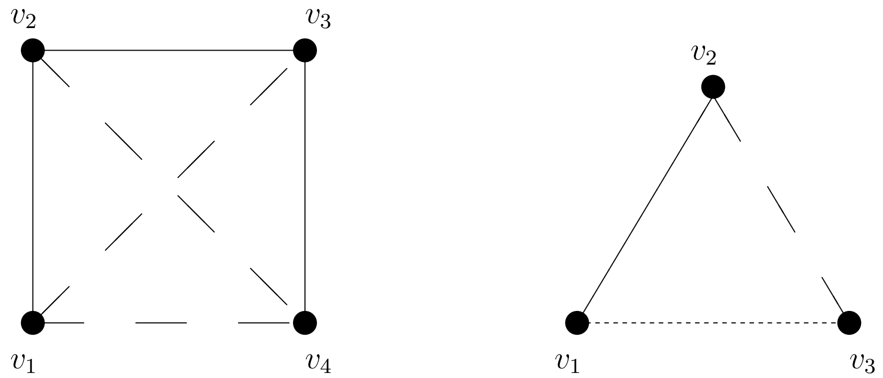

Figure 1: 2- and 3-colored d-graphs Π and Δ.

Remark 1 *Clearly, Π and Δ are respectively 2- and 3-colored d-graphs; both non-empty chromatic components of Π are isomorphic to P_4 and Δ is a three-colored triangle. Let us also notice that, formally, d-graphs $\Pi(d)$ (respectively, $\Delta(d)$) is defined for every fixed $d \geq 2$ (respectively, $d \geq 3$) and $d - 2$ (respectively, $d - 3$) of its chromatic components are empty. Yet, we will omit argument d assuming that it is a fixed parameter.*

Both were introduced in 1967 by Gallai in his seminal paper [11]. Numerous applications of Π- and Δ-free d-graphs in theory of positional games are considered in [14, 15, 16, 17]. A Δ-free d-graph is called Gallai d-graph.

Δ-Conjecture ([14]) Each CIS d-graph is a Gallai d-graph, or in other words, CIS property holds for no d-graph that contains a Δ.

Since 1978, this conjecture is open; partial results are given in [1], where in particular, Δ-conjecture for an arbitrary d is reduced to the case $d = 3$.

It is also shown in [1] that, modulo Δ-conjecture, the characterization of CIS d-graphs is reduced to the case $d = 2$, that is, to the characterization of CIS graphs; see Section 4 and also [1] Sections 1.6 and 1.7.

Let us note, however, that case $d = 2$ is still very difficult. The problem of characterization and recognition of CIS graphs was suggested in early 90s by Vasek Chvatal to his graduate student from RUTCOR Wenan Zang who published first partial results in [28]. Further results on CIS graphs were obtained in [8, 9, 1] and some related results (on almost CIS graphs) in [5, 27].

Let us also mention in passing that both P_4-free and CIS graphs are closed under complementation.

2 Main result

It is not difficult to verify that Π and Δ are not CIS d-graphs; moreover, they are minimal, that is, every sub-d-graph of Π or Δ is a CIS d-graph. As our main result, we will show that Π and Δ are the *only* minimal and, moreover, the only locally minimal, not CIS d-graphs.

Theorem 1 *Each non-CIS d-graph $\mathcal{G}$ distinct from Π and Δ, contains a vertex $v \in V$ such that the sub-d-graph $\mathcal{G}[V \setminus \{v\}]$ is still not CIS.*

In the next Section we will give a proof and in Section 4.4 show that this result easily follows from Δ-conjecture.

In contrast, minimal and locally minimal CIS d-graphs differ already for $d = 2$. Indeed, let $G = L(K_{3,3})$ be the line graph of the complete bipartite 3×3 graph; see Figure 1.4 in [1]. It is easy to verify that G is a CIS graph, while for each $v \in V$ the induced subgraph $G[V \setminus \{v\}]$ has no CIS property. By symmetry, it is enough to verify this claim for one arbitrary $v \in V$. Thus, G is a locally minimal CIS 2-graph. Yet, it is not minimal, since there is only one minimal CIS graph, the trivial one, which consists of a single vertex.

Remark 2 *Let us remark that $G = L(K_{3,3})$ is also a locally edge-minimal perfect graph, that is, G is perfect but it becomes imperfect whenever we delete an edge from it or add to G an edge (between two its already existing vertices) [3]. Yet, G is not an edge-minimal perfect graph, since it contains the edge-free graph on the same vertex-set. An infinite family of locally edge-minimal but not edge-minimal perfect graphs, so-called Rotterdam graphs, was introduced in [3]. However, no good characterization of locally edge-minimal perfect graphs or locally vertex-minimal CIS d-graphs is known yet.*

3 Proof of Theorem 1

3.1 Plan of the proof

Let us assume that $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a non-CIS d-graph, while $\mathcal{G}[V \setminus \{v\}]$ is a CIS sub-d-graph for every $v \in V$.

Furthermore, without loss of generality, we will also assume that $E_i \neq \emptyset$ for all $i \in [d] = \{1, \ldots, d\}$, or in other words, that our d-graph $\mathcal{G}$ is d-colored.

First, we shall prove several preliminary statements, then that $d \leq 3$ must hold, and next that $\mathcal{G}$ is either Δ (if $d = 3$) or Π (if $d = 2$).

3.2 Preliminary claims

By our assumption $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a non-CIS d-graph, that is, for every $i \in [d]$ one can choose a maximal stable set S_i in graph $G_i = (V, E_i)$ so that $\bigcap_{i=1}^{d} S_i = \emptyset$. Let us fix such a family $\mathcal{S} = \{S_i \mid i \in [d]\}$.

Lemma 1 *Family $\mathcal{S}$ is a vertex-cover, that is, $\bigcup_{i=1}^{d} S_i = V$.*

Proof. Assume indirectly that $\bigcup_{i=1}^{d} S_i \neq V$. Then, by eliminating a vertex $v \in V \setminus \bigcup_{i=1}^{d} S_i$, one gets a non-CIS sub-d-graph $\mathcal{G}' = \mathcal{G}[V \setminus \{v\}]$, in contradiction with our assumptions. □

Furthermore, by the same assumption, for each vertex $v \in V$ there is an index $i \in [d]$ such that the stable set $S_i \setminus \{v\}$ is not maximal in graph G_i.

Lemma 2 *In fact, there is exactly one such $i = i(v)$.*

Proof. Indeed, let us assume indirectly that there are two distinct $i, j \in [d]$ such that the stable sets $S_i \setminus \{v\}$ and $S_j \setminus \{v\}$ are not maximal in the corresponding graphs G_i and G_j, respectively. Then, obviously, $v \in S_i \cap S_j$. Furthermore, by our assumption, $\mathcal{G}[V \setminus \{v\}]$ is a CIS d-graph. Hence, there is a vertex $u \in V$ such that the sets $(S_i \setminus \{v\}) \cup \{u\}$ and $(S_j \setminus \{v\}) \cup \{u\}$ are stable in G_i and G_j, respectively. Obviously, $u \notin S_i \cup S_j$, since otherwise already $S_i \setminus \{v\}$ or $S_j \setminus \{v\}$ would be maximal. Moreover, $(u, w) \notin E_i$ (respectively, $(u, w) \notin E_j$) for every $w \in S_i \setminus \{v\}$ (respectively, $w \in S_j \setminus \{v\}$). Since S_i and S_j are maximal stable sets, the edge (v, u) must belong to E_i and E_j simultaneously, which is a contradiction. □

(i) Thus, to each vertex $v \in V$ we can assign a unique index $i = i(v) \in [d]$ such that the stable set $S_i \setminus \{v\}$ is not maximal in G_i. Hence, given $i \in [d]$, there is a unique subset $S'_i \subseteq S_i$ such that $i = i(v)$ if and only if $v \in S'_i$.

(ii) Furthermore, as the above proof shows, to every vertex $v \in S'_i$ we can associate a vertex $u = u(v) \notin S_i$ such that $(u, v) \in E_i$ and $(u, w) \notin E_i$ for all $w \in S_i \setminus \{v\}$. Namely, since $\mathcal{G}[V \setminus \{v\}]$ is CIS, for an arbitrary maximal stable set $Q \supset S_i \setminus \{v\}$ of G_i we must have a (unique) vertex $u \in Q \cap (\bigcap_{\ell \neq i} S_\ell)$.

(iii) Due to property (ii), we must have $u(v) \neq u(v')$ whenever v and v' are two distinct vertices of S'_i.

Thus, introducing the notation $X_i = \bigcap_{\ell\in[d] \mid \ell\neq i} S_\ell$ for $i \in [d]$ we must have by (ii) and (iii) above that

$$|S'_i| \leq |X_i| \quad \text{for all indices } \; i \in [d], \tag{3.1}$$

from which it follows that

$$|V| = \sum_{i=1}^{d} |S'_i| \leq \sum_{i=1}^{d} |X_i| \leq |V|, \tag{3.2}$$

where the last inequality follows from the fact that $\mathcal{G}$ is not a CIS d-graph and hence $\bigcap_{\ell=1}^{d} S_\ell = \emptyset$. Therefore, we must have equalities everywhere in (3.1) and (3.2). Consequently, every vertex $v \in V$ belongs to exactly $(d-1)$ maximal stable sets of the considered family $\mathcal{S} = \{S_i \mid i \in [d]\}$. In other words,

(iv) The sets X_i, $i \in [d]$ are pairwise disjoint, form a partition of V, and $S_i = V \setminus X_i$ holds for all $i \in [d]$.

Furthermore, by property (iii) for every $i \in [d]$ and $u \in X_i$ there is a unique edge $(v,u) \in E_i$ such that $v \in S'_i \subseteq S_i$, since $|X_i| = |S'_i|$, by (3.1) and (3.2). Note that $u = u(v)$ as introduced in (ii) above.

Let us introduce $Y_{ij} = X_i \cap S'_j$ for all $i, j \in [d]$. By this definition, $Y_{ii} = \emptyset$ and $Y_{ij} \cap Y_{k\ell} = \emptyset$ whenever $\{i,j\} \neq \{k,\ell\}$ since the sets S'_i, $i \in [d]$ are pairwise disjoint, by Lemma 2, and the sets X_i, $i \in [d]$ are pairwise disjoint by (iv). Thus, the following equalities define partitions of the sets X_i and S'_i, $i \in [d]$:

$$X_i = \bigcup_{j\in[d] \mid j\neq i} Y_{ij} \qquad \text{and} \qquad S'_i = \bigcup_{\ell\in[d] \mid \ell\neq i} Y_{\ell i}. \tag{3.3}$$

Corollary 1 *For any index $i \in [d]$ the edge set $E_i \cap (X_i \times S'_i)$ is a matching. Consequently, for any two distinct indices $i \neq j \in [d]$ only the vertices of Y_{ij} are connected by edges of E_j between the sets X_i and X_j, matching Y_{ij} to a subset of X_j.* □

Lemma 3 *For any two distinct indices $i \neq j$ we must have*

$$\textit{either} \qquad X_i \setminus Y_{ij} = \emptyset \qquad \textit{or} \qquad X_j \setminus Y_{ji} = \emptyset. \tag{3.4}$$

Proof. Assume indirectly that there are vertices $u \in X_i \setminus Y_{ij}$, $v \in X_j \setminus Y_{ji}$. We claim that the edge (u,v) does not belong to E_ℓ for all $\ell \in [d]$ contradicting the fact that $\mathcal{G}$ is a d-graph, and hence proving the statement of the lemma.

To see the claim, let us first note that $(u,v) \notin E_\ell$ whenever $\ell \notin \{i,j\}$, since $u, v \in S_\ell$. Let us note next that $(u,v) \notin E_i \cup E_j$ either by Corollary 1, since neither u nor v belongs to $Y_{ij} \cup Y_{ji}$. □

3.3 Case $d > 3$

Now, we shall show that inequality $d \leq 3$ must hold.

Proof. Let us assume indirectly that $d > 3$, i.e., at least four pairwise distinct color classes are not empty. Equivalently, without any loss of generality, we can assume that the sets X_1, X_2, X_3 and X_4 are all not empty, or in other words, that $S_i \neq V$ for $i = 1, 2, 3, 4$).

Let us then successively apply property (3.4) by setting (i, j) to $(1, 2)$, $(1, 3)$, $(1, 4)$ and $(3, 4)$.

If $(i, j) = (1, 2)$, then without loss of generality we can assume by (3.4) that $X_1 \setminus Y_{12} = \emptyset$ and thus $Y_{12} \neq \emptyset$ is implied, since $X_1 \neq \emptyset$. If $(i, j) = (1, 3)$ then $X_1 \setminus Y_{13} \supseteq Y_{12} \neq \emptyset$, and thus by (3.4) we must have $X_3 \setminus Y_{31} = \emptyset$. Furthermore, since $X_3 \neq \emptyset$ we also must have $Y_{31} \neq \emptyset$.

Similarly, if $(i, j) = (1, 4)$ then $X_1 \setminus Y_{14} \supseteq Y_{12} \neq \emptyset$, and hence by (3.4) we must have $X_4 \setminus Y_{41} = \emptyset$, and analogously, when $(i, j) = (3, 4)$ then $X_3 \setminus Y_{34} \supseteq Y_{31} \neq \emptyset$ implies by (3.4) that $X_4 \setminus Y_{43} = \emptyset$.

Thus, we arrive to the contradiction $\emptyset \neq X_4 \subseteq (X_4 \setminus Y_{41}) \cup (X_4 \setminus Y_{43}) = \emptyset$ implying that $d \leq 3$ must hold. □

To complete the proof of the theorem, we only need to consider the cases $d = 3$ and $d = 2$.

3.4 If $d = 3$ then $\mathcal{G}$ is a Δ.

Proof. Setting successively (i, j) to $(1, 2), (1, 3), (2, 3)$ and applying the same arguments using (3.4) as in the previous subcase we can conclude by (3.3) that

$$\text{either} \quad Y_{13} = Y_{21} = Y_{32} = \emptyset \quad \text{or} \quad Y_{12} = Y_{23} = Y_{31} = \emptyset.$$

Without loss of generality, we can assume that the former two equalities hold. Then we have $X_1 = Y_{12} = S'_2$, $X_2 = Y_{23} = S'_3$, and $X_3 = Y_{31} = S'_1$, and furthermore, $|Y_{12}| = |Y_{23}| = |Y_{31}| = y$ by (3.1).

Let us recall that by Corollary 1 those edges of E_i which connect X_i to S'_i form a matching, $i = 1, ..., 3$. Thus, if we can show that $y \leq 1$, then it follows that $y = 1$, since $\mathcal{G}$ is not empty and, therefore, $\mathcal{G}$ is indeed a Δ, as stated in the theorem.

To this end, let us assume indirectly that $y \geq 2$, that is by Corollary 1 that there are two distinct edges of color 1, say (u_1, u_2), $(v_1, v_2) \in E_1$, both between the sets $X_1 = Y_{12}$ and $S'_1 = X_3 = Y_{31}$.

We can prove then that no color in $[d] = \{1, 2, 3\}$ is feasible for edge (u_1, v_2).

First, let us notice that $(u_1, v_2) \notin E_1$, since for every vertex of X_1 there is a unique edge in E_1 connecting it to $S_1 = X_2 \cup X_3$ by Corollary 1 and, from vertex u_1, this unique edge is $(u_1, u_2) \in E_1$ according to our assumption.

Let us next note that $S_2 = X_1 \cup X_3$ and, hence, $(u_1, v_2) \notin E_2$, either.

Finally, let us note that the edges of E_3 connecting X_3 to $S_3 = X_1 \cup X_2$ are all incident with $Y_{23} = X_2$ by Corollary 1 and, hence, no edge between X_1 and X_3 can be of color 3.

Thus, edge (u_1, v_2) must be colored by a fourth color, in contradiction with our assumption that $\mathcal{G}$ is a 3-colored graph and, hence, $y \leq 1$ follows, concluding the proof of this case. □

3.5 If $d = 2$ then $\mathcal{G}$ is a Π

Proof. If $\mathcal{G} = (V; E_1, E_2)$ is a non-CIS 2-graph then for $i = 1, 2$ there is a maximal stable set S_i in graph $G_i = (V, E_i)$ such that $S_1 \cap S_2 = \emptyset$. Let us fix such a pair S_1, S_2. If $S_1 \cup S_2 \neq V$ then by eliminating any vertex $v \in V \setminus (S_1 \cup S_2)$ one gets a non-CIS sub-2-graph $\mathcal{G}[V \setminus \{v\}]$ and the theorem follows. Hence, our assumption implies that $S_1 \cup S_2 = V$.

For each $v_1 \in S_1$ the stable set $S_1 \setminus \{v_1\}$ is not maximal, since, by the assumption, $\mathcal{G}[V \setminus \{v_1\}]$ is already a CIS sub-d-graph. Hence, there is a vertex $v_2 \in S_2$ such that $(S_1 \setminus \{v_1\}) \cup \{v_2\}$ is a stable set in G_1; obviously, it is maximal. Hence, $(v_1', v_2) \in E_2$ for every $v_1' \in V_1$ distinct from v_1. Clearly, $(v_1, v_2) \in E_1$, since otherwise stable set S_1 itself would not be maximal. Furthermore, let us consider two vertices $v_1, w_1 \in S_1$ and the corresponding $v_2, w_2 \in S_2$. It is clear that $v_2 \neq w_2$ whenever $v_1 \neq w_1$. Indeed, $(v_1, v_2) \in E_1$, while $(v_1, w_2) \in E_2$. Hence, $|S_2| \geq |S_1|$. Yet, by symmetry, $|S_1| \geq |S_2|$ and we conclude that $|S_1| = |S_2|$. Moreover, the above construction defines two edge-disjoint perfect matchings between S_1 and S_2 in graphs G_1 and G_2, respectively. In addition, for every pair of vertices $v_1 \in V_1, v_2 \in V_2$ the edge (v_1, v_2) must belong to one of these two matchings. Obviously, this happens if and only if $|S_1| = |S_2| = 2$ and in this case $\mathcal{G} = \Pi$. □

4 Modular decomposition of Gallai d-graphs and its applications to CIS d-graphs

4.1 Substitution

Given two d-graphs $\mathcal{G}' = (V'; E_1', \ldots, E_d')$ and $\mathcal{G}'' = (V''; E_1'', \ldots, E_d'')$ such that $V' \cap V'' = \emptyset$, let us fix a vertex $v \in V'$, substitute the whole d-graph $\mathcal{G}''$ into $\mathcal{G}'$ for v and denote the obtained d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ by $\mathcal{G}'(v, \mathcal{G}'')$.

By this definition, $V = V'' \cup V' \setminus \{v\}$ and colors E_i for $i \in [d]$ are as follows:

if $v', v'' \in V''$ (respectively, $v', v'' \in V' \setminus v$) then edge (v', v'') is colored in $\mathcal{G}$ with the same color as it was colored in $\mathcal{G}''$ (respectively, in $\mathcal{G}'$);

if $v' \in V' \setminus \{v\}$ and $v'' \in V''$ then (v', v'') is colored in $\mathcal{G}$ with the same color as edge (v, v') was colored in $\mathcal{G}'$.

Let us remark that $\mathcal{G}$ contains as subgraphs both $\mathcal{G}'$ and $\mathcal{G}''$; the latter is called a *module*. In case $d = 2$ the above definition agrees with the standard concept of substitution for graphs. In general, substitution is defined in the same way for digraphs [21], Boolean functions [22], etc. [23].

We say that a family of d-graphs $\mathcal{F}$ (and, in particular, of graphs) is *exactly closed* with respect to substitution [1] if $\mathcal{G}'(v, \mathcal{G}'') \in \mathcal{F}$ if and only if $\mathcal{G}', \mathcal{G}'' \in \mathcal{F}$.

Proposition 1 *The CIS d graphs, as well as Gallai d-graphs, are exactly closed with respect to substitution.*

Proof. In case of Gallai d-graphs it is enough to verify that $\mathcal{G}$ contains a Δ if and only if $\mathcal{G}'$ or $\mathcal{G}''$ do; see, for example, [7, 6, 18, 1]. In case of CIS d-graphs proof is reduced to a tedious but simple case analysis; see [1], Section 4.1. □

4.2 Modular decomposition

Every more than 2-colored Gallai d-graph $\mathcal{G}$ can be decomposed into 2-colored Gallai d-graphs. This results immediately from the following claim.

Theorem 2 *(Cameron and Edmonds, [6]; Gyárfás and Simonyi, [18]).*
Let $\mathcal{G}$ be an at least 3-colored Gallai d-graph. Then $\mathcal{G} = \mathcal{G}'(v, \mathcal{G}'')$, where $\mathcal{G}'$ and $\mathcal{G}''$ are Gallai d-graphs distinct from $\mathcal{G}$.

Clearly, we can proceed with this decomposition until both $\mathcal{G}'$ and $\mathcal{G}''$ become 2-colored, since they remain Δ-free. Decomposing in such a way recursively, we will represent $\mathcal{G}$ by a binary tree $T(\mathcal{G})$ whose leaves are associated with 2-colored d-graphs. The following two properties of the Gallai d-graphs are instrumental for such decomposition.

Lemma 4 *[6, 18]. Let $\mathcal{G} = (V; E_1, \ldots, E_n)$ be a Gallai d-graph; at least one of its chromatic component, say $G_1 = (V, E_1)$, be disconnected and V_1' and V_1'' be the vertex-sets of two connected components of G_1. Then all edges between V_1' and V_1'' are homogeneously colored, more precisely, they all are of the same color $i \in [d]$ for some $i \neq 1$.*

Proof. Since V_1' and V_1'' are connected components of G_1, no edge between them can be of color 1. Assume indirectly that $(x', x'') \in E_2$ and $(y', y'') \in E_3$ for some $x', y' \in V_1'$ and $x'', y'' \in V_1''$. Since V_1' and V_1'' are connected, we can choose a path p' between x' and y' in C_1' and p'' between x'' and y'' in C_1''. Then we can get a contradiction by showing that the d-graph induced by $V(p') \cup V(p'')$ contains a Δ, namely, a triangle colored by $1, 2$ and 3. This is easy to show by induction on the lengths of p' and p''. □

Lemma 5 ([11], [6], and [18]) *Every Gallai d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ with at least 3 non-trivial chromatic components has a color $i \in [d] = \{1, \ldots, d\}$ that does not span V, that is, $G_i = (V, E_i)$ is not connected for some $i \in [d]$.*

Gyárfás and Simonyi remark that Lemma 5 "is essentially a content of Lemma (3.2.3) in [11]". Lemmas 4 and 5 imply Theorem 2. Indeed, let $\mathcal{G} = (V; E_1, \dots, E_d)$ be a Gallai d-graph. If it is 2-colored then we are done. Otherwise, by Lemma 5, there is a non-connected graph $G_i = (V, E_i)$. Let us decompose it into connected components and let $V = V_1 \cup \dots V_k$ be the corresponding partition of V. At least one of these sets, say V_1, is of cardinality at least 2, since $E_i \neq \emptyset$. By Lemma 4, for every two distinct vertex-sets $V_{j'}$ and $V_{j''}$ all edges between them are homogeneously colored, that is, there exists a color $i' \in [d]$ such that $i' \neq i$ and $(v', v'') \in E_{i'}$ for every $v' \in V_{j'}, v'' \in V_{j''}$. Thus, collapsing V_1 into one vertex v we obtain a non-trivial modular decomposition $\mathcal{G} = \mathcal{G}(\mathcal{G}', v, \mathcal{G}'')$, where "non-trivial" means that both $\mathcal{G}'$ and $\mathcal{G}''$ are distinct from $\mathcal{G}$.

It is well-known that decomposing a graph into connected components can be executed in linear time. Hence, given a Gallai d-graph $\mathcal{G}$, its decomposition tree $T(\mathcal{G})$ can be constructed in linear time, too.

4.3 Extending Cameron-Edmonds-Lovász' Theorem

Some nice properties of Gallai colorings easily results from Theorem 2.

Corollary 2 *A Gallai d-graph with n vertices contains at most $n-1$ non-trivial chromatic components.*

As it was mentioned in [18], this result by Erdős, Simonovits, and Sós [10] immediately follows from Theorem 2 by induction.

Corollary 3 *If all but one chromatic components of a Gallai d-graph are perfect graphs then the remaining one is a perfect graph too.*

This claim was proved by Cameron, Edmonds, and Lovász [7]. (Clearly, it turns into Lovász' Perfect Graph Theorem [19, 20] if $d = 2$.) Later, Cameron and Edmonds [6] strengthened this claim showing that the same statement holds not only for perfect graphs but, in fact, for any family of graphs that is closed under: (i) substitution, (ii) complementation, and (iii) taking induced subgraphs. In [1] the claim is strengthened further as follows.

Theorem 3 *[1]. Let $\mathcal{F}$ be a family of graphs closed under complementation and exactly closed under substitution and let $\mathcal{G} = (V; E_1, \dots, E_d)$ be a Gallai d-graph such that at least $d-1$ of its chromatic components, say $G_i = (V, E_i)$ for $i = 1, \dots, d-1$, belong to $\mathcal{F}$. Then*

(a) the last component $G_d = (V, E_d)$ is in $\mathcal{F}$ too, and moreover,

(b) all 2^d graphs associated with $\mathcal{G}$ belong to $\mathcal{F}$, that is, for each subset $J \subseteq [d] = \{1, \dots, d\}$ the graph $G_J = (V, \cup_{j \in J} E_j)$ is in $\mathcal{F}$.

Proof. Part (a). By Theorem 2, $\mathcal{G}$ is a modular decomposition of 2-colored d-graphs. Such a decomposition of $\mathcal{G}$ is given by a tree $T(\mathcal{G})$ whose leaves correspond to 2-colored d-graphs. It is easy to see that by construction each chromatic component $G_i = (V, E_i)$ of

$\mathcal{G}$ is decomposed by the same tree $T(\mathcal{G})$. Hence, all we have to prove is that both chromatic components of every 2-colored d-graph belong to $\mathcal{F}$. For colors $1, \dots, d-1$ this holds, since $\mathcal{F}$ is exactly closed under substitution, and for the last color d it holds, too, since $\mathcal{F}$ is also closed under complementation.

Part (b). It follows easily from part (a). Given a d-colored d-graph $\mathcal{G} = (V; E_1, \dots, E_d,$ let us identify the last two colors d and $d-1$ and consider the $(d-1)$-graph $\mathcal{G}' = (V; E_1, \dots, E_{\mathbf{d-1}},$ where $E_{\mathbf{d-1}} = E_{d-1} \cup E_d$. We assume that $\mathcal{G}$ is Δ-free and that $G_i = (V, E_i) \in \mathcal{F}$ for $i = 1, \dots, d-1$. Then $\mathcal{G}'$ is Δ-free too and it follows from part (a) that $G_{\mathbf{d-1}} = (V, E_{\mathbf{d-1}})$ is also in $\mathcal{F}$. Hence, the union of any two colors is in $\mathcal{F}$. From this by induction we derive that the union of any set of colors is in $\mathcal{F}$, too. □

This theorem and the following Lemma imply Cameron-Edmonds' Theorem.

Lemma 6 *Let $\mathcal{F}$ be a family of graphs closed with respect to substitution and taking induced subgraphs then $\mathcal{F}$ is exactly closed under substitution.*

Proof. If $G = G(G', v, G'')$ then both G' and G'' are induced subgraphs of $\mathcal{G}$. □

In particular, Theorem 3 is applicable to the family $\mathcal{F}$ of the CIS d-graphs, although Cameron-Edmonds' Theorem is not, because only conditions (i) and (ii) hold in this case but (iii) does not.

4.4 Theorem 1 results from Δ-conjecture

Again, let us assume that $\mathcal{G}$ is a non-CIS d-graph, while $\mathcal{G}[V \setminus \{v\}]$ is a CIS sub-d-graph for every $v \in V$ and prove that $\mathcal{G}$ is either Π or Δ.

Case 1: $\mathcal{G} = \Delta$. Then there is nothing to prove.

Case 2: $\mathcal{G}$ strictly contains a Δ. Then let us eliminate any vertex outside it. Obviously, the reduced d-graph $\mathcal{G}'$ still contains this Δ. Then, by Δ-conjecture, $\mathcal{G}'$ is a non-CIS d-graph.

Let us remark that Δ-conjecture gives us a lot of freedom: one can delete *any* vertex of $\mathcal{G}$ that does not belong to some Δ. In fact, Theorem 1 claims much less: it only says that there is *a* vertex $v \in V$ such that the reduced d-graph $\mathcal{G}[V \setminus \{v\}]$ is still a non-CIS d-graph.

Case 3: $\mathcal{G}$ is Δ-free, or in other words, is a Gallai d-graph.

Subcase 3a: $\mathcal{G}$ is 2-colored (that is, it contains only two non-empty chromatic components). In this case we have to borrow the proof from Section 3.5.

Subcase 3b: d-graph $\mathcal{G}$ is more than 2-colored. Since $\mathcal{G}$ is a Gallai d-graph, there is a modular decomposition of it into 2-colored d-graphs. Let us choose such a decomposition and consider the corresponding decomposition tree. At least one of its vertices is associated with a non-CIS 2-colored d-graph, since CIS property is exactly closed with respect to substitution. The rest easily follows from subcase 3a and structure of modular decomposition $\mathcal{G} = \mathcal{G}'(v, \mathcal{G}'')$. □

5 Complementary connected d-graphs

A d-graph $\mathcal{G} = (V, E_1, \ldots, E_d)$ is called *complementary connected (CC)* if for every $i \in [d] = \{1, \ldots, d\}$ graph $\overline{G}_i = (V, \bigcup_{j \in [d] \setminus \{i\}} E_j)$, complementary to $G_i = (V, E_i)$, is connected. In other words, for every $v, v' \in V$ and $i \in [d]$ there is a path from v to v' that contains no edge from E_i.

It is easy to verify that one edge ($|V| = 2$) is not a CC d-graph. Furthermore, a triangle ($|V| = 3$) colored by one or two colors is not a CC d-graph either.

Yet, Π and Δ are CC d-graphs; moreover, they are minimal, that is, every sub-d-graph of Π or Δ is already a non-CC d-graph.

Remark 3 *For the last claim to hold we have to postulate that the trivial d-graph, which consists of a single vertex, is* **not** *CC. Indeed, otherwise CC d-graphs Π and Δ would not be minimal.*

Furthermore, it appears that Π and Δ are the only minimal and, moreover, the only locally minimal, CC d-graphs.

Theorem 4 *Each CC d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$, distinct from Π and Δ, contains a vertex $v \in V$ such that the sub-d-graph $\mathcal{G}[V \setminus \{v\}]$ is still CC.*

Remark 4 *This statement was announced in [14, 16] and its proof was recently published in [3]. The case $d = 2$ is simpler than the general one, since Δ cannot exist when $d \leq 2$. This case was considered earlier, in [25, 26, 24, 13, 14, 16]. It was also suggested as a problem for Moscow Mathematical Olympiad in 1971 (Problem 72 in [12]) and was successfully solved by seven high school students.*

Interestingly, two distinct families, the CC and non-CIS d-graphs, have the same (locally) minimal elements: Π and Δ. However, it is easy to see that these two families are in general position already for $d = 2$. For example, the A-graph (or bull) is CC and CIS; adding a vertex of degree 4 to P_4 one obtains a non-CC and non-CIS graph; furthermore, a single edge is not CC but CIS graph; finally, P_4 is CC but not CIS graph; recall that Π and Δ are CC but not CIS d-graphs.

6 Not complementary connected d-graphs

As we just mentioned, a single edge is not a CC graph. In other words, it is not a CC d-graph for $d = 2$. Moreover, it is not a CC d-graph for every $d \geq 1$. However, this CC d-graph is still not minimal, since by convention, a single vertex is not CC either. Obviously, it is the only minimal CC d-graph. Moreover, it is the only locally minimal one.

Proposition 2 *Each non-CC d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ with $|V| \geq 2$ contains a vertex $v \in V$ such that the sub-d-graph $\mathcal{G}[V \setminus \{v\}]$ is still not CC.*

Proof. It is simple. Since $\mathcal{G}$ is not CC, there is an $i \in [d]$ such that graph $\overline{G}_i = (V, \overline{E_i})$ is not connected. Let us eliminate vertices $v \in V$ one by one in a way to keep this property. Obviously, V can be reduced to two vertices. Then, as the last step, we reduce V to one vertex. □

7 Weakly monotone Boolean functions

Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$ of n variables, a vector $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ is called a *true* (*false*) vector of f if $f(x) = 1$ (respectively, $f(x) = 0$). Furthermore, x is a *minimal* true vector of f if all its predecessors are false; in other words, $f(x) = 1$, while $f(x') = 0$ whenever $x' \leq x$ and $x' \neq x$. Finally, x is a *locally minimal* true vector of f if all it immediate predecessors are false; in other words, if $f(x) = 1$, while $f(x - e_i) = 0$ for all $i \in supp(x) = \{i \in [n] \mid x_i \neq 0\}$, where $[n] = \{1, \ldots, n\}$ and e_i is a Boolean vector whose ith coordinate is 1, while all others are equal to 0.

Let $T = T(f),\ M = M(f),\ L = L(f) \subseteq \{0,1\}^n$ denote the sets of true, minimal true, and locally minimal true vectors of f, respectively.

Containments $M \subseteq L \subseteq T$ hold for every f, by the above definitions.

Remark 5 *It is also obvious from the definitions that the local minimality in f can be verified in polynomial time whenever f is given by a polynomial oracle. In contrast, even in this case verifying minimality is exponential. Indeed, the number of immediate predecessors of a vector $x \in \{0,1\}^n$ is n, while the number of all predecessors of x is 2^k, where $k = k(x) = |supp(x)|$.*

Boolean function f is *monotone* if $f(x') = 1$ whenever $x' \leq x$ and $f(x) = 1$; furthermore, f is *weakly monotone* [2, 3] if every its *not minimal* true vector has an immediate true predecessor, or in other words, if $x \in T \setminus M$ implies that $x - e_i \in T$ for some $i \in [n]$, that is, $x \notin L$. The next two properties of weakly monotone functions immediately follow from the above definitions.

Claim 1 *A Boolean function f is weakly monotone if and only if the sets of its minimal and locally minimal true vectors coincide: $M(f) = L(f)$.* □

Claim 2 *Every monotone Boolean function is weakly monotone.* □

Weakly monotone Boolean functions frequently appear in combinatorics; see [3, 4] for their applications in graph and game theories.

All above statements related to CC and CIS d-graphs can be reformulated in terms of weak monotonicity as follows:

The families of CC and non-CIS d-graphs are weakly monotone, moreover, they have the same (local) minima: Π and Δ; see Theorems 1 and 4.

Family of the non-CC d-graphs is also weakly monotone; it has a unique minimal element, the trivial (single-vertex) d-graph; see Proposition 2.

Thus, both CC and non-CC d-graphs form weakly monotone (although not monotone) families.

In contrast, family of the CIS d-graphs is not weakly monotone already for $d = 2$. For example, $G = L(K_{3,3})$ is a CIS graph but the CIS property will be lost after eliminating any vertex.

As we already mentioned in Remark 2, the same graph G is a locally edge-minimal (but not edge-minimal) perfect graph too. Thus, the family of perfect graphs is not weakly monotone (with respect to edges) either. However, let us remark that, with respect to vertices, this family is monotone and, hence, weakly monotone too. Indeed, perfectness is a hereditary property, just by definition.

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