PORTA, NEOS, and Knapsack Covers

Cover Inequalities

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Today’s Outline

• Knapsack Cover inequalities
  ◦ Facets
  ◦ Lifting

• Why would we care?
Valid Inequalities for the Knapsack Problem

- We are interested in valid inequalities for the knapsack set $\text{KNAP}$

$$\text{KNAP} = \{x \in \mathbb{B}^n | \sum_{j \in N} a_j x_j \leq b\}$$

- $N = \{1, 2, \ldots n\}$

- A set $C \subseteq N$ is a cover if $\sum_{j \in C} a_j > b$

- A cover $C$ is a minimal cover if $C \setminus j$ is not a cover $\forall j \in C$

- If $C \subseteq N$ is a cover, then the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a valid inequality for $S$
I want to solve MIPs, why do I care about strong inequalities for the knapsack problem?

If \( P = \{ x \in \mathbb{B}^n \mid Ax \leq b \} \), then for any row \( i \),
\[ P_i = \{ x \in \mathbb{B}^n \mid a_i^T x \leq b_i \} \]
is a relaxation of \( P \).

- \( P \subseteq P_i \forall i = 1, 2, \ldots m \)
- \( P \subseteq \bigcap_{i=1}^{m} P_i \)

Any inequality valid for a relaxation of an IP is valid for the IP itself.

Generating valid inequalities for a relaxation is often easier.

If the intersection of the relaxations is a good approximation to the true problem, then the inequalities will be quite useful.

Crowder, Johnson, and Padberg is the seminal paper that shows this to be true.
Example

MYKNAP = \{ x \in \mathbb{B}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \}

• Some minimal covers are the following:

\[
\begin{align*}
x_1 + x_2 + x_3 & \leq 2 \\
x_1 + x_2 + x_6 & \leq 2 \\
x_1 + x_5 + x_6 & \leq 2 \\
x_3 + x_4 + x_5 + x_6 & \leq 3
\end{align*}
\]
Back to the Knapsack

- If $C \subseteq N$ is a cover, the extended cover $E(C)$ is defined as
  
  $E(C) = C \cup \{ j \in N \mid a_j \geq a_i \ \forall i \in C \}$

- If $E(C)$ is an extended cover for $S$, then the extended cover inequality
  
  $\sum_{j \in E(C)} x_j \leq |C| - 1,$

  is a valid inequality for $S$

- Note this inequality dominates the cover inequality if
  $E(C) \setminus C \neq \emptyset$

- (Example, cont.) The cover inequality $x_3 + x_4 + x_5 + x_6 \leq 3$
  is dominated by the extended cover inequality
  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$
In General...

- Order the variables so that $a_1 \geq a_2 \ldots \geq a_n$
- Let $C$ be a cover with $C = \{j_1, j_2, \ldots j_r\}$ ($j_1 < j_2 < \ldots < j_r$) so that $a_{j_1} \geq a_{j_2} \geq \ldots \geq a_{j_r}$. Let $p = \min\{j \mid j \in N \setminus E(C)\}$.
- If any of the following conditions hold, then
  \[ \sum_{j \in E(C)} x_j \leq |C| - 1 \]
  gives a facet of $\text{conv}(\text{KNAP})$
  - $C = N$
  - $E(C) = N$ and (*) $\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b$
  - $C = E(C)$ and (**) $\sum_{j \in C \setminus j_1} a_j + a_p \leq b$
  - $C \subset E(C) \subset N$ and (*) and (**).
Examples

• \( C = \{1, 2, 6\} \). \( E(C) = C \).
  ◦ If \( a_2 + a_6 + a_3 \leq b \), then \( x_1 + x_2 + x_6 \leq 2 \) is a facet of \( \text{conv(\text{MYKNAP})} \)
  ◦ \( 16 \leq 19 \). It is a facet!

• \( C = \{3, 4, 5, 6\} \). \( E(C) = \{1, 2, 3, 4, 5, 6\} \). \( C \subset E(C) \subset N \).
  \( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \) is a facet of \( \text{conv(\text{MYKNAP})} \) if...
  ◦ \( a_4 + a_5 + a_6 + a_7 \leq b \)? (Yes!)
  ◦ \( a_5 + a_6 + a_1 \leq b \) (No!),

• So \( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \) is not facet-defining for \( \text{conv(\text{MYKNAP})} \)
conv(MYKNAP)

\[
\begin{align*}
  x_j & \geq 0 & \forall j = 1, 2, \ldots, 7 \\
  x_j & \leq 1 & \forall j = 1, 2, \ldots, 7 \\
  x_1 + x_5 + x_6 & \leq 2 \\
  x_1 + x_4 + x_6 & \leq 2 \\
  x_1 + x_4 + x_5 & \leq 2 \\
  x_1 + x_3 + x_6 & \leq 2 \\
  x_1 + x_3 + x_5 & \leq 2 \\
  x_1 + x_3 + x_4 & \leq 2 \\
  x_1 + x_2 + x_6 & \leq 2 \\
  x_1 + x_2 + x_5 & \leq 2 \\
  x_1 + x_2 + x_4 & \leq 2 \\
  x_1 + x_2 + x_3 & \leq 2 \\
  2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 & \leq 3
\end{align*}
\]
Covers and Lifting

• Let $P_{1,2,7} = \text{MYKNAP} \cap \{x \in \mathbb{R}^7 \mid x_1 = x_2 = x_7 = 0\}$

• Consider the cover inequality arising from $C = \{3, 4, 5, 6\}$.

• $\sum_{j \in C} x_j \leq 3$ is facet defining for $P_{1,2,7}$

• If $x_1$ is not fixed at 0, can we strengthen the inequality?

• For what values of $\alpha_1$ is the inequality

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

valid for

$P_{2,7} = \{x \in \text{MYKNAP} \mid x_2 = x_7 = 0\}$?

◊ If $x_1 = 0$ then the inequality is valid for all values of $\alpha_1$
The Other Case

• If \( x_1 = 1 \), the inequality is valid if and only if

\[
\alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3
\]

is valid for all \( x \in \mathbb{B}^4 \) satisfying

\[
6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11
\]

• Equivalently, if and only if

\[
\alpha_1 + \max_{x \in \mathbb{B}^4} \{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\} \leq 3
\]

• Equivalently if and only if \( \alpha_1 \leq 3 - \gamma \), where

\[
\gamma = \max_{x \in \mathbb{B}^4} \{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}.
\]
Solving the Knapsack Problem

- In this case, we can “solve” the knapsack problem to see that $\gamma = 1$. Therefore $\alpha_1 \leq 2$.
- The inequality

$$2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

is a valid inequality for $P_{27}$

◊ Is it facet-defining?
• What we’ve done is called lifting. Lifting is a process in which a valid (and facet defining) inequality for \( S \cap \{ x \in \mathbb{B}^n \mid x_k = 0 \} \) is turned into a facet defining inequality for \( S \).

• **Theorem.** Let \( S \subseteq \mathbb{B}^n \), for \( \delta \in \{0, 1\} \), \( S^\delta = S \cap \{ x \in \mathbb{B}^n \mid x_1 = \delta \} \). Suppose

\[
\sum_{j=2}^n \pi_j x_j \leq \pi_0
\]

is valid for \( S^0 \).
• If $S^1 = \emptyset$, then $x_1 \leq 0$ is valid for $S$

• If $S^1 \neq \emptyset$, then

$$\alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

is valid for $S$ for any $\alpha_1 \leq \pi_0 - \gamma$, where

$$\gamma - \max\{\sum_{j=2}^{n} \pi_j x_j \mid x \in S^1\}.$$
Lifting Thm. (3)

• If $\alpha_1 = \pi_0 - \gamma$ and $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a face of dimension $k$ of $\text{conv}(S^0)$, then

$$\alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

defines a face of dimension at least $k + 1$ of $\text{conv}(S)$. 
You Can Also “DownLift”

- Let $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ be valid for $S^1$.
- If $S^0 = \emptyset$, $x_1 \geq 1$ is valid for $S$, otherwise
  $\xi_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0 + \xi_1$

is valid for $S$, for $\xi_i \geq \gamma - \pi_0$

$\diamond \quad \gamma = \max\{\sum_{j=2}^{n} \pi_j x_j \mid x \in S^0\}$.

- Similar facet/dimension results to uplifting if the lifting is maximum.
• Group exercise

• Find facets of the polyhedron:

\[ 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39 \]
Quiz Problem

(12) + x1 + x9 <= 1
(13) + x1 + x8 <= 1
(14) + x1 + x7 <= 1
(15) + x1 + x2 + x6 <= 1
(16) + x1 + x2 + x5 <= 1
(17) + x1 + x2 + x3 + x4 <= 1
(18) + 2x1 + x2 + x8 + x9 <= 2
(19) + x1 + x2 + x7 + x10 <= 2
(20) + 2x1 + x2 + x3 + x7 + x9 <= 2
(21) + 2x1 + x2 + x3 + x7 + x8 <= 2
(22) + 2x1 + 2x2 + x3 + x4 + x5 + x6 <= 2
(23) + x1 + x2 + x3 + x6 + x10 <= 2
(24) + x1 + x2 + x3 + x5 + x10 <= 2
(25) + 2x1 + x2 + x3 + x4 + x6 + x9 <= 2
(26) + 2x1 + x2 + x3 + x4 + x6 + x8 <= 2
(27) + 2x1 + x2 + x3 + x4 + x5 + x9 <= 2
(28) + 2x1 + x2 + x3 + x4 + x5 + x8 <= 2
(29) + 2x1 + x2 + x3 + x4 + x5 + x6 + x7 <= 2
(30) + 3x1 + 2x2 + 2x3 + 2x4 + x5 + x6 + x9 <= 3
(31) + 3x1 + 2x2 + 2x3 + 2x4 + x5 + x6 + x8 <= 3
(32) + 2x1 + 2x2 + 2x3 + x4 + x5 + x6 + x10 <= 3
(33) + 2x1 + x2 + x3 + x8 + x9 + x10 <= 3
(34) + 3x1 + 2x2 + 2x3 + x4 + x5 + x6 + x7 + x9 <= 3
(35) + 3x1 + 2x2 + 2x3 + x4 + x5 + x6 + x7 + x8 <= 3
(36) + 2x1 + x2 + x3 + x4 + x7 + x9 + x10 <= 3
(37) + 2x1 + x2 + x3 + x4 + x7 + x8 + x10 <= 3
\begin{align*}
(38) & \quad +2x1+2x2+ x3+ x4+ x5+ x6+ x7 +x10 <= 3 \\
(39) & \quad +2x1+ x2+ x3+ x4+ x5+ x6 +x8 +x10 <= 3 \\
(40) & \quad +3x1+2x2+ x3+ x4+ x5+ x6+ x7+ x8+ x9 <= 3 \\
(41) & \quad +3x1+2x2+ x3+ x4 +2x7+ x9+x10 <= 4 \\
(42) & \quad +3x1+3x2+2x3+ x4+ x5+2x6+ x7+ x8+ x9 <= 4 \\
(43) & \quad +3x1+3x2+2x3+ x4+2x5+ x6+ x7 +x10 <= 4 \\
(44) & \quad +3x1+3x2+2x3+ x4+ x5+2x6 +x7 +x10 <= 4 \\
(45) & \quad +3x1+2x2+2x3+ x4+ x5+2x6 + x8 +x10 <= 4 \\
(46) & \quad +3x1+2x2+2x3+ x4+2x5+ x6 +x8 +x10 <= 4 \\
(47) & \quad +4x1+3x2+2x3+2x4+ x5+2x6+ x7+ x8+ x9 <= 4 \\
(48) & \quad +4x1+3x2+2x3+2x4+2x5+ x6+ x7+ x8+ x9 <= 4 \\
(49) & \quad +4x1+2x2+2x3+ x4+ x5+ x6+2x7+ x8+ x9 <= 4 \\
(50) & \quad +3x1+2x2+2x3+2x4+ x5+ x6+ x7 + x9+x10 <= 4 \\
(51) & \quad +3x1+2x2+2x3+2x4+ x5+ x6+ x7+ x8 +x10 <= 4 \\
(52) & \quad +3x1+2x2+2x3+ x4+ x5+ x6+ x7+ x8+ x9+x10 <= 4 \\
(53) & \quad +4x1+4x2+3x3+2x4+2x5+2x6+ x7 +x10 <= 5 \\
(54) & \quad +5x1+3x2+3x3+2x4+2x5+2x6+2x7+ x8+ x9 <= 5 \\
(55) & \quad +5x1+4x2+3x3+3x4+2x5+2x6+ x7+ x8+ x9 <= 5 \\
(56) & \quad +4x1+3x2+3x3+2x4+2x5+2x6+ x7+ x8 +x10 <= 5 \\
(57) & \quad +4x1+3x2+3x3+2x4+ x5+2x6+ x7+ x8+ x9+x10 <= 5 \\
(58) & \quad +4x1+3x2+3x3+2x4+2x5+ x6+ x7+ x8+ x9+x10 <= 5 \\
(59) & \quad +4x1+3x2+2x3+2x4+ x5+ x6+2x7+ x8+ x9+x10 <= 5 \\
(60) & \quad +5x1+3x2+3x3+3x4+2x5+2x6+ x7+2x8 +x10 <= 6 \\
(61) & \quad +5x1+4x2+3x3+3x4+2x5+2x6+2x7+ x8+ x9+x10 <= 6 \\
(62) & \quad +5x1+3x2+3x3+2x4+2x5+2x6+ x7+2x8+ x9+x10 <= 6 \\
(63) & \quad +5x1+4x2+4x3+3x4+2x5+2x6+ x7+ x8+ x9+x10 <= 6 \\
(64) & \quad +5x1+3x2+3x3+2x4+ x5+ x6+2x7+ x8+2x9+x10 <= 6 \\
(65) & \quad +5x1+3x2+3x3+2x4+ x5+ x6+2x7+2x8+ x9+x10 <= 6 \\
(66) & \quad +6x1+5x2+4x3+3x4+3x5+3x6+2x7+ x8 +x10 <= 7 \\
(67) & \quad +6x1+4x2+4x3+3x4+2x5+2x6+2x7+ x8+2x9+x10 <= 7 \\
(68) & \quad +6x1+4x2+4x3+3x4+2x5+3x6+ x7+2x8+ x9+x10 <= 7 \\
(69) & \quad +6x1+4x2+4x3+3x4+3x5+2x6+ x7+2x8+ x9+x10 <= 7 \\
(70) & \quad +7x1+5x2+4x3+3x4+2x5+2x6+3x7+2x8+2x9+x10 <= 8
\end{align*}
(71) + 7x1 + 5x2 + 5x3 + 4x4 + 3x5 + 3x6 + 2x7 + 2x8 + x9 + x10 <= 8
(72) + 7x1 + 5x2 + 5x3 + 4x4 + 2x5 + 3x6 + 2x7 + x8 + 2x9 + x10 <= 8
(73) + 7x1 + 5x2 + 5x3 + 4x4 + 3x5 + 2x6 + 2x7 + x8 + 2x9 + x10 <= 8
(74) + 8x1 + 6x2 + 5x3 + 4x4 + 3x5 + 3x6 + 3x7 + 2x8 + 2x9 + x10 <= 9
(75) + 8x1 + 6x2 + 6x3 + 5x4 + 3x5 + 3x6 + 2x7 + x8 + 2x9 + x10 <= 9
(76) + 9x1 + 7x2 + 6x3 + 5x4 + 3x5 + 4x6 + 3x7 + 2x8 + 2x9 + x10 <= 10
(77) + 9x1 + 7x2 + 6x3 + 5x4 + 4x5 + 3x6 + 3x7 + 2x8 + 2x9 + x10 <= 10
(78) + 9x1 + 7x2 + 6x3 + 5x4 + 4x5 + 4x6 + 3x7 + 2x8 + x9 + x10 <= 10
(79) + 10x1 + 8x2 + 7x3 + 6x4 + 4x5 + 4x6 + 3x7 + 2x8 + 2x9 + x10 <= 11
(80) + 12x1 + 9x2 + 8x3 + 6x4 + 5x5 + 5x6 + 4x7 + 3x8 + 2x9 + x10 <= 13
Knapsack Separation

- So there are *lots* of inequalities. How do I find one that might be useful?
- First note that $\sum_{j \in C} x_j \leq |C| - 1$ can be rewritten as
  $$\sum_{j \in C} (1 - x_j) \geq 1.$$
- Separation Problem: Given a “fractional” LP solution $\hat{x}$, does $\exists C \subseteq N$ such that $\sum_{j \in C} a_j > b$ and $\sum_{j \in C} (1 - \hat{x}_j) < 1$?
- Is $\gamma = \min_{C \subseteq N} \{ \sum_{j \in C} (1 - \hat{x}_j) \mid \sum_{j \in C} a_j > b \} < 1$?
- Let $z_j \in \{0, 1\}$, $z_j = 1$ if $j \in C$, $z_j = 0$ if $j \notin C$.
- Is $\gamma = \min \{ \sum_{j \in N} (1 - \hat{x}_j) z_j \mid \sum_{j \in N} a_j z_j > b, z \in \mathbb{B}^n \} < 1$?
- If $\gamma \geq 1$, $\hat{x}$ satisfies all cover inequalities.
• If $\gamma < 1$ with optimal solution $z_R$, then $\sum_{j \in R} x_j \leq |R| - 1$ is a violated cover inequality.
Example

\[ \text{MYKNAP} = \{ x \in \mathbb{B}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \} \]

- \( \hat{x} = (0, 2/3, 0, 1, 1, 1, 1) \)

\( \gamma = \min \{ z_1 + 1/3z_2 + z_3 \mid 11z_1 + 6z_2 + 6z_3 + 5z_4 + 5z_5 + 4z_6 + z_7 \geq 20 \} \).

- \( \gamma = 1/3 \)

- \( z = (0, 1, 0, 1, 1, 1, 1) \)

- \( x_2 + x_4 + x_5 + x_6 + x_7 \leq 4 \)

- Minimal Cover: \( x_2 + x_4 + x_5 + x_6 \leq 3 \)

- Extended Cover: \( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \)

- To get the facet, you would have to start lifting from the
minimal cover, with $x_1, x_2, x_7$ fixed at 0.
General Lifting and SuperAdditivity

- \( K = \text{conv}(\{ x \in \mathbb{Z}_+^N, y \in \mathbb{R}_+^M \mid a^T x + g^T y \leq b, x \leq u \}) \)

- Partition \( N \) into \( [L, U, R] \)
  - \( L = \{ i \in N \mid x_i = 0 \} \)
  - \( U = \{ i \in N \mid x_i = u_i \} \)
  - \( R = N \setminus L \setminus U \)

- We will use the notation: \( x_R \) to mean the vector of variables that are in the set \( R \).
  - \( a_R^T x_R = \sum_{j \in R} a_j x_j \)

- \( K(L, U) = \text{conv}(\{ x \in \mathbb{Z}_+^N, y \in \mathbb{R}_+^M \mid a_R^T x + g^T y \leq d, x_R \leq u_R, x_i = 0 \ \forall i \in L, x_i = u_i \ \forall i \in U. \}) \)
  - So \( d = b - a_U^T x_U \)
Lifting

• Let $\pi^T x_R - \sigma^T y \leq \pi_0$ be a valid inequality for $K(L, U)$.

• Consider the lifting function $\Phi : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$
  
  ◦ $(\infty)$ if lifting problem is infeasible

\[
\Phi(\alpha) = \pi_0 - \max \{ \pi^T x_R + \sigma^T y \mid a^T_R x_R + g^T y \leq d - \alpha, x_R \leq u_R, x_R \in \mathbb{Z}_+^{\lvert R \rvert}, y \in \mathbb{R}_+^{\lvert M \rvert} \}
\]

• In words, $\Phi(\alpha)$ is the maximum value of the LHS of the valid inequality if the RHS in $K$ is reduced by $\alpha$. 
Why do we care about $\Phi$?

$$\pi_R^T x_R + \pi_L^T x_L + \pi_U^T (u_U - x_U) + \sigma^T y \leq \pi_0$$

is a valid inequality for $K$ if and only if

$$\pi_L^T x_L + \pi_U^T (u_U - x_U) \leq \Phi (a_L^T x_L + a_U^T (x_U - u_U)) \forall (x, y) \in K.$$ 

Proof.?
Example—Sequential Lifting

- Lifting one variable (at a time) in 0-1 IP (like we have done so far)...
- \( \alpha x_k + \pi^T_R x_R \leq \pi_0 \) is valid for \( P \leftrightarrow \alpha x_k \leq \Phi(a_k x_k) \forall x \in P \)
  - \( x_k = 0, \quad 0 \leq \Phi(0) \) is always true.
  - \( x_k = 1, \quad \Rightarrow \alpha \leq \Phi(a_l) \)
- If I “know” \( \Phi(q)(\forall q \in \mathbb{R}) \), I can just “lookup” the value of the lifting coefficient for variable \( x_k \)

* Note that if I have restricted more than one variable, then this “lookup” logic is not necessarily true
  - For lifting two (0-1) variables, I would have to look at four possible values.
  - In general, the lifting function changes with each new variable “lifted”.

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A function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is superadditive if

$$\phi(q_1) + \phi(q_2) \leq \phi(q_1 + q_2)$$

Superadditive functions play a significant role in the theory of integer programming. (See N&W page 229). (We’ll probably revisit them later).

Superadditive Fact:

$$\sum_{j \in N} \phi(a_j)x_j \leq \sum_{j \in N} \phi(a_jx_j) \leq \phi \left( \sum_{j \in N} a_jx_j \right).$$
"Multiple Lookup"—Superadditivity

- Suppose that \( \phi \) is a superadditive lower bound on \( \Phi \) that satisfies\( \pi_i = \phi(a_i) \forall i \in L \) and \( \pi_i = \phi(-a_i) \forall i \in U \)

\[
\sum_{i \in L} \phi(a_i)x_i + \sum_{i \in U} \phi(-a_i)(u_i - x_i) \leq \phi(a_L^T x_L + a_U^T (x_U - u_U)) \leq \Phi(a_L^T x_L + a_U^T (x_U - u_U))
\]

- So

\[
\pi_R^T x_R + \pi_L^T x_L + \pi_U^T (u_U - x_U) + \sigma^T y \leq \pi_0
\]

is a valid inequality for \( K \)
The Main Result

- If $\phi$ is a superadditive lower bound on $\Phi$, any inequality of the form $\pi^T_R x_R - \sigma^T y \leq \pi_0$, which is valid for $K(L,U)$, can be extended to the inequality

$$\pi^T_R x_R + \sum_{j \in L} \phi(a_j)x_j + \sum_{j \in U} \phi(-a_j)(u_j - x_j) + \sigma^T y \leq \pi_0$$

which is valid for $K$.

- If $\pi_i = \phi(a_i)$ $\forall i \in L$ and $\pi_i = \phi(-a_i)$ $\forall i \in U$ and $\pi^T x_R - \sigma^T y = \pi_0$ defines a $k$-dimensional face of $K(L,U)$, then the lifted inequality defines a face of dimension at least $k + |L| + |U|$.