30 YEARS OF DM
(A PERSONAL PERSPECTIVE)

JAROSLAV NEŠETRIL
CHARLES UNIVERSITY
PRAGUE

DIMACS 30
22/11/2019
30 YEARS OF DM
(A PERSONAL PERSPECTIVE)

JAROSLAV NEŠETŘIL
CHARLES UNIVERSITY
PRAGUE

DIMACS, 22/11/2019
30 YEARS OF DM (A PERSONAL PERSPECTIVE)

M. JAROSLAV IVES

CHARLES-UNIVERSITY PRAGUE
IT'S OVER

RECHS ARE FREE
Who would think 30 years ago that
Who would think 30 years ago that

structural graph theory moves towards stability theory
Who would think 30 years ago that

- structural graph theory moves towards stability theory
- Ramsey theory moves towards topological dynamics
WHO WOULD THINK 30 YEARS AGO THAT

— STRUCTURAL GRAPH THEORY MOVES TOWARDS STABILITY THEORY

— RAMSEY THEORY MOVES TOWARDS TOPOLOGICAL DYNAMICS

— COMPLEXITY OF COLORINGS WILL FIND SETTING IN UNIVERSAL ALGEBRA
WHO WOULD THINK 30 YEARS AGO THAT

— STUDY OF FINITE PROPERTIES WILL LEAD TO STUDY OF LIMITS, STRUCTURAL LIMITS
WHO WOULDN'T THINK 30 YEARS AGO THAT:

— STUDY OF FINITE PROPERTIES WILL LEAD TO STUDY OF LIMITS:
  STRUCTURAL LIMITS

... AND MUCH MORE
Just a few glimpses

I. Ramsey Theory
II. Structural Graph Theory
III. Limits
Finite Ramsey theorem

For every \( p, k, n \) there exists \( N = N(p, k, n) \) such that

\[
N \rightarrow (n)_k^p
\]

Ramsey class of structures \( \mathcal{C} \)

For every \( A, B \in \mathcal{C}, k \in \mathbb{N}, \)
there exists \( C \in \mathcal{C} \)

\[
C \rightarrow (B)_k^A
\]
FINITE RAMSEY THEOREM

For every $p, k, n$ there exists $N = N(p, k, n)$ such that

$$N \rightarrow (n)^p_k$$

\[\Downarrow\]

RAMSEY CLASS OF STRUCTURES $\mathcal{C}$

For every $A, B \in \mathcal{C}$, $k \in \mathbb{N}$, there exists $C \in \mathcal{C}$

$$C \rightarrow (B)^A_k$$

\[\Updownarrow\]

Explicitely:

For every partition

$$\left( \begin{array}{c} C \\ A \end{array} \right) = A_1, \ldots, A_k$$

there exists $B' \in (C)_B$ and $i_0$ such that

$$\left( \begin{array}{c} B' \\ A \end{array} \right) \subseteq A_{i_0}.$$
EXAMPLES OF RAMSEY CLASSES

- ALL COMPLETE GRAPHS \((FRT)\)
- ALL ORDERED GRAPHS
- ALL ORDERED \(K_r\)-FREE GRAPHS
- CLASSES OF COMPLEMENTARY GRAPHS
  \((n. \text{ Rödl})\)
EXAMPLES OF RAMSEY CLASSES
- All complete graphs (FRT)
- All ordered graphs
- All ordered $K_4$-free graphs
- Classes of complementary graphs
  (N. Rödl)

\[ \text{THM (N. 69)} \]

Every Ramsey class of structures is the age of an ultrahomogeneous structure

\[ \text{CORR} \]

The above are all Ramsey classes of graphs
THM (KECHRIS/DESTOU, TODORČEVIĆ 2006)

FOR A CLASS $\mathcal{C}$ WITH ORDER PROPERTY

1. $\mathcal{C}$ IS RAMSEY

2. $\mathcal{C}$ IS THE AGE OF AN ULTRAHOMOGENEOUS STRUCTURE $U$ SUCH THAT $\text{AUT}(U)$ IS EXTREMELY AMENABLE SUBGROUP OF $S_w$. 
THM (N. 2006)

ORDERED FINITE METRIC SPACES ARE RAMSEY
Ordered finite metric spaces are Ramsey.

Ordered graphs with isometric embeddings are Ramsey class (Della Monica, Rödl 2012).

Ramsey ordered $\mathcal{S}$-metric spaces are characterized (Hubička, n. 2019).
**THM (N. 2006)**

Ordered finite metric spaces are Ramsey

---

Ordered graphs with isometric embeddings are Ramsey class (Dellamonica, Rödl 2014)

---

Ramsey ordered \( S \)-metric spaces are characterized (Hubička, N. 2019)

---

All those Ramsey classes advances in Math. 2019
$\mathcal{M}$-METRIC SPACE
$(X, \mathcal{M})$

$l : (X, \mathcal{M}) \to \mathcal{M}$

$l([x,y]) \leq l([x,y]) \cup l([y,z])$

---

**THM** (HKN 2018)

The class of all finite ordered $\mathcal{M}$-metric spaces is Ramsey providing $\mathcal{M}$ is Archimedean and locally finite.

---

ARCHIMEDEAN:

For every $a, b \in \mathcal{M}$ there exists $n$ such that

$n \times a = a \oplus a \oplus a \ldots \oplus a \leq b$
II. STRUCTURAL GRAPH THEORY

- THEORY OF MINORS
  (ROBERTSON, SEYMOUR)

- SPARSITY
  ("MATHEMATICS OF SPARSITY"
  E. CANDÉS)
Sparsity is a property of classes (not individual) graphs.

**DEF**

Let $\mathcal{C}$ be a class of graphs. $\mathcal{C}$ is **nowhere dense** if for every $d > 0$ there exists $n(d)$ such that $K_{n(d)}^{(d)}$ fails to be a subgraph of any $G \in \mathcal{C}$. $\mathcal{C}$ is **somewhere dense** if it isn't nowhere dense.

---

$K_n^{(d)} \subseteq G$

SHALLOW TOPOLOGICAL MINOR AT DEPTH $d$

\[ \forall \]

SHALLOW MINOR AT DEPTH $d$

CONTRACTION OF CONNECTED SUBGRAPHS OF RADIUS $\leq d$

\((\text{Milleri, Vavasis, Teng, Thurston})\)

$G \preceq_d H$
**DEF**

\( \mathcal{C} \) **has bounded expansion**

If for every \( d \) the class \( \mathcal{C} \supseteq_d \) of all shallow minors at depth \( \leq d \) is degenerated.

---

\( \mathcal{C} \) **is \( d \)-degenerated**

If every \( G = (V, E) \in \mathcal{C} \) has an ordering \( \leq \) such that every \( n \in V \) has at most \( d \) neighbours \( n' \leq n \).
**Def**

\( \mathcal{C} \) has bounded expansion if for every \( d \) the class \( \mathcal{C} \updownarrow d \) of all shallow minors at depth \( \leq d \) is \( D(d) \)-degenerated.

\( \mathcal{C} \) is \( d \)-degenerated if every \( G = (V, E) \in \mathcal{C} \) has an ordering \( \leq \) such that every \( n \in V \) has at most \( d \) neighbours \( n' \leq n \).

\[ D(d) = \text{expansion function} \]
Every BE-class is a ND-class

\[ \{ G : \text{girth}(G) > \delta(G) \} \] is nowhere dense but not bounded expansion

"Erdős classes"
Sparsity

Graphs, Structures, and Algorithms

Springer
FOR A MONOTONE CLASS $\mathcal{C}$ OF STRUCTURES

1. $\mathcal{C}$ is nowhere dense,
2. $\mathcal{C}$ is stable
3. $\mathcal{C}$ is NIP
4. $\mathcal{C}$ has almost linear algorithm for model checking.
For a monotone class $\mathcal{C}$ of graphs:

1. $\mathcal{C}$ is nowhere dense,
2. $\mathcal{C}$ is stable \( \rightarrow \) Shelah, Podelski + Ziegler
3. $\mathcal{C}$ is NIP \( \rightarrow \) Adler + Adler
4. $\mathcal{C}$ has almost linear algorithm for model checking \( \rightarrow \) Grohe, Kreutzer, Siebertz

Counting universality of categories:

\( \vdots \)
III. LIMITS

FOR

GRAPHS OR

STRUCTURES

(Lovász, B. Szegedy,

J. Chayes, G. Elek, ...
STRUCTURAL LIMITS
(IN FULL GENERALITY)

\( \sigma \) LANGUAGE, \( \sigma \)-STRUCTURE A
\( \psi \) FIRST ORDER \( \sigma \)-FORMULA WITH \( p \) FREE VARIABLES

\( \text{FO}(\sigma) \) ALL \( \sigma \)-FO FORMULAS

\( \text{B}(\text{FO}(\sigma)) \) THE QUOTIENT OF \( \text{FO}(\sigma) \)
BY LOGICAL EQUIVALENCE

\[ \psi(A) = \{ (\nu_1, \ldots, \nu_p) ; A \models \psi(\nu_1, \ldots, \nu_p) \} \]

STONE PAIRING (OF \( \psi, A \))

\[ \langle \psi, A \rangle = \Pr(A \models \psi(\nu_1, \ldots, \nu_p)) \]
\[ = \frac{|\psi(A)|}{|A|^p} \]
Structural Convergence

Sequence \((A_n)\) of \(\sigma\)-structures is \(X\)-convergent if \(<\varphi, A_n>\) is convergent for every \(\varphi \in X\).

\(X\) a set of formulas (fragment)

Relevant fragments

- \(FO_0\): Sentences
- \(FO_p\): \(p\) free variables
- \(QF\): Quantifier free
- \(FO_{local}\): Local formulas
- \(FO\): First order formulas
For every $\varphi$ there exists $n_0(\varphi)$ such that either $n > n_0 \Rightarrow A_n = \varphi$ or $n > n_0 \Rightarrow A_n \neq \varphi$
**X Convergence for Special Fragments**

\[ F_0 \text{- convergence} = \text{elementary convergence} \]

\[ QF \text{- convergence} \approx \text{left convergence} \]

(Aldoüs, Hoover, Loehr's)
A formula $\psi(x_1, \ldots, x_p)$ is **local** if there exists $n$ such that

$$G = \psi(u_1, \ldots, u_p)$$

$$\iff$$

$$G[N_n(\{u_1, \ldots, u_p\})] \models \psi(u_1, \ldots, u_p)$$

For a sequence $(G_n)$ of graphs with degrees $\leq \alpha$

1. $(G_n)$ is local convergent; (Benjamini, Schramm)

2. $(G_n)$ is $\text{FO}^{\text{local}}_\alpha$-convergent;

3. $(G_n)$ is $\text{FO}^{\text{local}}$-convergent.
A FORMULA $\psi(x_1, \ldots, x_p)$ IS \textbf{LOCAL} IF THERE EXISTS $n$ SUCH THAT

$$G = \psi(u_1, \ldots, u_p)$$

$$\iff$$

$$G[N_n(\{u_1, \ldots, u_p\})] = \psi(u_1, \ldots, u_p)$$

\begin{itemize}
  \item FOR A SEQUENCE $(G_n)$ OF GRAPHS WITH DEGREES $\leq d$
    \begin{itemize}
      \item 1. $(G_n)$ IS LOCAL CONVERGENT; \quad (BENJAMINI, SCHRAMM')
      \item 2. $(G_n)$ IS $\text{FO}_{\text{local}}$-CONVERGENT;
      \item 3. $(G_n)$ IS $\text{FO}_{\text{local}}$-CONVERGENT.
    \end{itemize}
\end{itemize}

\begin{itemize}
  \item FOR A SEQUENCE $(G_n)$ OF GRAPHS
    \begin{itemize}
      \item 1. $(G_n)$ $\text{FO}$ CONVERGENT;
      \item 2. $(G_n)$ IS BOTH ELEMENTARY AND $\text{FO}_{\text{local}}$-CONVERGENT.
    \end{itemize}
\end{itemize}
What are limits of converging sequences?
What are limits of converging sequences?

Distributional

(Exchangeable random graph
Aldous, Hoover for left conv.
Unimodular distr.
Benjamini-Schramm)
What are limits of converging sequences?

- **Distributional**
  - Exchangeable random graph
  - Aldous, Hoover for left conv.
  - Unimodular distr.
  - Benjamini-Schramm

- **Non-standard**
  - Elek-Szegedy
  - Ultraproducts
  - Loeb measure
  - Construction
What are limits of converging sequences?

- **Distributional**
  - Exchangeable random graph
  - Aldous, Hoover for left conv.
  - Unimodular distr.
  - Benjamini-Schramm

- **Non-standard**
  - Elek-Szegedy
  - Ultraproducts + Loeb measure construction

- **Graphon graphing**
What are limits of converging sequences?

**DISTRIBUTIONAL**
- Exchangeable random graph
- Aldous, Hoover for left conv.
- Unimodular distr.
- Benjamini-Schramm

**NON-STANDARD**
- Elek-Szegedy
- Ultraproducts
- Loeb measure
- Construction

**GRAPHON**
**GRAPHING**

**MODELING**

Representation theorem for $X$-convergence

Sparsity
NICE LIMIT STRUCTURE?

MODELING $\text{def}$

STANDARD BOREL SPACE ON UNIVERSUM OF A

+ ALL FIRST ORDER DEFINABLE SETS OF TUPLES ARE BOREL (IN PRODUCT SPACE)
NICE LIMIT STRUCTURE?

MODELING \[ \text{DEF} \]

STANDARD BOREL SPACE ON UNIVERSUM OF A +

ALL FIRST ORDER DEFINABLE SETS OF TUPLES ARE BOREL (IN PRODUCT SPACE)

BORREL GRAPHS (KECHRIS SOLECKI TODORCEVIC)

TOTALLY BOREL STRUCTURE (H. FREEDMAN STEINHORN)

\[ <\psi, A> = v^\oplus_A (\psi(A)) \]
NICE LIMIT STRUCTURE?

MODELING $\equiv$

STANDARD BOREL SPACE ON UNIVERSUM OF A +
ALL FIRST ORDER DEFINABLE SETS OF TUPLES ARE BOREL (IN PRODUCT SPACE)

BOREL GRAPHS (KECHRIS)
SOLECKI
TODORCEVIC

TOTALY BOREL STRUCTURE (H. FREEDMAN)
STEINHORN

CONVERGENCE TO MODELING

$(A_n)$ CONVERGES TO $A$
IF FOR EVERY $\psi \in X$:

$\langle \psi, A_n \rangle \rightarrow \langle \psi, A \rangle$

WHERE

$\langle \psi, A \rangle = \nu^+_A(\psi(A))$
NICE LIMIT STRUCTURE?

MODELING $\overset{\text{def}}{=} \text{STANDARD BOREL SPACE ON UNIVERSUM OF A}$
$+\text{ ALL FIRST ORDER DEFINABLE SETS OF TUPLES ARE BOREL (IN PRODUCT SPACE)}$

WHEN MODELING EXISTS?
**THM**

Every FO\textsuperscript{1}\textsubscript{1} convergent sequence of \(\sigma\)-structures has a modeling \(\text{FO}_{\text{1}}\)-limit.

---

**THM**

Every residual FO-convergent sequence \((A_n)\) of \(\sigma\)-structures has modeling FO-limit.

A sequence \((A_n)\) is **residual** if for every integer \(d\) it holds

\[
\lim_{n \to \infty} \sup_{N \in A_n} \frac{B_d(A_n, N)}{|A_n|} = 0.
\]

(Equivalently:

\[
\lim_{n \to \infty} \langle \text{dist}(x_1, x_2) \leq d, A_n \rangle = 0.
\]
J.N., P.OSSONA DE MENDEZ,
CLUSTER ANALYSIS OF LOCAL CONVERGENT SEQUENCES OF STRUCTURES,
THM (2017)

Let $\mathcal{E}$ be nowhere dense class of $\mathcal{G}$-structures.

Then for every FO-converging sequence $(A_n)$, $A_n \in \mathcal{E}$, there exists modeling FO-limit $M$.

"$\mathcal{E}$ has FO-modelings"
THM \text{(2017)}

Let \( \mathcal{C} \) be nowhere dense class of \( \sigma \)-structures.

Then for every FO-converging sequence \((A_n)\), \(A_n \in \mathcal{C}\), there exists modeling FO-limit \(M\).

"\( \mathcal{C} \) has FO-modelings"

\[ \text{COROLLARY} \]

For monotone class \( \mathcal{C} \) of graphs

1. \( \mathcal{C} \) is nowhere dense;
2. \( \mathcal{C} \) has FO-modelings.

N. Possona de Mendez

J. Symb. Logic 2019

Memoires AMS 2020
SEE YOU
AT
DIMATIA 30
IN
2025
See you at DIMATIA 30 in 2025.

Thank you for your attention.