Balanced and Unbalanced Split Graphs

Ann Trenk

Wellesley College

Joint work with Karen Collins, Wesleyan University

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**Split Graphs Definition**

**Definition:** A *split graph* $G$ is a graph whose vertex set can be partitioned as $V(G) = K \cup S$ where $K$ is a clique and $S$ is a stable set.

Such a partition is a *KS*-partition.

**Definition:** A *KS*-partition of a split graph $G$ is *$K$-max* if $|K| = \omega(G)$ and *$S$-max* if $|S| = \alpha(G)$.
A split graph can model a social network consisting of 2 groups of people, $K$, $S$ so that

All pairs of people from $K$ are friends, and
No pair of people from $S$ are friends.
Example: $P_4$ and $K_{1,3}$ are split graphs

$P_4$ has a unique $KS$-partition. It is both $K$-max and $S$-max.

$K_{1,3}$ has two $KS$-partitions. One is $S$-max, the other is $K$-max.
Example: $C_4$ is NOT a split graph

The largest clique in $C_4$ is $K_2$.

The remaining vertices do not form a stable set.
Characterizing Split Graphs

**Theorem:** (Földes and Hammer, 1977) The following are equivalent for a graph $G$:

- $G$ is a split graph.
- $G$ and $\overline{G}$ are chordal.
- $G$ has no induced $2K_2$, $C_4$ or $C_5$. 
Two kinds of split graphs

**Theorem (Hammer, Simeone: 1977)** For any $KS$-partition of a split graph $G$, exactly one of the following holds.

1. $|K| = \omega(G)$ and $|S| = \alpha(G)$. \hfill (K-max, S-max)
2. $|K| = \omega(G) - 1$ and $|S| = \alpha(G)$. \hfill (S-max)
3. $|K| = \omega(G)$ and $|S| = \alpha(G) - 1$. \hfill (K-max)

Moreover, in
(1.) the partition is unique, in
(2.) there exists $s \in S$ so that $K \cup \{s\}$ is complete, and in
(3.) there exists $k \in K$ so that $S \cup \{k\}$ is a stable set.

In (2) and (3), we call vertices $s, k$ *swing vertices.*
**Definition:** A split graph is *balanced* if there exists a $KS$-partition that is both $K$-max and $S$-max, and *unbalanced* otherwise.

Terms $K$-max and $S$-max refer to a partition. Balanced/unbalanced refer to a graph.
Example: Balanced vs. Unbalanced

$P_4$  $K_{1,3}$

$K$-max, $S$-max  $S$-max  $K$-max
Balanced    Unbalanced    Unbalanced
Threshold Graphs

**Definition:** A graph is a *threshold graph* if there exists a threshold $t > 0$ and a positive weight $a_i$ assigned to each $v_i \in V(G)$ so that $S$ is a stable set if and only if $\sum_{i \in S} a_i \leq t$.

**Characterization Theorem (Chvátal and Hammer: 1977)** A graph is a threshold graph if and only if it does not contain $2K_2$, $C_4$, or $P_4$ as an induced subgraph.
Threshold Graphs are split graphs

\[ \text{split graph } \iff \text{no induced } 2K_2, C_4, \text{ or } C_5. \]

\[ \text{threshold graph } \iff \text{no induced } 2K_2, C_4, \text{ or } P_4. \]

Since \( P_4 \) is induced in \( C_5 \), all threshold graphs are split graphs.

Are they balanced or unbalanced?
Threshold graphs are unbalanced split graphs

**Proposition (Collins, Trenk: 2019)**
All balanced split graphs contain $P_4$ as an induced subgraph.

**Corollary:** Threshold graphs are **unbalanced** split graphs.
Background

Balanced split graphs contain a $P_4$ (proof)

Let $G$ be a balanced split graph. Fix a $KS$-partition.
For $v \in K$, let $N_S(v)$ be the set of nbrs of $v$ in $S$.
Let $x$ be a min degree vertex in $K$.

- No swing vertices, so $N_S(x) \neq \emptyset$
- Each $z \in N_S(x)$ has a non-nbr $y$ in $K$ (or $z$ would be a swing vertex)
Balanced split graphs contain a $P_4$ (proof con’t)

- Since $\text{deg}(x) \leq \text{deg}(y)$, there exists $w \in N_S(y) - N_S(x)$.
- Now $x, y, z, w$ induce a $P_4$ in $G$. 
Recognizing Split Graphs from degree sequences

**Theorem (Hammer, Simeone: 1977)** Split graphs can be characterized by their degree sequence.

**Theorem (Cheng, Collins, Trenk: 2016)** The classes of balanced and unbalanced split graphs can also be characterized by their degree sequence.
Recognition from degree sequences

**Theorem (Cheng, Collins, Trenk: 2016)** Let $G$ be a split graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ and let $m = \max\{i : d_i \geq i - 1\}$. Then $G$ is unbalanced if $d_m = m - 1$ and balanced if $d_m > m - 1$.

Proof: Let $d_i = \text{deg}(v_i)$, $K = \{v_1, v_2, \ldots, v_m\}$ and $S = \{v_{m+1}, \ldots, v_n\}$.

- The standard proof of split graph recognition from degree sequences shows that $K$ is a clique, $S$ is a stable set, and $|K| = m = \omega(G)$.
- If $d_m = m - 1$ then $v_m$ is a swing vertex and $G$ is unbalanced.
- If $d_m > m - 1$ then every $v \in K$ has a nbr in $S$, so there are no swing vertices and $G$ is balanced.
Example of recognition from degree sequences

\[ m = \max\{i : d_i \geq i - 1\} \]

**Example 1:** Degree Sequence 2, 2, 1, 1.

\[\begin{array}{c|c|c|c|c}
 d_i & 2 & 2 & 1 & 1 \\
i - 1 & 0 & 1 & 2 & 3 \\
\end{array}\]

\[ m = 2 \text{ and } d_2 > 2 - 1. \quad \text{Balanced} (P_4).\]

**Example 2:** Degree Sequence 3, 1, 1, 1.

\[\begin{array}{c|c|c|c|c}
 d_i & 3 & 1 & 1 & 1 \\
i - 1 & 0 & 1 & 2 & 3 \\
\end{array}\]

\[ m = 2 \text{ and } d_2 = 2 - 1. \quad \text{Unbalanced} (K_{1,3}).\]
Additional Information on Split Graphs

Split Graphs chapter in the forthcoming book:

**Topics in Algorithmic Graph Theory**  (Cambridge University Press)

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