Reflected resolvents in the Douglas–Rachford algorithm: Order of the operators and linear convergence

Speaker: Walaa M. Moursi
Electrical Engineering, Stanford University, USA

Based on joint works with

Heinz H. Bauschke
University of British Columbia
Canada

Lieven Vandenberghe
University of California Los Angeles
USA

DIMACS workshop on ADMM and Proximal Splitting Methods in Optimization
Rutgers University
Wednesday, June 13, 2018
11:30 AM
Monotone operators

Throughout this talk

\[ X \text{ is a real Hilbert space} \]

with inner product \( \langle \cdot, \cdot \rangle \), and induced norm \( \| \cdot \| \), e.g., \( \mathbb{R}^n \) or \( \ell^2 \).

- Recall that an operator \( A : X \rightrightarrows X \) is monotone if

\[
(x, u), (y, v) \in \text{gr } A \Rightarrow \langle x - y, u - v \rangle \geq 0.
\]

- Recall also that a monotone operator \( A \) is maximally monotone if \( A \) cannot be properly extended without destroying monotonicity.

- Examples: Matrices with positive semidefinite parts, subdifferential operators \( \partial f \) of convex functions and skew symmetric operators, e.g.,

\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}.
\]
The three worlds

World I

Monotone operators

\[ A : X \Rightarrow X \]

is max. monotone

\[ 0 \in A x \]

---

- Let \( T : X \to X \). Then \( T \) is nonexpansive if \( \| Tx - Ty \| \leq \| x - y \| \).  
- \( T \) is firmly nonexpansive if \((\forall x \in X)(\forall y \in X)\) \[ \| Tx - Ty \|^2 + \| (\text{Id} - T) x - (\text{Id} - T) y \|^2 \leq \| x - y \|^2 . \]
The three worlds

World I

Monotone operators

\[ A : X \Rightarrow X \]

is max. monotone \iff \[ 0 \in Ax \]

---

- Let \( T : X \to X \). Then \( T \) is nonexpansive if \[ \| Tx - Ty \| \leq \| x - y \| \]. \( T \) is firmly nonexpansive if \( (\forall x \in X)(\forall y \in X) \quad \| Tx - Ty \|^2 + \| (\text{Id} - T)x - (\text{Id} - T)y \|^2 \leq \| x - y \|^2 \).
The three worlds

World I

Monotone operators

\[ A : X \ni x \Rightarrow x \]

is max. monotone

\[ 0 \in Ax \]

World II

Resolvents

\[ J_A := (\text{Id} + A)^{-1} \]

is firmly nonexpansive

\[ x = J_A x \]

---

- Let \( T : X \to X \). Then \( T \) is nonexpansive if \( \| Tx - Ty \| \leq \| x - y \| \). • \( T \) is firmly nonexpansive if \( (\forall x \in X)(\forall y \in X) \quad \| Tx - Ty \|^2 + \| (\text{Id} - T)x - (\text{Id} - T)y \|^2 \leq \| x - y \|^2. \)
The three worlds

World I

Monotone operators

A: X \Rightarrow X
is max. monotone \iff 0 \in Ax

World II

Resolvents

J_A := (\text{Id} + A)^{-1}
is firmly nonexpansive \iff x = J_A x

\text{\textbullet{} Let } T: X \rightarrow X. \text{ Then } T \text{ is nonexpansive if } \|Tx - Ty\| \leq \|x - y\|. \text{ \textbullet{} T is firmly nonexpansive if } (\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2.
The three worlds

<table>
<thead>
<tr>
<th>World I</th>
<th>World II</th>
<th>World III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monotone operators</td>
<td>Resolvents</td>
<td>Reflected resolvents</td>
</tr>
<tr>
<td>( A : X \ni x \Rightarrow x ) is max. monotone ( 0 \in Ax )</td>
<td>( J_A := (\text{Id} + A)^{-1} ) is firmly nonexpansive ( x = J_Ax )</td>
<td>( R_A := 2J_A - \text{Id} ) is nonexpansive ( x = R_Ax )</td>
</tr>
</tbody>
</table>

- Let \( T : X \rightarrow X \). Then \( T \) is nonexpansive if \( \|Tx - Ty\| \leq \|x - y\| \). \( T \) is firmly nonexpansive if \((\forall x \in X)(\forall y \in X)\) \( \|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2 \).
### The three worlds

<table>
<thead>
<tr>
<th>World I</th>
<th>World II</th>
<th>World III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monotone operators</td>
<td>Resolvents</td>
<td>Reflected resolvents</td>
</tr>
<tr>
<td>( A : X \supseteq X ) is max. monotone ( 0 \in Ax )</td>
<td>( J_A := (\text{Id} + A)^{-1} ) is firmly nonexpansive ( x = J_A x )</td>
<td>( R_A := 2J_A - \text{Id} ) is nonexpansive ( x = R_A x )</td>
</tr>
</tbody>
</table>

This talk will focus on the **Third World**!

---

- Let \( T : X \rightarrow X \). Then \( T \) is **nonexpansive** if \( \|Tx - Ty\| \leq \|x - y\| \). • \( T \) is **firmly nonexpansive** if \( (\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 + \|\text{Id} - T\| x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2 \).
The zeros of the sum

Throughout the rest of the talk we assume that

\[ A \text{ and } B \text{ are maximally monotone operators on } X. \]

The problem:
Find \( x \in X \) such that

\[ x \in \text{zer}(A + B) = (A + B)^{-1}(0). \]

---

- \( \text{Id} : X \to X : x \mapsto x \)
- \( R_A := 2J_A - \text{Id} = 2(\text{Id} + A)^{-1} - \text{Id} \)
The zeros of the sum

Throughout the rest of the talk we assume that

\[ A \text{ and } B \text{ are maximally monotone operators on } X. \]

The problem:
Find \( x \in X \) such that

\[ x \in \text{zer}(A + B) = (A + B)^{-1}(0). \]

The Douglas–Rachford algorithm: One successful technique to find a zero of \( A + B \) is via iterating the Douglas–Rachford operator \( T_{A,B} \) defined for the ordered pair \((A, B)\) by

\[ T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A). \]

\[ \text{Id} : X \to X : x \mapsto x. \quad R_A := 2J_A - \text{Id} = 2(\text{Id} + A)^{-1} - \text{Id}. \]
PART I: On the Douglas–Rachford Algorithm and the order of operators

The results in this part appear in

Classical convergence results

Let \( x_0 \in X \). Recall that when
\[
\text{zer}(A + B) \neq \emptyset
\]
we have:

\[\begin{align*}
\rightarrow & \quad \text{Combettes (2004)} \quad J_A(\text{Fix } T_{A,B}) = \text{zer}(A + B).
\end{align*}\]
Classical convergence results

Let $x_0 \in X$. Recall that when

$$\text{zer}(A + B) \neq \emptyset$$

we have:

- **Combettes (2004)** $J_A(\text{Fix } T_{A,B}) = \text{zer}(A + B)$.

- **Krasnosel’skiǐ–Mann (1950s)**

  $x_n = T^n_{A,B}x_0 \xrightarrow{\text{weakly}} \text{some point } \bar{x} \in \text{Fix } T_{A,B} \neq \text{zer}(A + B)$ (in general).
Classical convergence results

Let $x_0 \in X$. Recall that when

$$\text{zer}(A + B) \neq \emptyset$$

we have:

- **Combettes (2004)** $J_A(\text{Fix } T_{A,B}) = \text{zer}(A + B)$.

- **Krasnosel’škiĭ–Mann (1950s)**

  $$x_n = T_{A,B}^n x_0 \xrightarrow{\text{weakly}} \text{some point } \bar{x} \in \text{Fix } T_{A,B} \neq \text{zer}(A + B) \text{ (in general)}.$$

- **Lions–Mercier (1979) and Svaiter (2011)**

  $$J_A T_{A,B}^n x \xrightarrow{\text{weakly}} \text{some point in } \text{zer}(A + B).$$
Classical convergence results

Let \( x_0 \in X \). Recall that when

\[
\text{zer}(B + A) = \text{zer}(A + B) \neq \emptyset
\]

we have:

- **Combettes (2004)** \( J_A(\text{Fix } T_{A,B}) = \text{zer}(A + B) \). Similarly, \( J_B(\text{Fix } T_{B,A}) = \text{zer}(A + B) \).

- **Krasnosel’ščiǐ–Mann (1950s)**

  \[ x_n = T_{A,B}^n x_0 \quad \xrightarrow{\text{weakly}} \quad \text{some point } \bar{x} \in \text{Fix } T_{A,B} \neq \text{zer}(A + B) \text{ (in general)}. \]

  \[ \hat{x}_n = T_{B,A}^n x_0 \quad \xrightarrow{\text{weakly}} \quad \text{some point } \hat{x} \in \text{Fix } T_{B,A} \neq \text{zer}(A + B) \text{ (in general)}. \]

- **Lions–Mercier (1979) and Svaiter (2011)**

  \[ J_A T_{A,B}^n x \quad \xrightarrow{\text{weakly}} \quad \text{some point in } \text{zer}(A + B). \]

  \[ J_B T_{B,A}^n x \quad \xrightarrow{\text{weakly}} \quad \text{some point in } \text{zer}(A + B). \]
Classical convergence results

Let $x_0 \in X$. Recall that when

$$\text{zer}(B + A) = \text{zer}(A + B) \neq \emptyset$$

we have:

- **Combettes (2004)** $J_A(\text{Fix } T_{A,B}) = \text{zer}(A + B)$. Similarly,
  $$J_B(\text{Fix } T_{B,A}) = \text{zer}(A + B).$$

- **Krasnosel’skiǐ–Mann (1950s)**
  
  $x_n = T^n_{A,B}x_0 \overset{\text{weakly}}{\longrightarrow} \text{some point } \bar{x} \in \text{Fix } T_{A,B} \neq \text{zer}(A + B) \text{ (in general)}.$

  $\hat{x}_n = T^n_{B,A}x_0 \overset{\text{weakly}}{\longrightarrow} \text{some point } \hat{x} \in \text{Fix } T_{B,A} \neq \text{zer}(A + B) \text{ (in general)}.$

- **Lions–Mercier (1979) and Svaiter (2011)**
  
  $J_A T^n_{A,B}x \overset{\text{weakly}}{\longrightarrow} \text{some point in } \text{zer}(A + B).$

  $J_B T^n_{B,A}x \overset{\text{weakly}}{\longrightarrow} \text{some point in } \text{zer}(A + B).$

- **Question:** What can we say about $T_{A,B}$ vs. $T_{B,A}$?
In general: $T_{A,B} \neq T_{B,A}$

Suppose that $(A, B) = (N_U, N_V)$ where $U$ is the ball and $V$ is the line.
In general: $T_{A,B} \neq T_{B,A}$

Suppose that $(A, B) = (N_U, N_V)$ where $U$ is the ball and $V$ is the line.
In general: \( T_{A,B} \neq T_{B,A} \)

Suppose that \((A, B) = (N_U, N_V)\) where \(U\) is the ball and \(V\) is the line.
In general: $T_{A,B} \neq T_{B,A}$

Suppose that $(A, B) = (N_U, N_V)$ where $U$ is the ball and $V$ is the line.
In general: $T_{A,B} \neq T_{B,A}$

Suppose that $(A, B) = (N_U, N_V)$ where $U$ is the ball and $V$ is the line.
In general: \( T_{A,B} \neq T_{B,A} \)

Suppose that \((A, B) = (N_U, N_V)\) where \(U\) is the ball and \(V\) is the line.
In general: \( T_{A,B} \neq T_{B,A} \)

Suppose that \((A, B) = (N_U, N_V)\) where \(U\) is the ball and \(V\) is the line.
In general: $T_{A,B} \neq T_{B,A}$

Suppose that $(A, B) = (N_U, N_V)$ where $U$ is the ball and $V$ is the line.
In general: $T_{A,B} \neq T_{B,A}$

Suppose that $(A, B) = (N_U, N_V)$ where $U$ is the ball and $V$ is the line.
In general: \( T_{A,B} \neq T_{B,A} \)

Suppose that \((A, B) = (N_U, N_V)\) where \(U\) is the ball and \(V\) is the line.
In general: $T_{A,B} \neq T_{B,A}$

Suppose that $(A, B) = (N_U, N_V)$ where $U$ is the ball and $V$ is the line.
Reflected resolvent and fixed points of $T_{A,B}$ and $T_{B,A}$

**Theorem (Bauschke–M ’16)**

$R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$.

\[ R_A \quad \text{and} \quad R_B \]

\[ \text{Fix } T_{A,B} \quad \leftrightarrow \quad \text{Fix } T_{B,A} \]
Reflected resolvent and fixed points of $T_{A,B}$ and $T_{B,A}$

**Theorem (Bauschke–M ’16)**

$R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Moreover, we have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Fix } T_{A,B} & \overset{R_A}{\longrightarrow} & \text{Fix } T_{B,A} \\
\downarrow & & \downarrow \\
S_{(A,B)} & \leftarrow & S_{(B,A)} \\
\end{array}
$$

Here $S_{(A,B)} := \{(z, -w) \in X \times X \mid -w \in Bz, w \in Az\}$ is the Kuhn–Tucker set (aka the extended solution set) and $\Delta : X \rightarrow X \times X : x \mapsto (x, x)$. 

Here \[ S_{(A,B)} := \{(z, -w) \in X \times X \mid -w \in Bz, w \in Az\} \text{ is the Kuhn–Tucker set (aka the extended solution set) and } \Delta : X \rightarrow X \times X : x \mapsto (x, x). \]
Proof

Goal: To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$.
Proof

Goal: To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A) \text{ and } T_{B,A} = \frac{1}{2}(\text{Id} + R_A R_B).$$
Goal: To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A) \quad \text{and} \quad T_{B,A} = \frac{1}{2} (\text{Id} + R_A R_B).$$

Let $x \in X$ and note that $\text{Fix } T_{A,B} = \text{Fix } R_B R_A$ and $\text{Fix } T_{B,A} = \text{Fix } R_A R_B$. 

Step 1: $R_A$ maps $\text{Fix } T_{A,B}$ into $\text{Fix } T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x \implies R_A x = R_A R_B R_A x \implies R_A x \in \text{Fix } R_A R_B = \text{Fix } T_{B,A}$. By interchanging $A$ and $B$ one sees that $R_B$ maps $\text{Fix } T_{B,A}$ into $\text{Fix } T_{A,B}$.

Step 2: $R_A$ maps $\text{Fix } T_{A,B}$ onto $\text{Fix } T_{B,A}$. The proof is similar in “flavour” to the proof of Step 1.

Step 3: Since $(\forall x \in \text{Fix } T_{A,B}) R_B R_A x = x$, this proves that $R_A$ is a bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$ with the desired inverse.

Step 4: $R_A : \text{Fix } T_{A,B} \to \text{Fix } T_{B,A}$ is an isometry. Indeed, note that $(\forall x \in \text{Fix } T_{A,B}) (\forall y \in \text{Fix } T_{A,B}) \| x - y \| = \| R_B R_A x - R_B R_A y \| \leq \| R_A x - R_A y \| \leq \| x - y \|.$
Proof

Goal: To show that $R_A$ is an isometric bijection from Fix $T_{A,B}$ to Fix $T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A) \text{ and } T_{B,A} = \frac{1}{2} (\text{Id} + R_A R_B).$$

Let $x \in X$ and note that Fix $T_{A,B} = \text{Fix } R_B R_A$ and Fix $T_{B,A} = \text{Fix } R_A R_B$.

▸ Step 1: $R_A$ maps Fix $T_{A,B}$ into Fix $T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x$
**Proof**

**Goal:** To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A) \text{ and } T_{B,A} = \frac{1}{2} (\text{Id} + R_A R_B).$$

Let $x \in X$ and note that $\text{Fix } T_{A,B} = \text{Fix } R_B R_A$ and $\text{Fix } T_{B,A} = \text{Fix } R_A R_B$.

▶ **Step 1:** $R_A$ maps $\text{Fix } T_{A,B}$ into $\text{Fix } T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x \Rightarrow R_A x = R_A R_B R_A x$.
Proof

**Goal:** To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A) \quad \text{and} \quad T_{B,A} = \frac{1}{2} (\text{Id} + R_A R_B).$$

Let $x \in X$ and note that $\text{Fix } T_{A,B} = \text{Fix } R_B R_A$ and $\text{Fix } T_{B,A} = \text{Fix } R_A R_B$.

▸ **STEP 1:** $R_A$ maps $\text{Fix } T_{A,B}$ into $\text{Fix } T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x \Rightarrow R_A x = R_A R_B R_A x \iff R_A x \in \text{Fix } R_A R_B = \text{Fix } T_{B,A}$. 


Proof

**Goal:** To show that $R_A$ is an isometric bijection from Fix $T_{A,B}$ to Fix $T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A) \quad \text{and} \quad T_{B,A} = \frac{1}{2} (\text{Id} + R_A R_B).$$

Let $x \in X$ and note that Fix $T_{A,B} = \text{Fix } R_B R_A$ and Fix $T_{B,A} = \text{Fix } R_A R_B$.

- **Step 1:** $R_A$ maps Fix $T_{A,B}$ into Fix $T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x \Rightarrow R_A x = R_A R_B R_A x \iff R_A x \in \text{Fix } R_A R_B = \text{Fix } T_{B,A}$. By interchanging $A$ and $B$ one sees that $R_B$ maps Fix $T_{B,A}$ into Fix $T_{A,B}$. 
  
- **Step 3:** Since $(\forall x \in \text{Fix } T_{A,B}) R_B R_A x = x$, this proves that $R_A$ is a bijection from Fix $T_{A,B}$ to Fix $T_{B,A}$ with the desired inverse.
Proof

**Goal:** To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A) \quad \text{and} \quad T_{B,A} = \frac{1}{2}(\text{Id} + R_A R_B).$$

Let $x \in X$ and note that $\text{Fix } T_{A,B} = \text{Fix } R_B R_A$ and $\text{Fix } T_{B,A} = \text{Fix } R_A R_B$.

- **Step 1:** $R_A$ maps $\text{Fix } T_{A,B}$ into $\text{Fix } T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x \Rightarrow R_A x = R_A R_B R_A x \iff R_A x \in \text{Fix } R_A R_B = \text{Fix } T_{B,A}$. By interchanging $A$ and $B$ one sees that $R_B$ maps $\text{Fix } T_{B,A}$ into $\text{Fix } T_{A,B}$.

- **Step 2:** $R_A$ maps $\text{Fix } T_{A,B}$ onto $\text{Fix } T_{B,A}$. The proof is similar in “flavour” to the proof of Step 1.
Proof

**Goal:** To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2}(\text{Id} + R_BR_A) \quad \text{and} \quad T_{B,A} = \frac{1}{2}(\text{Id} + R_AR_B).$$

Let $x \in X$ and note that $\text{Fix } T_{A,B} = \text{Fix } R_BR_A$ and $\text{Fix } T_{B,A} = \text{Fix } R_AR_B$.

- **Step 1:** $R_A$ maps $\text{Fix } T_{A,B}$ into $\text{Fix } T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_BR_Ax \Rightarrow R_Ax = R_AR_BR_Ax \iff R_Ax \in \text{Fix } R_AR_B = \text{Fix } T_{B,A}$. By interchanging $A$ and $B$ one sees that $R_B$ maps $\text{Fix } T_{B,A}$ into $\text{Fix } T_{A,B}$.

- **Step 2:** $R_A$ maps $\text{Fix } T_{A,B}$ onto $\text{Fix } T_{B,A}$. The proof is similar in “flavour” to the proof of Step 1.

- **Step 3:** Since $(\forall x \in \text{Fix } T_{A,B}) \ R_BR_Ax = x$, this proves that $R_A$ is a bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$ with the desired inverse.
Proof

**Goal:** To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A) \quad \text{and} \quad T_{B,A} = \frac{1}{2}(\text{Id} + R_A R_B).$$

Let $x \in X$ and note that $\text{Fix } T_{A,B} = \text{Fix } R_B R_A$ and $\text{Fix } T_{B,A} = \text{Fix } R_A R_B$.

- **Step 1:** $R_A$ maps $\text{Fix } T_{A,B}$ into $\text{Fix } T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x \Rightarrow R_A x = R_A R_B R_A x \iff R_A x \in \text{Fix } R_A R_B = \text{Fix } T_{B,A}$. By interchanging $A$ and $B$ one sees that $R_B$ maps $\text{Fix } T_{B,A}$ into $\text{Fix } T_{A,B}$.

- **Step 2:** $R_A$ maps $\text{Fix } T_{A,B}$ onto $\text{Fix } T_{B,A}$. The proof is similar in “flavour” to the proof of Step 1.

- **Step 3:** Since $(\forall x \in \text{Fix } T_{A,B}) R_B R_A x = x$, this proves that $R_A$ is a bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$ with the desired inverse.

- **Step 4:** $R_A : \text{Fix } T_{A,B} \to \text{Fix } T_{B,A}$ is an isometry.
Proof

Goal: To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A) \text{ and } T_{B,A} = \frac{1}{2} (\text{Id} + R_A R_B).$$

Let $x \in X$ and note that $\text{Fix } T_{A,B} = \text{Fix } R_B R_A$ and $\text{Fix } T_{B,A} = \text{Fix } R_A R_B$.

▶ **Step 1:** $R_A$ maps $\text{Fix } T_{A,B}$ into $\text{Fix } T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x \Rightarrow R_A x = R_A R_B R_A x \iff R_A x \in \text{Fix } R_A R_B = \text{Fix } T_{B,A}$. By interchanging $A$ and $B$ one sees that $R_B$ maps $\text{Fix } T_{B,A}$ into $\text{Fix } T_{A,B}$.

▶ **Step 2:** $R_A$ maps $\text{Fix } T_{A,B}$ onto $\text{Fix } T_{B,A}$. The proof is similar in “flavour” to the proof of Step 1.

▶ **Step 3:** Since $(\forall x \in \text{Fix } T_{A,B}) \ R_B R_A x = x$, this proves that $R_A$ is a bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$ with the desired inverse.

▶ **Step 4:** $R_A : \text{Fix } T_{A,B} \to \text{Fix } T_{B,A}$ is an isometry. Indeed, note that $(\forall x \in \text{Fix } T_{A,B}) \ (\forall y \in \text{Fix } T_{A,B})$ we have

$$\|x - y\| = \| R_B R_A x - R_B R_A y \|$$
Proof

Goal: To show that \( R_A \) is an isometric bijection from Fix \( T_{A,B} \) to Fix \( T_{B,A} \), with isometric inverse \( R_B \). Recall that

\[
T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A) \quad \text{and} \quad T_{B,A} = \frac{1}{2} (\text{Id} + R_A R_B).
\]

Let \( x \in X \) and note that Fix \( T_{A,B} = \text{Fix} R_B R_A \) and Fix \( T_{B,A} = \text{Fix} R_A R_B \).

- **Step 1:** \( R_A \) maps Fix \( T_{A,B} \) into Fix \( T_{B,A} \). Indeed, \( x \in \text{Fix} \ T_{A,B} \iff x = R_B R_A x \Rightarrow R_A x = R_A R_B R_A x \iff R_A x \in \text{Fix} R_A R_B = \text{Fix} T_{B,A} \). By interchanging \( A \) and \( B \) one sees that \( R_B \) maps Fix \( T_{B,A} \) into Fix \( T_{A,B} \).

- **Step 2:** \( R_A \) maps Fix \( T_{A,B} \) onto Fix \( T_{B,A} \). The proof is similar in “flavour” to the proof of Step 1.

- **Step 3:** Since \( (\forall x \in \text{Fix} T_{A,B}) \ R_B R_A x = x \), this proves that \( R_A \) is a bijection from Fix \( T_{A,B} \) to Fix \( T_{B,A} \) with the desired inverse.

- **Step 4:** \( R_A : \text{Fix} T_{A,B} \rightarrow \text{Fix} T_{B,A} \) is an isometry. Indeed, note that \( (\forall x \in \text{Fix} T_{A,B}) \ (\forall y \in \text{Fix} T_{A,B}) \) we have

\[
\| x - y \| = \| R_B R_A x - R_B R_A y \| \leq \| R_A x - R_A y \|
\]
Proof

**Goal:** To show that $R_A$ is an isometric bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$, with isometric inverse $R_B$. Recall that

$$T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A) \quad \text{and} \quad T_{B,A} = \frac{1}{2}(\text{Id} + R_A R_B).$$

Let $x \in X$ and note that $\text{Fix } T_{A,B} = \text{Fix } R_B R_A$ and $\text{Fix } T_{B,A} = \text{Fix } R_A R_B$.

- **Step 1:** $R_A$ maps $\text{Fix } T_{A,B}$ into $\text{Fix } T_{B,A}$. Indeed, $x \in \text{Fix } T_{A,B} \iff x = R_B R_A x \implies R_A x = R_A R_B R_A x \iff R_A x \in \text{Fix } R_A R_B = \text{Fix } T_{B,A}$. By interchanging $A$ and $B$ one sees that $R_B$ maps $\text{Fix } T_{B,A}$ into $\text{Fix } T_{A,B}$.

- **Step 2:** $R_A$ maps $\text{Fix } T_{A,B}$ onto $\text{Fix } T_{B,A}$. The proof is similar in “flavour” to the proof of Step 1.

- **Step 3:** Since $(\forall x \in \text{Fix } T_{A,B}) \quad R_B R_A x = x$, this proves that $R_A$ is a bijection from $\text{Fix } T_{A,B}$ to $\text{Fix } T_{B,A}$ with the desired inverse.

- **Step 4:** $R_A$: $\text{Fix } T_{A,B} \to \text{Fix } T_{B,A}$ is an isometry. Indeed, note that $(\forall x \in \text{Fix } T_{A,B})$ $(\forall y \in \text{Fix } T_{A,B})$ we have

$$\|x - y\| = \|R_B R_A x - R_B R_A y\| \leq \|R_A x - R_A y\| \leq \|x - y\|. $$
When $A$ is normal cone of a closed affine subspace

**Theorem (Bauschke–M ’16)**

Suppose that $U$ is a closed affine subspace and that $A = N_U$. Then $(\forall n \in \mathbb{N})$ we have

$$T_{B,A}^n = R_A T_{A,B}^n R_A \quad \text{and} \quad T_{A,B}^n = R_A T_{B,A}^n R_A.$$

---

- $T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A)$. 
When $A$ is normal cone of a closed affine subspace

Theorem (Bauschke–M ’16)

Suppose that $U$ is a closed affine subspace and that $A = N_U$. Then $(\forall n \in \mathbb{N})$ we have

$$T_{B,A}^n = R_A T_{A,B}^n R_A \quad \text{and} \quad T_{A,B}^n = R_A T_{B,A}^n R_A.$$ 

Proof.

► **Step 1:** $R_A^2 = \text{Id}$ (details omitted).

• $T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A)$. 

When $A$ is normal cone of a closed affine subspace

Theorem (Bauschke–M ’16)

Suppose that $U$ is a closed affine subspace and that $A = N_U$. Then $(\forall n \in \mathbb{N})$ we have

$$T_{B,A}^n = R_A T_{A,B}^n R_A \quad \text{and} \quad T_{A,B}^n = R_A T_{B,A}^n R_A.$$ 

Proof.

- **Step 1:** $R_A^2 = \text{Id}$ (details omitted).
- **Step 2:** The proof follows by induction. We verify the base case.

$\bullet$ $T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A)$. 
When $A$ is normal cone of a closed affine subspace

Theorem (Bauschke–M ’16)

Suppose that $U$ is a closed affine subspace and that $A = N_U$. Then $(\forall n \in \mathbb{N})$ we have

$$T_{B,A}^n = R_A T_{A,B}^n R_A \quad \text{and} \quad T_{A,B}^n = R_A T_{B,A}^n R_A.$$

Proof.

- **Step 1:** $R_A^2 = \text{Id}$ (details omitted).
- **Step 2:** The proof follows by induction. We verify the base case.

$$R_A T_{A,B} - T_{B,A} R_A = 2J_A T_{A,B} - J_A - J_A R_B R_A \quad \text{(needs $A$ and $B$ max. mono.)}$$

- $T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A)$. 
When $A$ is normal cone of a closed affine subspace

Theorem (Bauschke–M ’16)

Suppose that $U$ is a closed affine subspace and that $A = N_U$. Then $(\forall n \in \mathbb{N})$ we have

$$T_{B,A}^n = R_A T_{A,B}^n R_A \quad \text{and} \quad T_{A,B}^n = R_A T_{B,A}^n R_A.$$ 

Proof.

- **Step 1:** $R_A^2 = \text{Id}$ (details omitted).
- **Step 2:** The proof follows by induction. We verify the base case.

$$R_A T_{A,B} - T_{B,A} R_A = 2J_A T_{A,B} - J_A - J_A R_B R_A \quad \text{(needs } A \text{ and } B \text{ max. mono.)}$$

$$= J_A (2T_{A,B} - \text{Id}) - J_A R_B R_A \quad (J_A \text{ is affine})$$

- $T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A)$. 


When \( A \) is normal cone of a closed affine subspace

**Theorem** (Bauschke–M ’16)

Suppose that \( U \) is a closed affine subspace and that \( A = N_U \). Then (\( \forall n \in \mathbb{N} \)) we have

\[
T^n_{B,A} = R_A T^n_{A,B} R_A \quad \text{and} \quad T^n_{A,B} = R_A T^n_{B,A} R_A.
\]

**Proof.**

- **Step 1:** \( R_A^2 = \text{Id} \) (details omitted).
- **Step 2:** The proof follows by induction. We verify the base case.

\[
R_A T_{A,B} - T_{B,A} R_A = 2J_A T_{A,B} - J_A - J_A R_B R_A \quad \text{(needs } A \text{ and } B \text{ max. mono.)}
\]
\[
= J_A (2T_{A,B} - \text{Id}) - J_A R_B R_A \quad (J_A \text{ is affine})
\]
\[
= J_A (2(\frac{1}{2}(\text{Id} + R_B R_A)) - \text{Id}) - J_A R_B R_A
\]

- \( T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A) \).
When $A$ is normal cone of a closed affine subspace

Theorem (Bauschke–M ’16)

Suppose that $U$ is a closed affine subspace and that $A = N_U$. Then $(\forall n \in \mathbb{N})$ we have

$$T^n_{B,A} = R_A T^n_{A,B} R_A \quad \text{and} \quad T^n_{A,B} = R_A T^n_{B,A} R_A.$$ 

Proof.

- **Step 1:** $R^2_A = \text{Id}$ (details omitted).
- **Step 2:** The proof follows by induction. We verify the base case.

$$R_A T_{A,B} - T_{B,A} R_A = 2J_A T_{A,B} - J_A - J_A R_B R_A \quad \text{(needs $A$ and $B$ max. mono.)}$$

$$= J_A (2T_{A,B} - \text{Id}) - J_A R_B R_A \quad (J_A \text{ is affine})$$

$$= J_A (2(\frac{1}{2}(\text{Id} + R_B R_A)) - \text{Id}) - J_A R_B R_A$$

$$= J_A R_B R_A - J_A R_B R_A = 0.$$
When $A$ is normal cone of a closed affine subspace

Theorem (Bauschke–M ’16)

Suppose that $U$ is a closed affine subspace and that $A = N_U$. Then $(\forall n \in \mathbb{N})$ we have

$$T_{B,A}^n = R_A T_{A,B}^n R_A \quad \text{and} \quad T_{A,B}^n = R_A T_{B,A}^n R_A.$$

Proof.

▶ **Step 1:** $R_A^2 = \text{Id}$ (details omitted).

▶ **Step 2:** The proof follows by induction. We verify the base case.

$$R_A T_{A,B} - T_{B,A} R_A = 2J_A T_{A,B} - J_A - J_AR_B R_A \quad \text{(needs $A$ and $B$ max. mono.)}$$

$$= J_A(2T_{A,B} - \text{Id}) - J_AR_B R_A \quad \text{($J_A$ is affine)}$$

$$= J_A(2\left(\frac{1}{2}(\text{Id} + R_B R_A)\right) - \text{Id}) - J_AR_B R_A$$

$$= J_AR_B R_A - J_AR_B R_A = 0.$$

▶ **Step 3:** Now apply $R_A$ and use **Step 1**.

- $T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A)$.
Example: Convex feasibility

**LEFT:** Two closed convex sets in $\mathbb{R}^2$, $L$ is a linear subspace (red line) and $B$ (the ball). Shown are also the first few terms of the sequences $(T_{L,B}^n R_L x_0)_{n \in \mathbb{N}}$ (cyan points) and $(T_{B,L}^n x_0)_{n \in \mathbb{N}}$ (red points).
Example: Convex feasibility

**Left:** Two closed convex sets in $\mathbb{R}^2$, $L$ is a linear subspace (red line) and $B$ (the ball). Shown are also the first few terms of the sequences $(T_{L,B}^n R_L x_0)_{n \in \mathbb{N}}$ (cyan points) and $(T_{B,L}^n x_0)_{n \in \mathbb{N}}$ (red points). **Right:** Two closed convex sets in $\mathbb{R}^2$, $E$ (the ellipse) and $B$ (the ball). Shown are also the first few terms of the sequences $(T_{E,B}^n R_E x_0)_{n \in \mathbb{N}}$ (cyan points) and $(T_{B,E}^n x_0)_{n \in \mathbb{N}}$ (red points).
Applications

Application to the parallel splitting algorithm: The results are of interest when the Douglas–Rachford algorithm is applied to find the zero of the sum of $m$ operators, $m > 2$. 

Recently Bauschke and Dao prove the finite convergence of the Douglas–Rachford algorithm for the case when $A = N_U$, $B = N_V$, $U$ is a closed affine subspace and $V$ is a polyhedral set such that Slater’s condition $U \cap \text{int} V \neq \emptyset$ holds.
Applications

- Application to the parallel splitting algorithm: The results are of interest when the Douglas–Rachford algorithm is applied to find the zero of the sum of $m$ operators, $m > 2$. In this case one can apply Douglas–Rachford in the product space where one operator is the normal cone operator of the diagonal subspace (extended coupling space).
Applications

- Application to the parallel splitting algorithm: The results are of interest when the Douglas–Rachford algorithm is applied to find the zero of the sum of $m$ operators, $m \geq 2$. In this case one can apply Douglas–Rachford in the product space where one operator is the normal cone operator of the diagonal subspace (extended coupling space).

- The result remains true if $J_B$ is replaced by any operator $Q_B : X \to X$ (and $R_B$ is replaced by $2Q_B - \text{Id}$, of course).
Applications

- Application to the parallel splitting algorithm: The results are of interest when the Douglas–Rachford algorithm is applied to find the zero of the sum of $m$ operators, $m > 2$. In this case one can apply Douglas–Rachford in the product space where one operator is the normal cone operator of the diagonal subspace (extended coupling space).

- The result remains true if $J_B$ is replaced by any operator $Q_B : X \to X$ (and $R_B$ is replaced by $2Q_B - \text{Id}$, of course). This is interesting because in nonconvex feasibility settings $Q_B$ is chosen to be a selection of the (set-valued) projector onto a set $V$ that is not convex.
Applications

- Application to the parallel splitting algorithm: The results are of interest when the Douglas–Rachford algorithm is applied to find the zero of the sum of $m$ operators, $m > 2$. In this case one can apply Douglas–Rachford in the product space where one operator is the normal cone operator of the diagonal subspace (extended coupling space).

- The result remains true if $J_B$ is replaced by any operator $Q_B : X \to X$ (and $R_B$ is replaced by $2Q_B - \text{Id}$, of course). This is interesting because in nonconvex feasibility settings $Q_B$ is chosen to be a selection of the (set-valued) projector onto a set $V$ that is not convex.

- Recently Bauschke and Dao prove the finite convergence of the Douglas–Rachford algorithm for the case when $A = N_U$, $B = N_V$, $U$ is a closed affine subspace and $V$ is a polyhedral set such that Slater’s condition $U \cap \text{int } V \neq \emptyset$ holds.
Applications

- Application to the parallel splitting algorithm: The results are of interest when the Douglas–Rachford algorithm is applied to find the zero of the sum of $m$ operators, $m > 2$. In this case one can apply Douglas–Rachford in the product space where one operator is the normal cone operator of the diagonal subspace (extended coupling space).

- The result remains true if $J_B$ is replaced by any operator $Q_B : X \to X$ (and $R_B$ is replaced by $2Q_B - \text{Id}$, of course). This is interesting because in nonconvex feasibility settings $Q_B$ is chosen to be a selection of the (set-valued) projector onto a set $V$ that is not convex.

- Recently Bauschke and Dao prove the finite convergence of the Douglas–Rachford algorithm for the case when $A = N_U$, $B = N_V$, $U$ is a closed affine subspace and $V$ is a polyhedral set such that Slater’s condition $U \cap \text{int } V \neq \emptyset$ holds. Observe that Slater’s condition is nonsymmetric and therefore, in view of our results, we obtain a novel sufficient condition for finite convergence.
PART II: On linear convergence of Douglas–Rachford method

The results in this part appear in

Recall that we are interested in solving:

Find $x \in X$ such that $x \in \text{zer}(A + B)$. 
Linear rates of convergence: Known results

Recall in Giselsson’s talk we saw:

### DR contraction factors

<table>
<thead>
<tr>
<th>#</th>
<th>Properties for $A$</th>
<th>Properties for $B$</th>
<th>Reference</th>
<th>Sharp</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>O1</td>
<td>$\partial f$, $f$: str. cvx &amp; smooth</td>
<td>$\partial g$</td>
<td>[1,2]</td>
<td>✔</td>
<td></td>
</tr>
<tr>
<td>O2</td>
<td>$\partial f$, $f$: str. cvx</td>
<td>$\partial g$, $g$: smooth</td>
<td>[3]</td>
<td>✗</td>
<td>1.</td>
</tr>
<tr>
<td>M1</td>
<td>str. mono. &amp; cocoercive</td>
<td>-</td>
<td>[3]</td>
<td>✔</td>
<td></td>
</tr>
<tr>
<td>M3</td>
<td>str. mono.</td>
<td>cocoercive</td>
<td>[3]</td>
<td>✗</td>
<td></td>
</tr>
</tbody>
</table>

1. sharp rates for some parameter choices in [3]
2. Lions and Mercier [5] provided conservative rate in this setting
3. sharp rate when $B$ is in addition linear in [4]

---

The case of $A$ Lipschitz continuous and $B$ strongly monotone

Example
Suppose that $X = \mathbb{R}^2$ and define

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = N\{0\}.$$
The case of $A$ Lipschitz continuous and $B$ strongly monotone

Example
Suppose that $X = \mathbb{R}^2$ and define

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = N\{0\}.$$  

Then

$\triangleright$ $A$ is monotone and nonexpansive (hence 1-Lipschitz continuous).
The case of $A$ Lipschitz continuous and $B$ strongly monotone

Example
Suppose that $X = \mathbb{R}^2$ and define

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \mathcal{N}\{0\}.$$ 

Then
- $A$ is monotone and nonexpansive (hence 1-Lipschitz continuous).
- $B$ is maximally monotone and $\mu$-strongly monotone for every $\mu > 0$. 
The case of $A$ Lipschitz continuous and $B$ strongly monotone

Example
Suppose that $X = \mathbb{R}^2$ and define

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = N\{0\}.$$  

Then

- $A$ is monotone and nonexpansive (hence 1-Lipschitz continuous).
- $B$ is maximally monotone and $\mu$-strongly monotone for every $\mu > 0$.
- $R_A = -A$ and $R_B = -\text{Id}$, i.e., $R_A$ is “barely” nonexpansive.
The case of $A$ Lipschitz continuous and $B$ strongly monotone

**Example**

Suppose that $X = \mathbb{R}^2$ and define

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = N\{0\}.$$  

Then
- $A$ is monotone and nonexpansive (hence 1-Lipschitz continuous).
- $B$ is maximally monotone and $\mu$-strongly monotone for every $\mu > 0$.
- $R_A = -A$ and $R_B = -\text{Id}$, i.e., $R_A$ is “barely” nonexpansive.
- Hence, $-R_B R_A = -A$ which is not averaged.
Ingredient 1: Reflected resolvents of Lipschitz continuous operators are hypomonotone!

Lemma (M–Vandenberghe '18)

Suppose that $A: X \rightarrow X$ is monotone and $\beta$-Lipschitz continuous with $\beta > 0$. Let $(x, y) \in X \times X$. Then

$$\langle x - y, RAx - RAy \rangle \geq -\lambda \|x - y\|^2$$

where $\lambda = \left(1 - \frac{1}{(1+\beta)^2} - \frac{1}{1+\beta^2}\right) \in \mathbb{R}$. 

Ingredient 1: Reflected resolvents of Lipschitz continuous operators are hypomonotone!

Lemma (M–Vandenberghe ’18)

Suppose that $A: X \rightarrow X$ is monotone and $\beta$-Lipschitz continuous with $\beta > 0$. Let $(x, y) \in X \times X$. Then

$$\langle x - y, R_A x - R_A y \rangle \geq -\lambda \|x - y\|^2$$

where $\lambda = \left(1 - \frac{1}{(1+\beta)^2} - \frac{1}{1+\beta^2}\right) \in ]-1, 1[.$

If, in addition, $A: X \rightarrow X$ is linear and skew, i.e., $A = -A^*$, then the inequality is satisfied with the sharp bound

$$\lambda = 1 - \frac{2}{1+\beta^2}.$$
Proposition (M–Vandenberghe ’18)

Let $R : X \to X$ be such that $-R$ is $\alpha$-averaged, with $\alpha \in [0, 1]$. Let $M : X \to X$ be nonexpansive such that $(\forall (x, y) \in X \times X)$

$$\langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2, \text{ with } \lambda \in ]-1, 1[.$$

Define

$$T = \frac{1}{2} (\text{Id} + RM)$$

Then $T$ is Lipschitz continuous with constant

$$\frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha} \right) < 1.$$ 

Hence, $T$ is a Banach contraction and Fix $T$ is a singleton.
Proof

**Step 1:** $\text{Id} + (\alpha - 1)M$ is Lipschitz continuous with constant
\[
\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2.
\]

---

**Proposition**

- Let $R: X \rightarrow X$ be such that $-R$ is $\alpha$-averaged, with $\alpha \in [0, 1[$.
- Let $M: X \rightarrow X$ be nonexpansive such that $\langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2$, with $\lambda \in ]-1, 1[$.
- Define $T = \frac{1}{2} (\text{Id} + RM)$.

Then $T$ is Lipschitz continuous with constant $\frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha} \right) < 1$. Hence, $T$ is a Banach contraction and Fix $T$ is a singleton.
Proof

**Step 1:** $\text{Id} + (\alpha - 1)M$ is Lipschitz continuous with constant 
$$\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2.$$ 
Indeed, set $S = \text{Id} + (\alpha - 1)M$ and let $(x, y) \in X \times X$.

---

**Proposition**

- Let $R : X \to X$ be such that $-R$ is $\alpha$-averaged, with $\alpha \in [0, 1]$.
- Let $M : X \to X$ be nonexpansive such that $\langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2$, with $\lambda \in ]-1, 1[$.
- Define $T = \frac{1}{2}(\text{Id} + RM)$.

Then $T$ is Lipschitz continuous with constant $\frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha} \right) < 1$. Hence, $T$ is a Banach contraction and $\text{Fix } T$ is a singleton.
Proof

**Step 1:** \( \text{Id} + (\alpha - 1)M \) is Lipschitz continuous with constant
\[
\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2.
\]
Indeed, set \( S = \text{Id} + (\alpha - 1)M \) and let \((x, y) \in X \times X\). Then
\[
\|Sx - Sy\|^2 = \|x - y\|^2 + (1 - \alpha)^2 \|Mx - My\|^2 - 2(1 - \alpha) \langle x - y, Mx - My \rangle
\]

---

**Proposition**

- Let \( R: X \to X \) be such that \(-R\) is \(\alpha\)-averaged, with \(\alpha \in [0, 1]\).
- Let \( M: X \to X \) be nonexpansive such that \( \langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2 \), with \(\lambda \in ]-1, 1[\).
- Define \( T = \frac{1}{2} (\text{Id} + RM) \).

Then \( T \) is Lipschitz continuous with constant \( \frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha} \right) < 1 \). Hence, \( T \) is a Banach contraction and \( \text{Fix} T \) is a singleton.
Proof

**Step 1:** $\text{Id} + (\alpha - 1)M$ is Lipschitz continuous with constant
\[
\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2.
\]
Indeed, set $S = \text{Id} + (\alpha - 1)M$ and let $(x, y) \in X \times X$. Then
\[
\|Sx - Sy\|^2 = \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 - 2(1 - \alpha)\langle x - y, Mx - My \rangle
\]
\[
\leq \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 + 2\lambda(1 - \alpha)\|x - y\|^2
\]

---

**Proposition**
- Let $R: X \to X$ be such that $-R$ is $\alpha$-averaged, with $\alpha \in [0, 1]$.
- Let $M: X \to X$ be nonexpansive such that $\langle x - y, Mx - My \rangle \geq -\lambda\|x - y\|^2$, with $\lambda \in [-1, 1]$.
- Define $T = \frac{1}{2}(\text{Id} + RM)$.

Then $T$ is Lipschitz continuous with constant $\frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} + \alpha \right) < 1$. Hence, $T$ is a Banach contraction and $\text{Fix } T$ is a singleton.
Proof

**Step 1:** \( \text{Id} + (\alpha - 1)M \) is Lipschitz continuous with constant 
\[
\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2.
\]
Indeed, set \( S = \text{Id} + (\alpha - 1)M \) and let \((x, y) \in X \times X\). Then
\[
\|Sx - Sy\|^2 = \|x - y\|^2 + (1 - \alpha)^2 \|Mx - My\|^2 - 2(1 - \alpha) \langle x - y, Mx - My \rangle \\
\leq \|x - y\|^2 + (1 - \alpha)^2 \|Mx - My\|^2 + 2\lambda(1 - \alpha) \|x - y\|^2 \\
\leq (1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)) \|x - y\|^2.
\]

---

**Proposition**
- Let \( R: X \to X \) be such that \(-R\) is \(\alpha\)-averaged, with \(\alpha \in [0, 1[\).  
- Let \( M: X \to X \) be nonexpansive such that \( \langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2 \), with \(\lambda \in ]-1, 1[\).  
- Define \( T = \frac{1}{2} (\text{Id} + RM) \).

Then \( T \) is Lipschitz continuous with constant  
\[
\frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha} \right) < 1.
\]
Hence, \( T \) is a Banach contraction and \( \text{Fix} \ T \) is a singleton.
Proof

**Step 1:** \( \text{Id} + (\alpha - 1)M \) is Lipschitz continuous with constant
\[
\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2.
\]
Indeed, set \( S = \text{Id} + (\alpha - 1)M \) and let \((x, y) \in X \times X\). Then
\[
\|Sx - Sy\|^2 = \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 - 2(1 - \alpha)\langle x - y, Mx - My \rangle
\leq \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 + 2\lambda(1 - \alpha)\|x - y\|^2
\leq (1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha))\|x - y\|^2.
\]

**Step 2:** \(-R\) is \(\alpha\)-averaged \(\Rightarrow R = (\alpha - 1)\text{Id} + \alpha N\) for some nonexpansive \(N\).

---

**Proposition**
- Let \( R: X \to X \) be such that \(-R\) is \(\alpha\)-averaged, with \(\alpha \in [0, 1]\).
- Let \( M: X \to X \) be nonexpansive such that \(\langle x - y, Mx - My \rangle \geq -\lambda\|x - y\|^2\), with \(\lambda \in ]-1, 1[\).
- Define \( T = \frac{1}{2}(\text{Id} + RM) \).

Then \( T \) is Lipschitz continuous with constant \(\frac{1}{2} \left(\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha}\right) < 1\). Hence, \( T \) is a Banach contraction and \( \text{Fix } T \) is a singleton.
Proof

STEP 1: \( \text{Id} + (\alpha - 1)M \) is Lipschitz continuous with constant
\[
\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2.
\]
Indeed, set \( S = \text{Id} + (\alpha - 1)M \) and let \((x, y) \in X \times X\). Then
\[
\|Sx - Sy\|^2 = \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 - 2(1 - \alpha)\langle x - y, Mx - My \rangle
\leq \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 + 2\lambda(1 - \alpha)\|x - y\|^2
\leq (1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha))\|x - y\|^2.
\]

STEP 2: \(-R\) is \(\alpha\)-averaged \(\Rightarrow\) \(R = (\alpha - 1)\text{Id} + \alpha N\) for some nonexpansive \(N\). Hence \(T = \frac{1}{2}(\text{Id} + (\alpha - 1)M + \alpha NM)\). Therefore

---

Proposition

- Let \(R: X \to X\) be such that \(-R\) is \(\alpha\)-averaged, with \(\alpha \in [0, 1]\).
- Let \(M: X \to X\) be nonexpansive such that \(\langle x - y, Mx - My \rangle \geq -\lambda\|x - y\|^2\), with \(\lambda \in ]-1, 1[\).
- Define \(T = \frac{1}{2}(\text{Id} + RM)\).

Then \(T\) is Lipschitz continuous with constant \(\frac{1}{2} \left(\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} + \alpha\right) < 1\). Hence, \(T\) is a Banach contraction and \(\text{Fix } T\) is a singleton.
**Proof**

**Step 1:** $\text{Id} + (\alpha - 1)M$ is Lipschitz continuous with constant $
\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2.$

Indeed, set $S = \text{Id} + (\alpha - 1)M$ and let $(x, y) \in X \times X$. Then

$$
\|Sx - Sy\|^2 = \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 - 2(1 - \alpha)\langle x - y, Mx - My \rangle
\leq \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 + 2\lambda(1 - \alpha)\|x - y\|^2
\leq (1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha))\|x - y\|^2.
$$

**Step 2:** $-R$ is $\alpha$-averaged $\Rightarrow R = (\alpha - 1)\text{Id} + \alpha N$ for some nonexpansive $N$. Hence $T = \frac{1}{2}(\text{Id} + (\alpha - 1)M + \alpha NM)$. Therefore

$$
\|Tx - Ty\| \leq \frac{1}{2} \left( \|(\text{Id} + (\alpha - 1)M)x - (\text{Id} + (\alpha - 1)M)y\| + \alpha\|NMx - NMy\| \right).
$$

---

**Proposition**

- Let $R: X \rightarrow X$ be such that $-R$ is $\alpha$-averaged, with $\alpha \in [0, 1]$.
- Let $M: X \rightarrow X$ be nonexpansive such that $\langle x - y, Mx - My \rangle \geq -\lambda\|x - y\|^2$, with $\lambda \in ]-1, 1[$.
- Define $T = \frac{1}{2}(\text{Id} + RM)$.

Then $T$ is Lipschitz continuous with constant $\frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha} \right) < 1$. Hence, $T$ is a Banach contraction and $\text{Fix } T$ is a singleton.
Proof

**Step 1:** \( \text{Id} + (\alpha - 1)M \) is Lipschitz continuous with constant
\[ \sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2. \]
Indeed, set \( S = \text{Id} + (\alpha - 1)M \) and let \((x, y) \in X \times X\). Then
\[ \|Sx - Sy\|^2 = \|x - y\|^2 + (1 - \alpha)^2 \|Mx - My\|^2 - 2(1 - \alpha) \langle x - y, Mx - My \rangle \]
\[ < \|x - y\|^2 + (1 - \alpha)^2 \|Mx - My\|^2 + 2\lambda(1 - \alpha) \|x - y\|^2 \]
\[ \leq (1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)) \|x - y\|^2. \]

**Step 2:** \(-R\) is \(\alpha\)-averaged \(\Rightarrow\) \(R = (\alpha - 1)\text{Id} + \alpha N\) for some nonexpansive \(N\). Hence \(T = \frac{1}{2}(\text{Id} + (\alpha - 1)M + \alpha NM)\). Therefore
\[ \|Tx - Ty\| \leq \frac{1}{2} (\| (\text{Id} + (\alpha - 1)M)x - (\text{Id} + (\alpha - 1)M)y \| + \alpha \|NMx - NMy\|) \]
\[ \leq \frac{1}{2} (\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} + \alpha) \|x - y\| \leq \frac{1}{2} (2 - \alpha + \alpha) = 1 \]

---

**Proposition**

- Let \( R: X \rightarrow X \) be such that \(-R\) is \(\alpha\)-averaged, with \(\alpha \in [0, 1]\).
- Let \( M: X \rightarrow X \) be nonexpansive such that \(\langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2\), with \(\lambda \in ]-1, 1[\).
- Define \( T = \frac{1}{2}(\text{Id} + RM) \).

Then \( T \) is Lipschitz continuous with constant \( \frac{1}{2} (\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} + \alpha) < 1 \). Hence, \( T \) is a Banach contraction and \(\text{Fix } T\) is a singleton.
Application to Douglas–Rachford method

**Theorem**

Suppose that $A: X \to X$ is monotone and $\beta$-Lipschitz continuous with $\beta > 0$, and that $B: X \rightrightarrows X$ is maximally monotone and $\mu$-strongly monotone with $\mu > 0$. Let $x_0 \in X$, let $T = \frac{1}{2} \left( \text{Id} + R_B R_A \right)$. Then the following hold:

1. $(x_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}$ converges strongly to some $\bar{x} \in X$, with a linear rate $r$, where

   $$r = \frac{1}{2(1+\mu)} \left( \sqrt{2\mu^2 + 2\mu + 1 + 2 \left( 1 - \frac{1}{(1+\beta)^2} - \frac{1}{1+\beta^2} \right) \mu(1+\mu)+1} \right) < 1.$$

2. $(J_A x_n)_{n \in \mathbb{N}}$ converges strongly to $J_A \bar{x}$ with a linear rate $r$. 


Application to Douglas–Rachford method

Theorem
Suppose that $A: X \to X$ is monotone and $\beta$-Lipschitz continuous with $\beta > 0$, and that $B: X \rightrightarrows X$ is maximally monotone and $\mu$-strongly monotone with $\mu > 0$. Let $x_0 \in X$, let $T = \frac{1}{2}(\text{Id} + R_B R_A)$. Then the following hold:

1. $(x_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}$ converges strongly to some $\bar{x} \in X$, with a linear rate $r$, where

$$r = \frac{1}{2(1+\mu)} \left( \sqrt{2\mu^2 + 2\mu + 1} + 2 \left( 1 - \frac{1}{(1+\mu)^2} - \frac{1}{1+\beta^2} \right) \mu(1+\mu+1) \right) < 1.$$ 

2. $(J_A x_n)_{n \in \mathbb{N}}$ converges strongly to $J_A \bar{x}$ with a linear rate $r$.

Moreover, $\text{Fix} \ R_B R_A = \text{Fix} \ T = \{\bar{x}\}$, and $\text{zer}(A + B) = \{J_A \bar{x}\}$.
Application to Douglas–Rachford method

Theorem
Suppose that $A : X \to X$ is monotone and $\beta$-Lipschitz continuous with $\beta > 0$, and that $B : X \rightrightarrows X$ is maximally monotone and $\mu$-strongly monotone with $\mu > 0$. Let $x_0 \in X$, let $T = \frac{1}{2} \left( \text{Id} + R_B R_A \right)$. Then the following hold:

1. $(x_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}$ converges strongly to some $\bar{x} \in X$, with a linear rate $r$, where

   $$r = \frac{1}{2(1+\mu)} \left( \sqrt{2\mu^2 + 2\mu + 1 + 2 \left( 1 - \frac{1}{(1+\beta)^2} - \frac{1}{1+\beta^2} \right) \mu(1+\mu)+1} \right) < 1.$$

2. $(J_A x_n)_{n \in \mathbb{N}}$ converges strongly to $J_A \bar{x}$ with a linear rate $r$.

Moreover, $	ext{Fix } R_B R_A = \text{Fix } T = \{ \bar{x} \}$, and $\text{zer}(A + B) = \{ J_A \bar{x} \}$. If, in addition, $A$ is linear and skew, then (i) and (ii) hold with the sharp rate

$$r = \frac{1}{2(1+\mu)} \left( \sqrt{2\mu^2 + 2\mu + 1 + 2 \left( 1 - \frac{2}{1+\beta^2} \right) \mu(1+\mu)+1} \right) < 1.$$
Proof

▶ Using ingredient 1: \( A \) is \( \beta \)-Lipschitz continuous \( \Rightarrow \) \( R_A \) satisfies

\[
\langle R_Ax - R_Ay \rangle \geq -\lambda \|x - y\|^2, \quad \text{and} \quad \lambda = \lambda(\beta) \in ]-1, 1[.
\]

---

Proposition

▶ Let \( M: X \to X \) be nonexpansive such that \( \langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2, \) with \( \lambda \in ]-1, 1[. \)

▶ Let \( R: X \to X \) be such that \( -R \) is \( \alpha \)-averaged, with \( \alpha \in [0, 1[. \)

▶ Define \( T = \frac{1}{2} (\text{Id} + RM) \).

Then \( T \) is Lipschitz continuous with constant \( \frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2} + 2\lambda(1 - \alpha) + \alpha \right) < 1. \)

Hence, \( T \) is a Banach contraction and \( \text{Fix} \ T \) is a singleton.
Proof

▶ Using ingredient 1: $A$ is $\beta$-Lipschitz continuous $\Rightarrow R_A$ satisfies

$$\langle R_Ax - R_Ay \rangle \geq -\lambda \|x - y\|^2, \text{ and } \lambda = \lambda(\beta) \in ]-1,1[.\]
Proof

- **Using ingredient 1:** $A$ is $\beta$-Lipschitz continuous $\Rightarrow R_A$ satisfies

\[
\langle R_Ax - R_Ay \rangle \geq -\lambda \|x - y\|^2, \text{ and } \lambda = \lambda(\beta) \in ]-1, 1[.
\]

- **(Gisselson '17)** $B$ is $\mu$-strongly monotone $\Rightarrow -R_B$ is $\frac{1}{1+\mu}$-averaged.

---

**Proposition**

- Let $M: X \to X$ be nonexpansive such that $\langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2$, with $\lambda \in ]-1, 1[. \checkmark$

- Let $R: X \to X$ be such that $-R$ is $\alpha$-averaged, with $\alpha \in [0, 1[.$

- Define $T = \frac{1}{2} (\text{Id} + RM)$.

Then $T$ is Lipschitz continuous with constant $\frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2} + 2\lambda(1 - \alpha) + \alpha \right) < 1$.

Hence, $T$ is a Banach contraction and $\text{Fix } T$ is a singleton.
Proof

- Using ingredient 1: $A$ is $\beta$-Lipschitz continuous $\Rightarrow RA$ satisfies

\[
\langle RAx - RAFy \rangle \geq -\lambda \|x - y\|^2, \quad \text{and} \quad \lambda = \lambda(\beta) \in ]-1,1[.
\]

- (Gisselson '17) $B$ is $\mu$-strongly monotone $\Rightarrow -RB$ is $\frac{1}{1+\mu}$-averaged.

---

Proposition

- Let $M : X \rightarrow X$ be nonexpansive such that $\langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2$, with $\lambda \in ]-1,1[$. ✓
- Let $R : X \rightarrow X$ be such that $-R$ is $\alpha$-averaged, with $\alpha \in [0,1[$. ✓
- Define $T = \frac{1}{2} (\text{Id} + RM)$.

Then $T$ is Lipschitz continuous with constant $\frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2} + 2\lambda(1 - \alpha) + \alpha \right) < 1$. Hence, $T$ is a Banach contraction and $\text{Fix } T$ is a singleton.
Proof

- Using ingredient 1: \( A \) is \( \beta \)-Lipschitz continuous \( \Rightarrow R_A \) satisfies

\[
\langle R_Ax - R_Ay \rangle \geq -\lambda \|x - y\|^2, \quad \text{and} \quad \lambda = \lambda(\beta) \in ] - 1, 1 [.
\]

- (Gisselson ’17) \( B \) is \( \mu \)-strongly monotone \( \Rightarrow -R_B \) is \( \frac{1}{1+\mu} \)-averaged.

- Using ingredient 2:

\[
T = \frac{1}{2} (\text{Id} + R_B R_A)
\]

is a Banach contraction and the conclusion follows by applying the earlier Proposition with \((R, M)\) replaced by \((R_B, R_A)\).

Proposition

- Let \( M: X \to X \) be nonexpansive such that \( \langle x - y, Mx - My \rangle \geq -\lambda \|x - y\|^2, \) with \( \lambda \in ] - 1, 1 [\). ✓

- Let \( R: X \to X \) be such that \(-R\) is \( \alpha \)-averaged, with \( \alpha \in [0, 1] \). ✓

- Define \( T = \frac{1}{2} (\text{Id} + RM) \).

Then \( T \) is Lipschitz continuous with constant \( \frac{1}{2} \left( \sqrt{1 + (1 - \alpha)^2} + 2\lambda(1 - \alpha) + \alpha \right) < 1 \). Hence, \( T \) is a Banach contraction and \( \text{Fix } T \) is a singleton.
Sharpness of the contraction factor

Example
Let $\beta > 0$ and let $\mu > 0$. Suppose that $X = \mathbb{R}^2$,

\[
A = \beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \mu \text{Id} + \mathcal{N}\{0\} \times \mathbb{R}.
\]
Sharpness of the contraction factor

Example
Let $\beta > 0$ and let $\mu > 0$. Suppose that $X = \mathbb{R}^2$,

$$A = \beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \mu \text{Id} + \underbrace{N_{\{0\} \times \mathbb{R}}}_{\text{max. mono.}}.$$

Then $A$ is $\beta$-Lipschitz continuous and monotone, $B$ is $\mu$-strongly monotone and
Sharpness of the contraction factor

Example

Let $\beta > 0$ and let $\mu > 0$. Suppose that $X = \mathbb{R}^2$,

$$A = \beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \mu \text{Id} + \max\text{ mono.}$$

Then $A$ is $\beta$-Lipschitz continuous and monotone, $B$ is $\mu$-strongly monotone and

$$R_A = \begin{bmatrix} 2 \frac{1}{\beta^2 + 1} - 1 & - \frac{2\beta}{\beta^2 + 1} \\ \frac{2\beta}{\beta^2 + 1} & 2 \frac{1}{\beta^2 + 1} - 1 \end{bmatrix}, \quad R_B = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1-\mu}{1+\mu} \end{bmatrix}.$$
Sharpness of the contraction factor

Example
Let $\beta > 0$ and let $\mu > 0$. Suppose that $X = \mathbb{R}^2$,

$$A = \beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \mu \text{Id} + N_{\{0\} \times \mathbb{R}} \max. \text{ mono.}.$$  

Then $A$ is $\beta$-Lipschitz continuous and monotone, $B$ is $\mu$-strongly monotone and

$$R_A = \begin{bmatrix} \frac{2}{\beta^2+1} - 1 & -\frac{2\beta}{\beta^2+1} \\ \frac{2\beta}{\beta^2+1} & \frac{2}{\beta^2+1} - 1 \end{bmatrix}, \quad R_B = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1-\mu}{1+\mu} \end{bmatrix}.$$

Therefore,

$$T = \frac{1}{2} (\text{Id} + R_B R_A) = \frac{1}{\beta^2+1} \begin{bmatrix} \frac{\beta^2}{\beta(1-\mu)} & \frac{\beta}{1+\beta^2\mu} \\ \frac{\beta^2}{\beta(1-\mu)} & \frac{1-\beta^2\mu}{1+\mu} \end{bmatrix}.$$
Sharpness of the contraction factor

Example
Let $\beta > 0$ and let $\mu > 0$. Suppose that $X = \mathbb{R}^2$, 
\[ A = \beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \mu \text{Id} + \mathcal{N}_{\{0\} \times \mathbb{R}}. \]

Then $A$ is $\beta$-Lipschitz continuous and monotone, $B$ is $\mu$-strongly monotone and 
\[ RA = \begin{bmatrix} \frac{2}{\beta^2+1} - 1 & -\frac{2\beta}{\beta^2+1} \\ \frac{2\beta}{\beta^2+1} & \frac{2\beta^2}{\beta^2+1} - 1 \end{bmatrix}, \quad RB = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1-\mu}{1+\mu} \end{bmatrix}. \]

Therefore, 
\[ T = \frac{1}{2} (\text{Id} + R_B R_A) = \frac{1}{\beta^2+1} \begin{bmatrix} \beta^2 & \beta \\ \beta(1-\mu) & 1+\beta^2 \mu \end{bmatrix}, \]
\[ \|T\| = \frac{1}{2(1+\mu)} \left( \sqrt{2\mu^2+2\mu+1+2\left(1-\frac{2}{1+\beta^2}\right)\mu(1+\mu)+1} \right). \]
Application to primal-dual Douglas–Rachford splitting

- $L : X \to Y$ is nonzero and linear.
- $f : X \to ]-\infty, +\infty]$ is $\sigma$-strongly convex and closed,
- $g : X \to \mathbb{R}$ is convex and $\nabla g$ is $\beta$-Lipschitz continuous for some $\beta > 0$. 

Consider the monotone inclusion:

Find $(x, y) \in X \times Y$ such that $0 \in A(x, y) + B(x, y)$,

where

$A : X \times Y \to X \times Y : (x, y) \mapsto (L^* y, -L x)$, and

$B : X \times Y \Rightarrow X \times Y : (x, y) \mapsto \partial f(x) \times \partial g^*(y)$.

Then $A$ is Lipschitz continuous with the sharp constant $\|L\| = 0$.

$B$ is maximally monotone and $\mu$-strongly monotone and $\mu = \min\{\sigma, 1/\beta\}$.

The above inclusion arises in primal-dual optimality conditions of the primal problem $(P)$ and its Fenchel–Rockafellar dual $(D)$ given by:

$(P)$ minimize $x \in X$ $f(x) + g(Lx)$

$(D)$ minimize $y \in Y$ $f^*(-L^* y) + g^*(y)$,

under appropriate assumptions on $f$, $g$ and $L$. 

Application to primal-dual Douglas–Rachford splitting

- $L: X \to Y$ is nonzero and linear.
- $f: X \to \mathbb{R}$ is $\sigma$-strongly convex and closed,
- $g: X \to \mathbb{R}$ is convex and $\nabla g$ is $\beta$-Lipschitz continuous for some $\beta > 0$.
- Consider the monotone inclusion:

$$\text{Find } (x, y) \in X \times Y \text{ such that } 0 \in A(x, y) + B(x, y),$$

where $A: X \times Y \to X \times Y: (x, y) \mapsto (L^*y, -Lx)$, and $B: X \times Y \nRightarrow X \times Y: (x, y) \mapsto \partial f(x) \times \partial g^*(y)$.

- Then $A$ is Lipschitz continuous with the sharp constant $\|L\| \neq 0$.
- $B$ is maximally monotone and $\mu$-strongly monotone and $\mu = \min \{\sigma, 1/\beta\}$. 
Application to primal-dual Douglas–Rachford splitting

- \( L: X \to Y \) is nonzero and linear.
- \( f: X \to ]-\infty, +\infty[ \) is \( \sigma \)-strongly convex and closed,
- \( g: X \to \mathbb{R} \) is convex and \( \nabla g \) is \( \beta \)-Lipschitz continuous for some \( \beta > 0 \).
- Consider the monotone inclusion:

\[
\text{Find } (x, y) \in X \times Y \text{ such that } 0 \in A(x, y) + B(x, y),
\]

where \( A: X \times Y \to X \times Y: (x, y) \mapsto (L^* y, -Lx) \), and \( B: X \times Y \rightrightarrows X \times Y: (x, y) \mapsto \partial f(x) \times \partial g^*(y) \).

- Then \( A \) is Lipschitz continuous with the sharp constant \( \|L\| \neq 0 \).
- \( B \) is maximally monotone and \( \mu \)-strongly monotone and \( \mu = \min \{\sigma, 1/\beta\} \).
- The above inclusion arises in primal-dual optimality conditions of the primal problem (P) and its Fenchel–Rockafellar dual (D) given by:

\[
\text{(P)} \quad \min_{x \in X} f(x) + g(Lx)
\]

\[
\text{(D)} \quad \min_{y \in Y} f^*(-L^* y) + g^*(y),
\]

under appropriate assumptions on \( f, g \) and \( L \).
Douglas and Rachford

Lions and Mercier

P.L. Lions
Fields Medal (1994)

B. Mercier


THANK YOU!!