Problems to Explore
Rutgers Young Scholars Program
Summer 2017

PROBLEM 1: AUTOMOBILE TRIP

A man goes on a 100 mile automobile trip. For the first hour he averages 50 miles per hour. He decides that he wants to average 60 mile per hour for the whole trip.

What must his average speed be for the remainder of the trip?

PROBLEM 2: POWER OUTAGE

When the power returns after a power outage, an analog (traditional round) electric clock continues measuring time from the moment the power returns. On the other hand, a digital clock returns to 12:00 and measures the amount of time that elapses after that.

When a man leaves his house at 8:00 in the morning, his watch, round electric clock, and digital clock all agree. When he returns in the afternoon, his watch gives the time as 3:20, his round electric clock reads 2:40, and his digital clock shows 1:30.

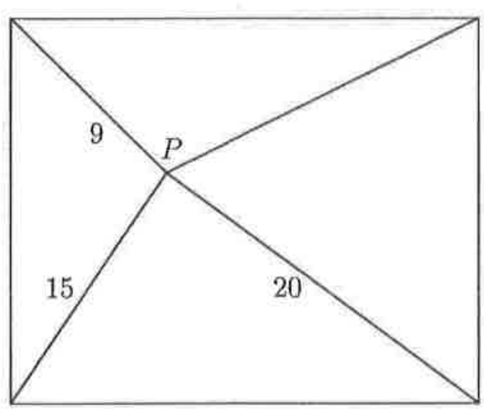
At what time did the power go off, and when did the power come back on?

PROBLEM 3: SWIMMING LAPS

Two swimmers, starting from opposite ends of the pool at the same time, swim laps. It takes the first swimmer 30 seconds to go from one end of the pool to the other and takes the other swimmer 45 seconds. If they swim for 12 minutes, how many time do they pass each other? (Count passes if the swimmers are going in the same direction, opposite directions, or turning around.)

PROBLEM 4: DISTANCES FROM THREE CORNERS OF A RECTANGLE

The distance of the point P from each of the three corners of the rectangle is as given. What is the distance of P from the fourth corner?



PROBLEM 5: TWO BICYCLISTS

The distance from the town of Poole to the town of Robards is 18 miles. Two bicyclists leave Poole at the same time, both heading to Robards, where each will turn around and return to Poole. Cyclist A has average speed that is 4 miles per hour faster than cyclist B.

Cyclist A gets to Robards and turns around heading back to Poole. He meets cyclist B 3 miles outside of Robards.

What is the average speed of each cyclist?

PROBLEM 6: SPANISH ADDITION

$$\begin{array}{r} \mathrm{T\ R\ E\ S} \\ \mathrm{S\ E\ I\ S} \\ \hline \mathrm{N\ U\ E\ V\ E} \end{array}$$

The addition at the left is correct in Spanish, since *tres* is three, *seis* is six, and *nueve* is nine. It will still be correct if the letters are replaced by some the digits 1, 2, 3, 4, 5, 6, 7, 8, 9. Assume that different letters represent different digits. Find all possible additions having this pattern.

PROBLEM 7: ATTACKING QUEENS

Recall that in chess a queen attacks any square that is on a straight line — horizontally, vertically, or diagonally — from the square on which the queen stands. Regard the queen as also attacking the square which she occupies. What is the smallest number of queens required to attack all of the squares of a 6×6 chessboard?

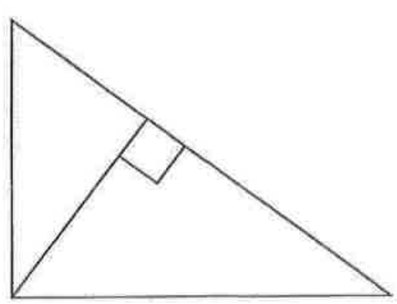
PROBLEM 8: TRIPLED VOLUMES

A 1×2 rectangle becomes a 2×3 rectangle if the length of each side is increased by 1. Then, the area triples from 2 to 6.

Find the dimensions of all rectangular boxes with edges of integer length so that if the length of each edge increases by 1, the volume triples.

PROBLEM 9: TRIANGLES FROM PERIMETER AND HEIGHT

A right triangle has perimeter 390 and height 60 measured by a perpindicular from the right angle to the hypotenuse, as shown.



Find the lengths of its sides.

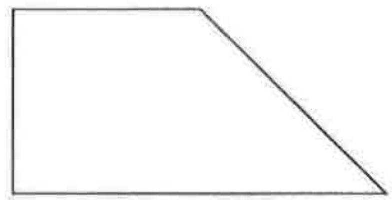
PROBLEM 10: CHANGING SQUARES

The square, S, of a non-zero positive integer remains a perfect square if it is increased by 455 and if it is increased by 2640.

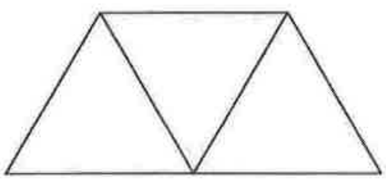
Find all possibilities for the square, S.

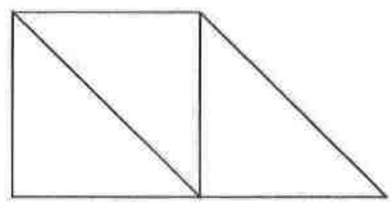
PROBLEM 11: DISSECTING TRAPEZOIDS

Two trapezoids are shown below.



The first is composed of three equilateral triangles, as shown below. The second is found by cutting a square into two triangles and placing a triangle congruent to one of the two triangles on the side of the square, also as shown.

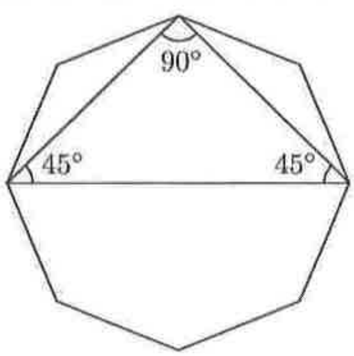


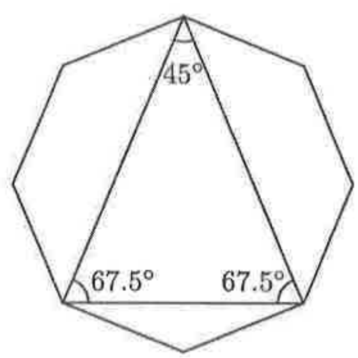


Dissect each trapezoid into four congruent trapezoids.

PROBLEM 12: TRIANGLES IN A NONAGON

There are two different ways to form a triangle using three of the vertices of a regular octagon so that the octagon and triangle have no common edge. The first triangle has angles 90°, 45°, 45°, as shown, and the second angles 45°, 67.5°, 67.5°, also shown.



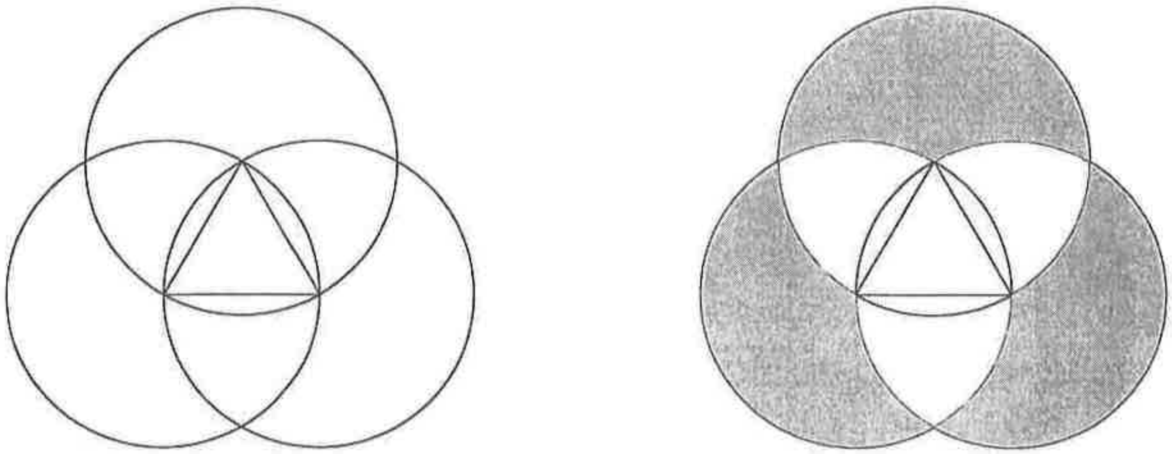


Determine all of the different ways to form a triangle using three of the vertices of a regular nonagon so that the triangle and nonagon have no common edge, and find the angles of these triangles, as above. (Recall that a regular nonagon is a nine sided polygon with all edges of the same length and all internal angles of the same size.)

PROBLEM 13: A SHADED AREA

The central triangle is equilateral with sides of length 1. Each of the circles has radius 1 and has center on one of the vertices of the equilateral triangle.

In the picture on the right, some of the area has been shaded. Find the shaded area.



PROBLEM 14: DIOPHANTINE EQUATIONS AND THE FIBONACCI SEQUENCE

A Diophantine equation is an equation whose solutions are required to be integers. A well-known example is $x^2 + y^2 = z^2$, which has solutions $(x, y, z) = (3, 4, 5)$ and (5,12,13). Until fairly recently one of the most famous unsolved problems in mathematics (known as Fermat's Last Theorem) was to show that the equation $x^n + y^n = z^n$ has no integer solutions (x, y, z) with x, y, z non-zero and n an integer with $n \geq 3$. This was proved by Andrew Wiles in 1993.

In this problem, we consider the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$. It follows easily that if any of the numbers x, y, or z is 0, then $xyz = 0$ and $x = y = z = 0$.

(a) Verify that if $(x, y, z) = (a, b, c)$ is a solution, so also are $(x, y, z) = (-a, -b, c), (-a, b, -c)$, and $(a, -b, -c)$. Explain why this implies that we may assume that $a > 0$, $b > 0$, $c > 0$.

(b) The Fibonacci sequence $f_1, f_2, f_3, f_4, \ldots, f_n, \ldots$ is defined by $f_1 = 1$, $f_2 = 1$, and if $n \geq 1$, by the recursion relation $f_{n+2} = f_{n+1} + f_n$. In other words, after the first two terms, each term of the sequence is the sum of the two previous terms. The resulting sequence is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 \ldots.$$

A new sequence $g_1, g_2, g_3, g_4, \ldots, g_m, \ldots$ is found by deleting the 2nd, 4th, 6th, 8th, 10th terms of the Fibonaccii sequence to obtain $1, 2, 5, 13, 34, 89, 233, 610 \ldots.$

Since these are the terms: $f_1, f_3, f_5, f_7, f_9, \ldots$, it follows that $g_m = f_{2m-1}$ for $m \geq 1$.

Find constants A, B so that, if $m \geq 1$, the terms g_m can be calculated from the recursion relation $g_{m+2} = Ag_{m+1} + Bg_m$.

(c) Verify that $(x, y, z) = (1, 1, 2), (1, 2, 5), (1, 5, 13), (1, 13, 34)$ are all solutions of $x^2 + y^2 + z^2 = 3xyz$.

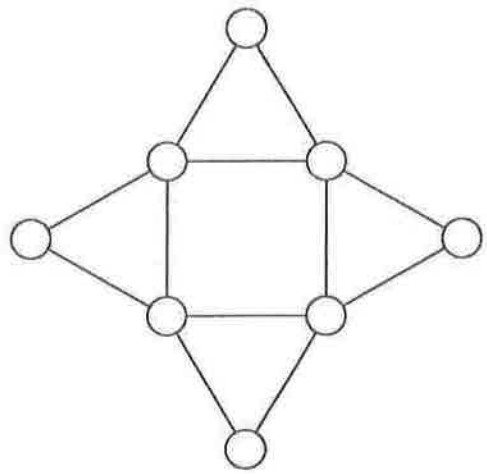
(d) Verify that if $g_1, g_2, g_3, g_4, \ldots, g_m, \ldots$ is the sequence of (b), then $(1, g_m, g_{m+1})$ is a solution of $x^2 + y^2 + z^2 = 3xyz$.

(e) There is a sequence $h_1, h_2, h_3, h_4, \ldots, h_m, \ldots$ with $h_1 = 5$, $h_2 = 29$ such that for all $m \geq 1$, $h_m < h_{m+1}$ and $(2, h_m, h_{m+1})$ is a solution of $x^2 + y^2 + z^2 = 3xyz$. Find h_3, h_4, h_5, h_6, h_7.

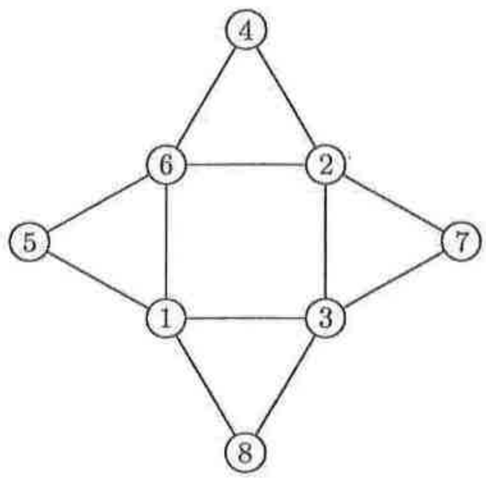
(f) As in (c) above, there are numbers U, V so that $h_{m+2} = Uh_{m+1} + Vh_m$ for all $m \geq 1$. Find U and V.

PROBLEM 15: NUMBERING A FOUR-POINTED STAR

The diagram represents a planar graph. The small circles are called the vertices of the graph and the lines are called the edges. The connected regions into which the graph divides the plane are called the faces. For this graph there are 8 vertices, 12 edges, and 6 faces (including the outside region, called the outer face).



The objective of the problem is to number the vertices 1,2,3,4,5,6,7,8 so that the sum of the numbers in each triangular face is the same number T. In the numbering below the number T is 12.



Find all of the possible values of T so that the graph can be numbered with $1, \ldots, 8$ with the sum of the numbers in each triangular face the same number T. For each value of T give an expicit numbering to show that it is possible.